On σ-convexity and σ-monotonicity

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International Workshop Variational Analysis and Optimization II: May 30-31, 2024, Milan, Catholic University of Milan

Outline

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- Subdifferential and σ-Subdifferential
- conjugate and (σ,y)-conjugate
- Monotone and σ-Monotone operators

Convex functions

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The function f is called *proper* if $\operatorname{dom} f \neq \emptyset$. In addition, f is said to be *convex* when for all $x, y \in X$ and for each $t \in [0, 1]$,

$$f\left((1-t)x + ty\right) \leq (1-t) f(x) + t f(y).$$

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If $f(x) \in \mathbb{R}$, then the *subdifferential* of f at x is denoted by $\partial f(x)$ and is defined as the set of all $x^* \in X^*$ satisfying

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We say that f is subdifferentiable at x if $\partial f(x) \neq \emptyset$.

Definition (M.H.A, Roohi, 2017)

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a map σ form dom f to $\mathbb{R}_+$, we say that f is σ-*convex* if

$$f\left(tx + (1-t)\,y\right) \leq tf\,(x) + (1-t)\,f\,(y) + t\,(1-t)\min\{\sigma\,(x)\,, \sigma\,(y)\}||x-y|| \qquad (1)$$

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There are σ-convex functions which are not ε-convex for any $\varepsilon \ge 0$, as shown in the following example.

Example (M.H.A, Roohi, 2017)

Consider the functions $\varphi, f, \sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x\sin^2 x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

$$\sigma(x) = \max\left\{\varphi(x), \max_{z \leq x} \varphi(z) - \varphi(x)\right\}$$

$$f(x) = \int_0^x \varphi(t)dt.$$

This function f is σ-convex, but it is not ϵ-convex for any $\epsilon > 0$.

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Note that if f is a σ-convex function, then $\operatorname{dom} f$ is a convex set.

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Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = -\|x\|$. Then f is σ-convex for $\sigma \equiv 2$, but it does not have an affine minorant.

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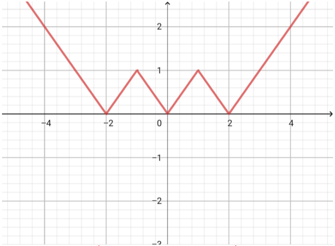
Example (M.H.A, Hosseinabadi, 2023)

Fixed $a \in X$ and define $f_0(x) = \big|\|x\| - \|a\|\big|$. For each $n \in \mathbb{N}$, define f_n recursively by $f_n(x) = \big|f_{n-1}(x) - \|a\|\big|$. Then f_n for all $n \in \mathbb{N} \cup \{0\}$ is σ-convex with $\sigma \equiv 2$, and $y = 0$ is an affine minorant of it.



$$\left|\,\big||x|-1\big|-1\,\right|$$
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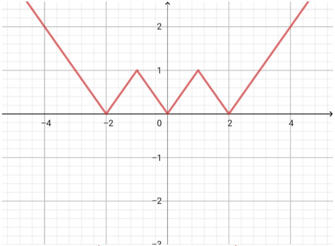

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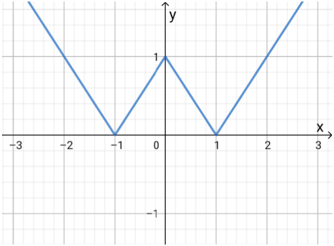

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Elementary properties

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Proposition

(i) If f *is* σ*-convex and* $\sigma \leq \sigma'$*, then* f *is* σ'*-convex.*
(ii) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a function. Then* f *is* σ*-convex if and only if for all* $x, y \in X$*, and* $t \in]0, 1[$,

$$f\left(tx + (1-t)\,y\right) \leq tf\,(x) + (1-t)\,f\,(y) + t\,(1-t)\,\sigma\,(x)\,||x-y|| \qquad (2)$$

(iii) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a function. Then* f *is convex if and only if it is* σ*-convex for every* $\sigma : \operatorname{dom} f \to \mathbb{R}_+$.

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we define the map $\sigma_f : \operatorname{dom} f \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\sigma_f(x) = \inf\{a \in \mathbb{R}_+ : \frac{f\left(tx + (1-t)y\right) - tf(x) - (1-t)f(y)}{t(1-t)} \leq a\|x-y\|, \forall y \in \operatorname{dom} f, t \in]0,1[\}.$$

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It should be noticed that if f is σ'-convex for some $\sigma' : \operatorname{dom} f \to \mathbb{R}_+$, then

$$\sigma_f = \inf\{\sigma : f \text{ is } \sigma\text{-convex}\}. \tag{3}$$

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In this case, σ_f is finite and f is σ_f-convex. Note that σ_f is the minimal σ such that f is σ-convex.

