A (maybe open) question on indecomposable Banach spaces

C. Zanco (INPS of Italy, former Università degli Studi - Milano, Italy), talk connected with a joint work with V.P. Fonf (former Ben-Gurion University - Beer-Sheva, Israel), S. Lajara (Universidad Complutense - Madrid, Spain), S. Troyanski (Bulgarian Academy of Science - Sofia, Bulgaria)

Infinite-dimensional convexity and geometry of Banach spaces
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A Banach space E is said *indecomposable* (I) if E cannot be obtained as the direct sum of two of its infinite-dimensional closed subsapaces. E is said *hereditarely indecomposable* (HI) whenever each of its infinite-dimensional subspaces is (I).

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Clearly E is (HI) iff the angle between any two closed infinite-dimensional subspaces X and Y of E, i.e. $\mathrm{dist}(S_X, S_Y)$, is 0.

First example

T. Gowers, B. Maurey: *The unconditional basic sequence problem* (1993).

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Consequences on the operator theory

Every (linear bounded) operator acting on a (HI) Banach space may be obtained as a strictly singular perburbation of a multiple of the identity (T. Gowers and B. Maurey: *Banach spaces with small spaces of operators*, 1997). In particular, any such operator must be either strictly singular or Fredholm with index 0. As a consequence, a (HI) space is not isomorphic to any of its proper subspaces, in particular to any of its hyperplanes.

Every (linear bounded) operator acting on a (HI) Banach space may be obtained as a compact perburbation of a multiple of the identity (S. Argyros and R. Haydon, 2011).

Basic facts on density character

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Assume the generalized continuum hypothesis. For every cardinal κ there is an (I) Banach space of density character bigger than κ. In particular, it has no infinite-dimensional complemented subspace of density smaller than κ. The spaces are Banach algebras of the form $C(K)$ with "few operators" where K is compact Hausdorff and connected (P. Koszmider, S. Shelah and M. Świetek, 2016).

More information

Any separable reflexive space is a quotient of a reflexive (HI) space (S. Argyros and T.Raikoftsalis, 2012).

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The Gowers space G without any reflexive subspace and not containing c_0 or ℓ_1 must contain some (HI) space, so (James + Gowers dichotomy theorem) there exist (HI) spaces free of reflexive (infinite-dimensional) subspaces. It seems a still open problem whether G is (HI) or at least (I).

Terminology

Let E be a Banach space, let X be a subspace of E and Z be a subspace of E^* (the dual space to E). We say that Z *is norming for* X if the formula

$$|||x||| = \sup_{f \in B_Z} |f(x)|, \;\; x \in X$$

defines an equivalent norm on X (where B_Z denotes the unit ball of Z). It is clear that if Z is norming for X, then Z is total over X (that is, $X \cap Z_\perp = \{0\}$, where
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Analogously, if X *is norming for* Z (namely, if the image of X through the canonical maping $\pi : E \to E^{**}$ is norming for Z), then X is total over Z (that is, $X^\perp \cap Z = \{0\}$, where
$X^\perp = \{f \in E^* : f(x) = 0 \text{ for every } x \in X\}$).

The two characterizations

A Banach space E is HI *if, and only if, for any closed subspace $X \subset E$ with* $\dim(X) = \infty$ *and any w^*-closed subspace $Z \subset E^*$ such that Z is norming for X, we have* $\operatorname{codim}(Z) < \infty$ (V.D. Milman).

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The key

Proposition

(V. Fonf, S. Lajara, S. Troyanski and C.Z., 2019)

E a Banach space, X a closed subspace of E, Z a w^*-closed subspace of E^*. Then

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Proof.

$\Longrightarrow$ part.

Let $\lambda \in (0,1]$ be a number satisfying
$\sup_{f \in B_Z} |f(x)| \geq \lambda \|x\|$ for every $x \in X$.

Fix $x \in X$ and pick $f \in B_Z$ such that $f(x) \geq \lambda \|x\|/2$. Therefore, for each $y \in Z_\perp$ we have

$$\|x - y\| \geq f(x - y) = f(x) \geq \lambda \|x\|/2.$$

Consider the quotient map $Q : E \to E/Z_\perp$: consequently,

$$\|Qx\| = \inf \{\|x - y\| : \, y \in Z_\perp\} \geq \lambda\|x\|/2,$$

hence the restriction map $Q_{|X}$ is an isomorphic embedding.
It suffices to show that $\inf\{\|x - y\| : \, x \in S_X, \, y \in S_{Z_\perp}\} > 0$.
Assume the contrary: then there exist sequences $(x_n)_n \subset S_X$ and $(y_n)_n \subset S_{Z_\perp}$ such that $\|x_n - y_n\| \to 0$. Thus, $\|Qx_n - Qy_n\| \to 0$ that implies $\|Qx_n\| \to 0$, contradicting the fact that the restriction map $Q_{|X}$ is an isomorphic embedding.

Proof.

$\Longleftarrow$ part.

