Minimal sublinear functions, recessive sets and applications to Cut Generating Functions

Alberto Zaffaroni

Università di Modena e Reggio Emilia

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Motivations

$$X = \{x \in \mathbb{R}^p_+ : \ Rx \in S\}, \qquad R = \left[r^1 | \dots | r^p\right]$$

$$S \subset \mathbb{R}^q \text{ closed}, \quad 0 \notin S \implies 0 \notin \operatorname{cl}\operatorname{conv} X$$

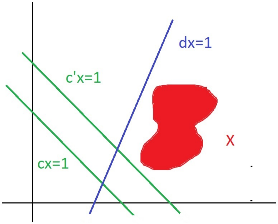
Definition

cut:

$c \in \mathbb{R}^p : \ c^T x \geq 1, \ \forall x \in X;$

dominant cut:

$c' \leq c \ \wedge \ c' \text{ cut} \ \Rightarrow \ c \text{ cut};$



Cut Generating Function

$\rho : \mathbb{R}^q \to \mathbb{R}$, sublinear, $\ \rho(r^i) = c_i,$

$\sum_{i=1}^p \rho(r^i) x_i \geq 1, \quad \forall x = (x_1, ..., x_p) \in X$

Cut Generating Functions

Main Reference

Conforti M., Cornuéjols G., Daniilidis A., Lemaréchal C., Malick J.:

Cut Generating Functions and S-free sets, M.O.R., 2015.

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CGF's and S-free sets

A sublinear $\rho : \mathbb{R}^q \to \mathbb{R}$ is a CGF for S if and only if $V = [\rho \leq 1]$ is S-free:

$$\text{int } V \cap S = [\rho < 1] \cap S = \emptyset.$$

Sublinear functions as representations

Given $\rho : \mathbb{R}^q \to \mathbb{R}$ sublinear, then $V = [\rho \leq 1]$ is a closed, convex neighbourhood of 0. And ρ represents V if $V = [\rho \leq 1]$

Minkowski gauge

Given $V \subset \mathbb{R}^q$ a closed, convex neighbourhood of 0, then

$$\mu_V(v) = \inf\{t > 0 : \; v \in tV\}$$

is a (sublinear, continuous) representation of V (the greatest!) Moreover

$$\mu_V(x) = \sup\{g^T x : \; g \in V^\circ\} = \sigma_{V^\circ}(x)$$

Minimal representation of V (Basu et al. 2010, Zaffaroni 2013)

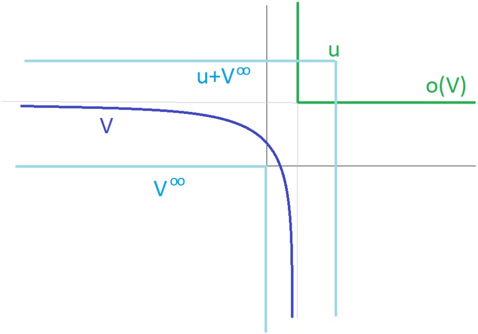
There exists a least representation $\gamma_V : \mathbb{R}^q \to \mathbb{R}$.
In both cases γ_V is the support function of a special subset $V^\bullet$ of V° (least prepolar of V).

The least prepolar

$$V^\bullet \stackrel{B}{=} \mathrm{cl}\,\mathrm{conv}\,\{g \in V^\circ : \exists \bar{v} \in V,\ g^T \bar{v} = 1\} \stackrel{Z}{=} V^\circ \cap (o(V))^\oplus$$

$$W^\oplus = \{g \in \mathbb{R}^q : g^T w \geq 1,\ \forall w \in W\}$$ reverse polar

$$o(V) = \{u \in \mathbb{R}^q : V \subseteq u + V^\infty\}$$ recession bounds

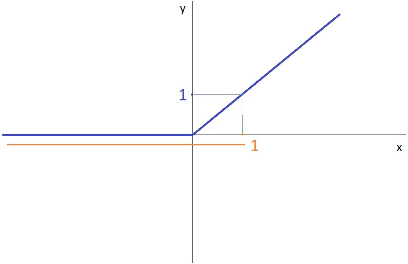


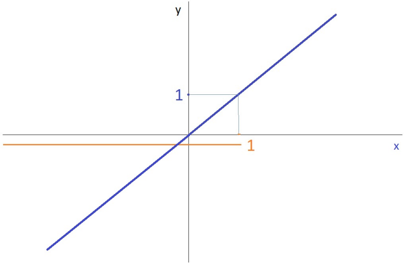
Sublinear functions as representations

Given $V = (-\infty, 1] \subset \mathbb{R}$, we have

$$\mu_V(x) = \sigma_{V^\circ} = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \gamma_V(x) = \sigma_{V^\bullet} = x$$

$$C^\circ = [0, 1] \qquad V^\bullet = \{1\}$$





Minimal CGF's

Definition

A sublinear function $\rho : \mathbb{R}^q \to \mathbb{R}$ is a minimal CGF if it is minimal among all sublinear functions which represent S-free sets.