Explicit formula

Proposition (M.H.A, 2020)

Suppose that f is σ-convex for some σ. Then

$$\sigma_f(x) = \max\left\{0, \sup_{t\in]0,1[} \sup_{y\in \operatorname{dom} f\setminus\{x\}} \frac{f\left(tx+(1-t)y\right) - tf(x) - (1-t)f(y)}{t(1-t)\|x-y\|}\right\}. \quad (4)$$

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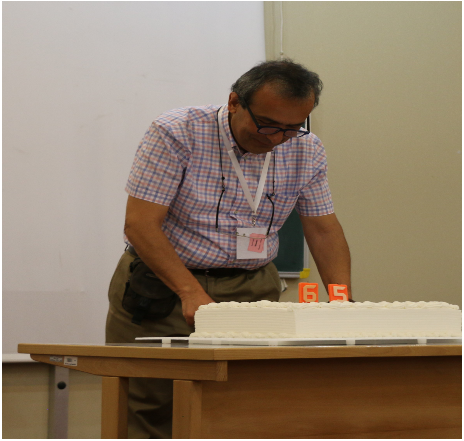
Property B

We introduce the following assumption:



We say that the function σ has the property B, if for every $x \in \operatorname{int} \operatorname{dom} f$ and every $\varepsilon > 0$ sufficiently small, σ is bounded on the sphere $S(x, \varepsilon) = \{y \in X : \|x - y\| = \varepsilon\}$.

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Theorem (M.H.A, 2020)

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a σ-convex function. Assume that f is locally bounded from above in the interior of its domain. If σ satisfies property B, then f is locally Lipschitz in the interior of its domain.

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Corollary

Every proper, σ-convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its domain.

Clarke-Rockafellar Directional Derivative

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For a proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ the Clarke-Rockafellar generalized directional derivative at x in a direction $z \in X$ is defined by

$$f^{\uparrow}(x, z) = \sup_{\delta > 0} \limsup_{(y,\alpha) \xrightarrow{f} x, \lambda \searrow 0} \inf_{u \in B(z,\delta)} \frac{f(y + \lambda u) - \alpha}{\lambda}$$

where $(y, \alpha) \xrightarrow{f} x$ means that $y \to x, \alpha \to f(x)$ and $\alpha \geq f(y)$.

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where $(y,\alpha) \xrightarrow{f} x$ means that $y \to x, \alpha \to f(x)$ and $\alpha \geq f(y)$.
If f is lsc at x, the above definition coincides with

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{y\xrightarrow{f}x,\lambda\searrow 0} \inf_{u\in B(z,\delta)} \frac{f(y+\lambda u)-f(y)}{\lambda}.$$

Here, $y \xrightarrow{f} x$ means that $y \to x$ and $f(y) \to f(x)$.

Clarke-Rockafellar Subdifferential

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$$\partial^{CR} f(x) = \left\{ x^* \in X^* : \langle x^*, z \rangle \leq f^{\uparrow}(x, z) \quad \forall z \in X \right\}.$$

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In the following we introduce the notion of σ-subdifferential.

Definition (M.H.A, 2020)

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper function. The σ-subdifferential of f is the multivalued operator $\partial^{\sigma} f : X \to 2^{X^*}$ defined by

$$\partial^{\sigma} f(x) := \left\{ x^* : \langle x^*, z \rangle \leq f(x+z) - f(x) + \min\left\{\sigma(x), \sigma(z+x)\right\} ||z|| \quad \forall z \in X \right\}$$

if $x \in \operatorname{dom} f$; otherwise it is empty.

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It follows from the above definition that $\partial f \subset \partial^{\sigma} f$ and so $D(\partial f) \subset D(\partial^{\sigma} f) \subset \operatorname{dom} f$.

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It follows from the above definition that $\partial f \subset \partial^{\sigma} f$ and so $D(\partial f) \subset D(\partial^{\sigma} f) \subset \operatorname{dom} f$. In the next proposition, we find a relationship between $\partial^{CR} f(x)$ and $\partial^{\sigma} f(x)$.

Proposition (M.H.A, 2020)

Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is lsc and σ-convex. Then $\partial^{CR} f(x) \subset \partial^{\sigma} f(x)$.

Example

Note that the function $f(x) = -|x|$ is σ-convex with $\sigma \equiv 2$. Then $\partial f(0) = \emptyset$, and $\partial^{CR}(f(0)) = [-1, 1]$

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On the other hand, if we take $\sigma' \equiv 4$, then f is σ'-convex and $\partial^{\sigma'} f(0) = [-3, 3]$.
Thus the inclusion in the above proposition can be equality or strict.