Set $U = X \oplus Z_\perp$ and let M and N denote the annihilator subspaces of X and $Z_\perp$ relative to U, that is, $M = \left\{f \in U^* : f_{|X} = 0\right\}$ and $N = \left\{g \in U^* : g_{|Z_\perp} = 0\right\}$. Since U is closed, it follows that $U^* = M \oplus N$. In particular, there exists $\alpha > 0$ such that

$$\alpha\left(\|f\| + \|g\|\right) \le \|f + g\| \le \|f\| + \|g\|$$

whenever $f \in M$ and $g \in N$. Choose a vector $x \in X$ with $\|x\| = 1$, take $\varphi \in U^*$ with $\varphi(x) = \|\varphi\| = 1$ and let functionals $f \in M$ and $g \in N$ such that $\varphi = f + g$. It is clear that $g(x) = \varphi(x) = 1$ and, because of the previous inequality, we have $\|g\| \le \alpha^{-1}$. Therefore, the functional $\psi = \alpha g$ belongs to B_N and $\psi(x) \ge \alpha$. Now, let $\widehat{\psi} \in E^*$ be such that $\widehat{\psi}_{|U} = \psi$ and $\|\widehat{\psi}\| = \|\psi\|$. Then, $\widehat{\psi} \in B_Z$ and $\widehat{\psi}(x) \ge \alpha$. Consequently, Z is norming for X.

□

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Let X be an infinite-dimensional closed subspace of E and Z a w^*-closed subspace of E^* which is norming for X. By the Proposition, we have $X \cap Z_\perp = \{0\}$ and $X \oplus Z_\perp$ is closed. E being HI implies $\dim(Z_\perp) < \infty$, so $\operatorname{codim}(Z) < \infty$.

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Conversely, if E is not HI there exist infinite-dimensional closed subspaces X and Y of E such that $X \oplus Y$ is closed in E. Take $Z = Y^\perp$: Z is w^*-closed so, by the Proposition, is norming for X. From $Y = Z_\perp$ we get $\operatorname{codim}(Z) = \dim(Y) = \infty$.

Theorem

(V. Fonf, S. Lajara, S. Troyanski and C.Z., 2019)
E a Banach space, X a closed subspace of E, Z a w^*-closed subspace of E^*. Then

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Proof.

$\Longrightarrow$ part.

Clearly $X \cap Z_\perp = \{0\}$. We claim that the direct sum $X \oplus Z_\perp$ is dense in E. Indeed, since Z is w^*-closed, the adjoint operator of the map $Q_{|X} : X \to E/Z_\perp$ can be identified with the restriction map $q^*_{|Z} : Z \to X^*$. It is clear that $\ker q^*_{|Z} = X^\perp \cap Z$. Bearing in mind that X is total over Z, it follows that $q^*_{|Z}$ is one-to-one. Hence, the operator $Q_{|X}$ has dense range, and using the Hahn-Banach theorem we deduce that the manifold $X \oplus Z_\perp$ is dense in E. On the other hand, as Z is norming for X, Proposition guarantees that $X \oplus Z_\perp$ is closed. Consequently, $E = X \oplus Z_\perp$. □

$\Longleftarrow$ part.

Taking into account that Z is w^*-closed and the sum $X \oplus Z_\perp (= E)$ is closed in E, according to Proposition we have that Z is norming for X. Moreover, it is a standard exercise to get $E^* = X^\perp \oplus (Z_\perp)^\perp = X^\perp \oplus Z$: this implies that X is total over Z.

Getting characterization of I spaces

A Banach space E is I *if, and only if, for every closed subspace $X \subset E$ with* $\dim(X) = \infty$ *and every w^*-closed subspace $Z \subset E^*$ such that Z is norming for X and X is total over Z, we have* $\operatorname{codim}(Z) < \infty$ (V. Fonf, S. Lajara, S. Troyanski and C.Z.).

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Let X be an infinite-dimensional closed subspace of E and Z a w^*-closed subspace of E^* which is norming for X. By the Theorem, we have $X \cap Z_\perp = \{0\}$ and $X \oplus Z_\perp = E$. E being I implies $\dim(Z_\perp) < \infty$, so $\mathrm{codim}(Z) < \infty$.

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Conversely, if E is not I there exist infinite-dimensional closed subspaces X and Y of E such that $X \oplus Y = E$. Take $Z = Y^\perp$: Z is w^*-closed so, by the Theorem, is norming for X. From $Y = Z_\perp$ we get $\operatorname{codim}(Z) = \dim(Y) = \infty$.

Theorem

Corollary (V. Fonf, S. Lajara, S. Troyanski and C.Z., 2019)
Let E be a Banach space, let X be a closed subspace of E and Z be a closed subspace of E^*. If X is reflexive then the following conditions are equivalent:

1. *X is norming for Z and Z is total over X.*
2. *Z is norming for X and X is total over Z.*
3. *X is norming for Z and Z is norming for X.*
4. *Z is w^*-closed and $E = X \oplus Z_\perp$.*
5. *Z is reflexive and $E = X \oplus Z_\perp$.*

Reference

This talk has been based on

V. P. Fonf, S. Lajara, S. Troyanski and C. Zanco, *Norming subspaces of Banach spaces*, Proc. Amer. Math. Soc. **147** (2019), n. 7, 3039-3045.

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$\mathbb{T}$*HANK YOU FOR YOUR ATTENTION*!