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a) If ρ is a minimal CGF, then it is the least representation of $V = [\rho \leq 1]$;
b) If V is a maximal, S-free, closed, convex neighbourhood of 0, then γ_V is a minimal CGF.
c) If ρ is a minimal CGF, then $V = [\rho \leq 1]$ is asymptotically maximal, i.e. $V \subseteq W$, int $W \cap S = \emptyset$, then $W^\infty = V^\infty$.

Main goals

1) Recession minimality

A sublinear function $\rho : \mathbb{R}^q \to \mathbb{R}$ is recession minimal, if it is minimal among sublinear functions ρ' with $[\rho' \leq 0] = [\rho \leq 0]$.

Here the set S is not considered.

Main goals

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Three stages for goal 1

1a - Sublinearity by lower level sets;

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1a - Sublinearity by lower level sets;

1b - Larger sublevels and recession bounds;

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Three stages for goal 1

1a - Sublinearity by lower level sets;

1b - Larger sublevels and recession bounds;

1c - Recession hull, recessive sets and recession minimality.

Sublinearity by lower level sets

Consider $q : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, positively homogeneous.
Let $L^+ = [q \leq 1]$ and $L^- = [q \leq -1]$, with $L^- \subseteq L^+$.
The pair (L^+, L^-) completely characterize q (the other sublevels are homotetic).
If q is also quasiconvex and lower semicontinuous, then L^+ is closed, convex, radiant, and L^- is closed, convex, coradiant.

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Theorem

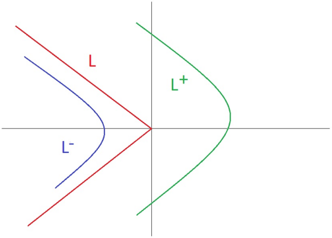
Under the above assumptions, then q is sublinear and continuous provided either $L^- = \emptyset$, or:

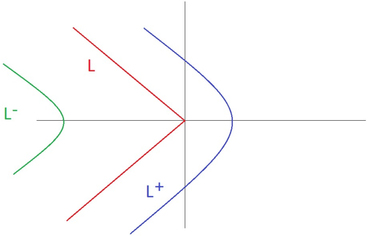
1. $0 \in \mathit{int}\, L^+$ *and* $0 \in \mathit{int}\, o(L^-)$ *(Lipschitz continuity);*
2. $(L^+)^\infty = (L^-)^\infty = L \equiv [q \leq 0]$;
3. *(balancing)*

$$L^+ + L^- \subseteq L.$$

Sublinearity by lower level sets

Balancing of sublevels





A quasiconvex function

A sublinear function

Larger sublevels and recession bounds

Given L^+ we look for the largest sublevel L^-_{max} in order that $L^+ + L^-_{max} \subseteq L$

$$
\begin{aligned}
L^-_{max} &= \{u \in \mathbb{R}^n : \ u + L^+ \subseteq L\} = L \overset{\star}{-} L^+ \\
&= \{u \in \mathbb{R}^n : \ L^+ \subseteq L - u\} \\
&= -o(L^+)
\end{aligned}
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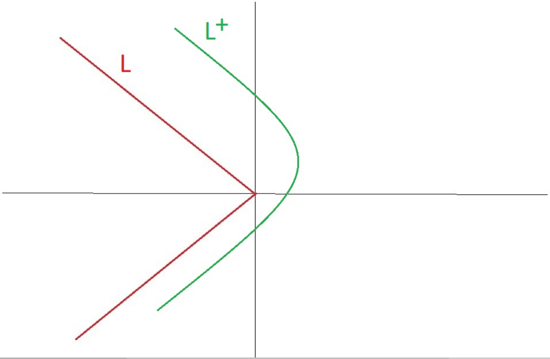

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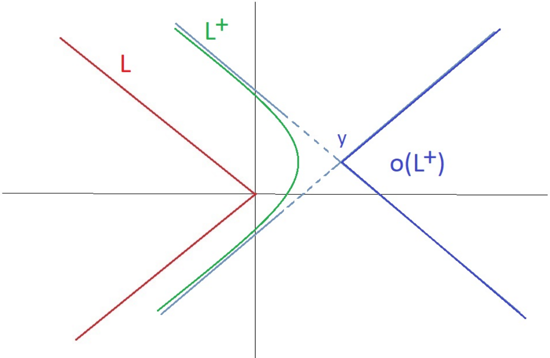