Proposition (M.H.A, 2021)

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper and convex function. Suppose that* f *and* $\sigma : \operatorname{dom} f \to \mathbb{R}_+$ *are Lipschitz near* x *and* $f(\cdot) + \sigma(\cdot)\|\cdot - x\|$ *is convex. Then*

$$\partial^{\sigma} f(x) \subset \partial f(x) + \sigma(x) B^*.$$

Proposition (M.H.A, 2021)

Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper σ-convex function. If f is Gateaux differentiable at $x \in X$, then $f'(x) \in \partial^{\sigma} f(x)$ i.e., $\partial^{\sigma} f(x)$ is nonempty.

Proposition (M.H.A, 2021)

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Proposition

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ *is proper. If* $\partial^{\sigma} f(x) \neq \emptyset$ *and* $\limsup_{y \to x} \sigma(y) < \infty$*, then* f *is lsc at* x*.*

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Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper* σ*-convex function and* $x \in \operatorname{dom} f$*. If* $\operatorname{int} \operatorname{dom} f \neq \emptyset$ *and* $x \in \operatorname{bd}(\operatorname{dom} f)$*, then* $\partial^\sigma f(x)$ *is either empty or unbounded.*

Proposition

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper* σ*-convex function and* $x_0 \in \operatorname{dom} f$ *be a local minimizer of* f*. Set*

$$\varphi(x) := f(x) + \min\{\sigma(x), \sigma(x_0)\}||x - x_0||. \tag{5}$$

Then φ *attains its global minimum at* x_0*.*

Proposition (M.H.A., Zanjani, 2024)

Suppose $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *are proper, lsc and* σ*−convex functions and* σ *has the property* $\boldsymbol{B}$*. If* $\bar{x} \in \operatorname{dom} g \cap \operatorname{int}(\operatorname{dom} f)$ *is a local minimum point of the function* $f + g - \langle x^*, \cdot \rangle$ *for all* $x^* \in \partial^{2\sigma}(f+g)(\bar{x})$*, then*

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$$\partial^{\sigma} f(x) + \partial^{\sigma} g(x) = \partial^{2\sigma}(f+g)(x).$$

Sigma-subdifferential of sum

Proposition (M.H.A., Zanjani, 2024)

Suppose $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *are proper, lsc and* σ*−convex functions and* σ *has the property* $\boldsymbol{B}$*. If* $\bar{x} \in \operatorname{dom} g \cap \operatorname{int}(\operatorname{dom} f)$ *is a local minimum point of the function* $f + g - \langle x^*, \cdot \rangle$ *for all* $x^* \in \partial^{2\sigma}(f+g)(\bar{x})$*, then*

$$\partial^{\sigma} f(x) + \partial^{\sigma} g(x) = \partial^{2\sigma}(f+g)(x).$$

Lemma

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper and convex function. Then* $\partial^{\sigma} f(x) = \partial(f + \sigma(\cdot)\|\cdot - x\|)(x)$*. If, in addition,* $f(\cdot) + \sigma(\cdot)\|\cdot - x\|$ *is convex, then* $\partial^{\sigma} f(x) = \partial^{CR}(f + \sigma(\cdot)\|\cdot - x\|)(x)$*.*

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$$\partial^{\sigma} f(x) + \partial^{\sigma} g(x) = \partial^{2\sigma}(f+g)(x).$$

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Theorem

Suppose that $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *are proper, lsc and* σ*-convex functions. Assume that* σ *is Lipschitz,* $\sigma(0) = 0$*,* $x_0 \in \operatorname{int}(\operatorname{dom} f) \cap \operatorname{dom} g$ *and* $\sigma(x_0 + z) = \sigma(z)$ *for all* $z \in X$*. Then*

$$\partial^{\sigma} f(x_0) + \partial^{\sigma} g(x_0) = \partial^{2\sigma}(f+g)(x_0).$$

Definition (M.H.A., 2021)

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a σ-convex function and $y \in X$ is fixed. Then the map $f^*_{\sigma,y} : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

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$$f^*_{\sigma,y}\left(x^*\right) = \sup_{x \in X} \left\{ \langle x^*, x \rangle - f(x) - \sigma(x) \|x - y\| \right\}, \qquad \forall x^* \in X^* \tag{6}$$

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As for the convex case, the function $f^*_{\sigma,y} : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f^{**}_{\sigma,y}(x) = \sup_{x^* \in X^*} \left\{ \langle x^*, x \rangle - f^*_{\sigma,y}(x^*) \right\}, \qquad \forall x \in X$$

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Proposition

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a σ-convex function. Then for every $x \in \operatorname{dom} f$, $x^* \in X^*$,

$$x^* \in \partial^\sigma f(x) \iff f^*_{\sigma,x}(x^*) + f(x) = \langle x^*, x \rangle. \tag{7}$$

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Proposition

Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ *is a* σ*-convex function. Then for every* $x \in \operatorname{dom} f$, $x^* \in X^*$,

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In particular, $\operatorname{gr}\left(\partial^\sigma f\right) \subset \operatorname{dom} f \times \operatorname{dom} f^*_{\sigma,x}$.