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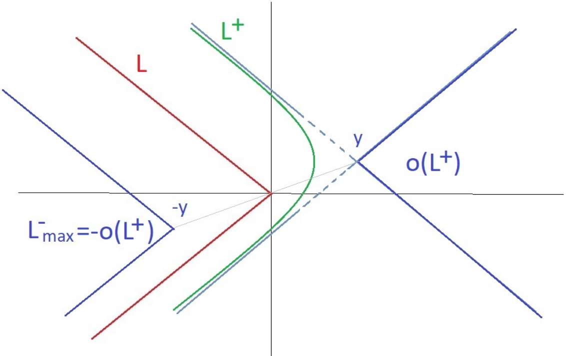

Larger sublevels and recession bounds

Simmetrically: given $L^- \neq \emptyset$ find L^+_{max} such that $L^+_{max} + L^- \subseteq L$. It holds

$$L^+_{max} = \{u \in \mathbb{R}^n : \ u + L^- \subseteq L\} = L \stackrel{\star}{-} L^- = -o(L^-).$$

Larger sublevels and recession bounds

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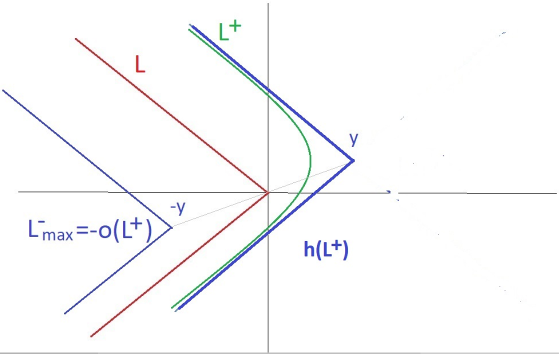
Two-steps procedure: start from L^+, find $L^-_{max} = -o(L^+)$, and then

$$L^+_{max} = -o(L^-_{max}) = -o(-o(L^+)) = o(o(L^+)) \equiv h(L^+).$$



Recession hull, recessive sets, recession minimality

Given $V \subset \mathbb{R}^n$ we call recession hull of V the set

$$h(V) = o(o(V)) = \bigcap_{z \in o(V)} z + V^{\infty}.$$

The set V is recessive if $V = h(V)$.

Theorem

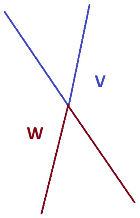
The sublinear function $q : \mathbb{R}^n \to \mathbb{R}$ is recession minimal if and only if $[q \leq 1] = L^+$ is recessive and $L^- = -o(L^+)$.

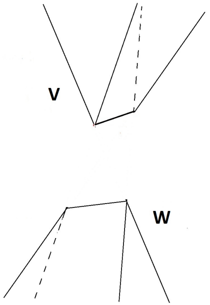
Recessive pairs and Dedekind cuts

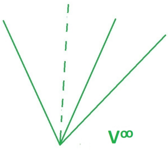
Recessive pairs: $(V, W) : V = o(W)$ and $W = o(V)$ (so that $V = h(V)$).

$$\text{It holds } o(V) = \bigcap_{v \in V} v - V^{\infty} = \text{ set of lower bounds of } V \text{ w.r.t. } V^{\infty}$$

Dedekind cuts in a partially ordered space (X, K): a pair (U, L) such that U is the set of upper bounds for L, and L is the set of lower bounds for U. [Ernst and Zaffaroni 2017, 2018]







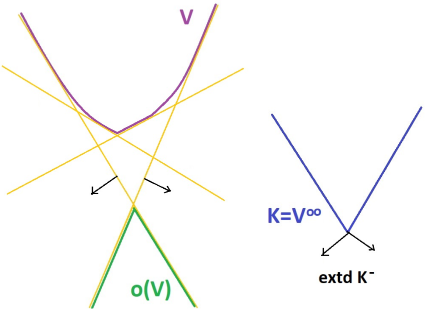
Recession hull, recessive sets

1. Supporting halfspaces
2. Extreme halfpaces
3. Support function
4. Polar sets

Supporting halfspaces

$$o(V) = \{z \in \mathbb{R}^n : g^T z \geq g^T v, \forall v \in V, g \in K^-\}$$
$$= \{z \in \mathbb{R}^n : g^T z \geq \sigma_V(g), \forall g \in K^-\}$$

where $\sigma_V(g) = \sup_{v \in V} g^T v$ is the support function of V.



Extremal halfspaces

Suppose that $K = V^\infty$ is polyhedral, i.e.

$$V^\infty = \{u \in \mathbb{R}^n : \; g^T u \leq 0, \; g \in E\}$$

where $E = \{g_1, g_2, ..., g_m\} = \operatorname{extd}(K^-)$ is finite (and minimal w.r.t. inclusion).

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Then

$$h(V) = \{w \in \mathbb{R}^n : \; g^T w \leq \sigma_V(g), \, \forall g \in E\}.$$

Morever for every function $\tau : E \to \mathbb{R}$ we can obtain a recessive pair (V, W), with $V^\infty = K$, by this formula:

$$V = \{z \in \mathbb{R}^n : \; g^T z \leq \tau(g)\} \qquad W = \{z \in \mathbb{R}^n : \; g^T z \geq \tau(g)\}$$

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These characterizations fails if V^∞ is not polyhedral.

Counterexample

$K = \{(r, s, t) \in \mathbb{R}^3 : \ t \geq \sqrt{r^2 + s^2}\}$

$h = (1, 0, 1) \in \text{extd}\,(K^+)$

$H = \{z \in \mathbb{R}^3 : \ h^T z \leq -1\}$

$V = -K \cap H$

$V^\infty = -K$

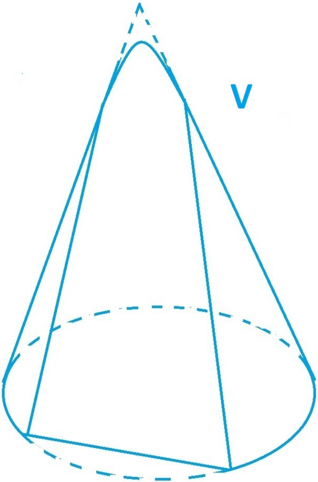
$B = \{k \in K^+ : \ k^T e = 1\}$,

with $e = (0, 0, 1) \in \text{int}\, K$

$E = \text{ext}\, B$

$h(V) = -K \neq V$ despite

$V = \{w \in \mathbb{R}^n : \ g^T w \leq \sigma_V(g), \forall g \in E\}$



Recession hull, recessive sets

In the example above $\sigma_V(h) = -1$ and $\sigma_V(g) = 0$, for all $g \in E \setminus \{h\}$. Hence σ_V is lower semicontinuous, but not normal lower semicontinuous.

Recession hull, recessive sets

In the example above $\sigma_V(h) = -1$ and $\sigma_V(g) = 0$, for all $g \in E \setminus \{h\}$. Hence σ_V is lower semicontinuous, but not normal lower semicontinuous.

Theorem

Let V be convex, $\operatorname{int} V^\infty \neq \emptyset$, and let E be the set of extreme points of a base B of $(V^\infty)^-$. It holds

$$o(V) = \{z \in \mathbb{R}^n : \; g^T z \geq s_V(g), \; \forall g \in E\}$$
$$h(V) = \{z \in \mathbb{R}^n : \; g^T z \leq s_V(g), \; \forall g \in E\}.$$

where s_V is the normal l.s.c. regularization of σ_V on E.

Support function

If V is the traslate of a convex cone, i.e. $V = y + V^\infty$ for some $y \in \mathbb{R}^n$, then it holds

$$\sigma_V(h) = \begin{cases} \langle y, h \rangle & \text{if } h \in (V^\infty)^- \\ +\infty & \text{otherwise} \end{cases}$$

that is σ_V is linear on its effective domain, and its graph is flat there.

Theorem

Let V be convex, with $\operatorname{int} V^\infty \neq \emptyset$. Then V is recessive if and only if its support function σ_V is normal lower semicontinuous, with $\operatorname{dom} \sigma_V = (V^\infty)^*$ and satisfies the following condition:

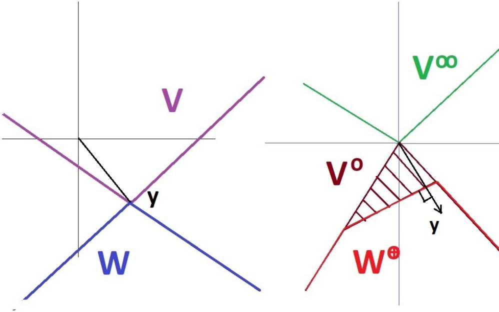
$$g^T y \leq \sigma_V(g), \quad \forall g \in E \quad \Longrightarrow \quad h^T y \leq \sigma_V(h), \quad \forall h \in (V^\infty)^-$$

Thus if V is recessive then the graph of its support function is in some way as flat as possible.