Theorem (M.H.A., 2021)

Suppose that $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are lsc. Let f be σ-convex, g be σ'-convex. Then (i) for every $y \in X$ and $x^* \in X^*$ one has

$$(f+g)^*_{\sigma+\sigma',y}\left(x^*\right) \leq \left(f^*_{\sigma,y}\left(\cdot\right) \Box g^*_{\sigma',y}\left(\cdot\right)\right)\left(x^*\right); \tag{8}$$

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(ii) if σ *satisfies property B,* $y \in \operatorname{int}\operatorname{dom} f$ *and it is a local minimum point of the function* $f + g - \langle x^*, \cdot \rangle$*, then the equality holds.*

Let T be a set-valued map from X to X^*. The domain and the graph of T are, respectively, defined by

$$D(T) = \{x \in X : T(x) \neq \emptyset\},$$

$$\operatorname{gr} T = \Big\{(x, x^*) \in X \times X^* : x \in D(T), \text{ and } x^* \in T(x)\Big\}.$$

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For two set-valued operators T and S, we write $T \subseteq S$ if S is an extension of T, i.e., $\operatorname{gr} T \subseteq \operatorname{gr} S$.

We recall that T is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

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Definition

(i) Given an operator $T : X \to 2^{X^*}$ and a map $\sigma : D(T) \to \mathbb{R}_+$, T is said to be *σ-monotone* if for every $x, y \in D(T)$, $x^* \in T(x)$ and $y^* \in T(y)$,

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(ii) A σ-monotone operator T is called *maximal σ-monotone*, if for every operator T' which is σ'-monotone with $\operatorname{gr} T \subseteq \operatorname{gr} T'$ and σ' an extension of σ, one has $T = T'$.

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The operator T is called premonotone, if it is σ-monotone for some σ.

Theorem (Rockafellar for σ-monotonicity, M.H.A., Hadjisavvas, Roohi, 2012)

Suppose that X is a Banach space and $T : X \to 2^{X^*}$ is a premonotone operator. Then T is locally bounded at every point of $\operatorname{int} D(T)$.

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Theorem (Libor Veselý for σ-monotonicity, M.H.A., Hadjisavvas, Roohi, 2012)

Suppose that T is maximal σ-monotone, σ is usc and $x_0 \in \overline{D(T)}$. If T is locally bounded at x_0, then $x_0 \in D(T)$. If in addition $\overline{D(T)}$ is convex, then $x_0 \in$ $\operatorname{int} D(T)$.

We recall that an operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is *bounded on bounded sets* if $\cup_{x \in B} T(x)$ is bounded for all bounded set $B \subset \mathbb{R}^n$.

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For a bounded set $B \subset \mathbb{R}^n$ and any positive $\varepsilon \in \mathbb{R}$, we define $B^\varepsilon \subset \mathbb{R}^n$ as

$$B^\varepsilon = \{x \in \mathbb{R}^n : \mathrm{dist}(x, B) \leq \varepsilon\}. \tag{10}$$

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Theorem (M.H.A. Iusem, Sosa, 2024)

Consider an operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, with convex domain which is bounded on bounded sets and monotone outside some bounded set $B \subset \mathbb{R}^n$. Then T is premonotone, with

$$\sigma(y) = \alpha + \sup_{v \in T(y)} \|v\|,$$

where

$$\alpha = \sup_{u \in T(x), x \in B^\varepsilon} \|u\|.$$

for an arbitrary $\varepsilon > 0$.

Proposition

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be bounded on bounded sets and σ-monotone. Consider a bounded set $B \subset \mathbb{R}^n$ and an operator $\bar{T} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ with convex domain which is bounded on bounded sets and such that $\bar{T}(x) = T(x)$ for all $x \notin B$. Then $\bar{T}$ is $\bar{\sigma}$-monotone, with

$$\bar{\sigma}(y) = \max\{\sigma(y), \widehat{\sigma}(y)\},$$

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Corollary

If $T:\mathbb{R}^n \to \mathbb{R}^n$ is point to point, continuous and monotone outside some bounded set $B \subset \mathbb{R}^n$, then T is σ-monotone, with $\sigma(y) = \alpha + \left\|T(y)\right\|$, where $\alpha = \sup_{x \in B^{\varepsilon}} \left\|T(x)\right\|$.

Corollary

If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree, then p is σ-monotone.

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Conjecture (Iusem Sosa, 2020, JNVA): Every maximal premonotone operator contains a maximal monotone one.

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