Polar sets

If $V = y + V^\infty$, and $y \in -\text{int}\, V^\infty$, then it holds
$V^\circ = \{g \in (V^\infty)^- : \langle y, g\rangle \leq 1\}$.

Thus V° is the smallest convex radiant set compatible with the values along extreme directions of the polar cone $(V^\infty)^-$.



Theorem

Let (V, W) be a recessive pair, with $0 \in int\, V$. Then

$$V^\circ = cl\, conv\{\ell \in extd\,(V^\infty)^- : \sigma_V(\ell) \leq 1\},$$

$$W^\oplus = cl\, conv\{\ell \in extd\,(V^\infty)^- : \iota_V(\ell) \geq 1\}.$$

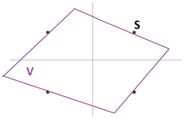
Minimal CGF

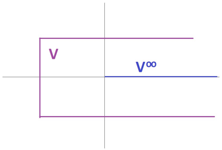
Only one representation, the Minkowski gauge.

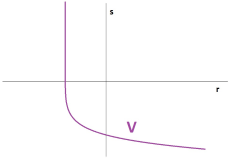
1. V is bounded;
2. V unbounded with $\operatorname{int} V^\infty = \emptyset$;
3. $\operatorname{int} V^\infty \neq \emptyset$, but $o(V) = \emptyset$.

Interesting cases:
$\operatorname{int} V^\infty \neq \emptyset$ and $o(V) \neq \emptyset$.

In some situations it is known that ρ is a minimal CGF if and only if $V(\rho)$ is a maximal S-free convex neighbourhood of 0, and $\rho = \gamma_V$. Tipically when S is (a subset of) $\mathbb{Z}^q - b$.







Minimal Cut Generating Functions

Necessary conditions, sufficient conditions

a) If ρ is a minimal CGF, then it is the least representation of $V = [\rho \leq 1]$;

b) If V is a maximal, S-free, closed, convex neighbourhood of 0, then γ_V is a minimal CGF.

c) If ρ is a minimal CGF, then $V = [\rho \leq 1]$ is asymptotically maximal, i.e. $V \subseteq W$, $\operatorname{int} W \cap S = \emptyset$, then $W^\infty = V^\infty$.

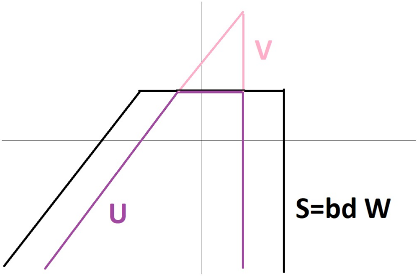
Theorem

The sublinear function $\rho : \mathbb{R}^n \to \mathbb{R}$ is a minimal CGF if and only if it is the least gauge of a (convex neighbourhood of 0) V such that V is asymptotically maximal and V is maximal S-free in $h(V)$.

Minimal CGF

Theorem

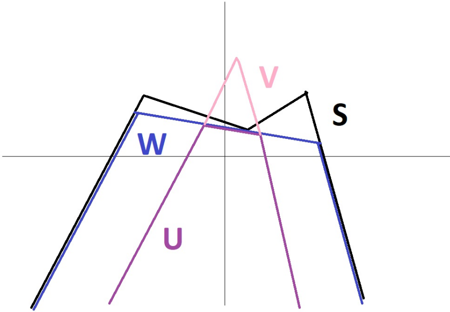
Suppose that there exists only one maximal S-free convex neighbourhood W of 0. And that $\operatorname{int} W^\infty \neq \emptyset$ and $o(W) \neq \emptyset$.
Then ρ is a minimal CGF if and only if $\rho = \gamma_U$, $U = W \cap V$, and V is a recessive neighbourhood of 0, with $V^\infty = W^\infty$.



Minimal CGF - 2

Theorem

Suppose that all maximal S-free convex neighbourhood W of 0 have the same (solid!) recession cone K. Then ρ is a minimal CGF if and only if $\rho = \gamma_V$, V is a neighbourhood of 0, which is maximal S-free in $h(V)$, with $V^\infty = K$.



Minimal CGF

In order to get rid of the assumption of asymptotic maximality, more general cases should be considered individually. For their analysis the following cone is relevant:

$$T = \{d \in \mathbb{R}^q : \mathbb{R}_+ d \cap S = \emptyset\}$$

If T has a maximal convex S-free component which is closed, with nonempty interior, it can be used as the recession cone of some set $U = W \cap V$, W maximal, V recessive, whose least representation is minimal.