Remarks on projected solutions for generalized Nash equilibrium problems

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Preliminaries

Generalized convexity

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be

- *convex* if, for all $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$

$$f(tx + (1 - t)t) \leq tf(x) + (1 - t)f(y);$$

- *quasiconvex* if, for all $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}$$

- *semi strictly quasiconvex* if, it is quasiconvex and for all $x, y \in \mathbb{R}^n$ such that $f(x) < f(y)$ we have

$$f(tx + (1 - t)y) < f(y), \text{ for all } t \in]0, 1[.$$

Clearly, any convex function is semi stricly quasiconvex.

The classical Nash equilibrium problem (NEP)

A **Nash equilibrium problem**, [1], consists of p players.

- Each player i controls the decision variable $x_i \in C_i$ where C_i is a subset of $\mathbb{R}^{n_i}$.
- The "total strategy vector" is x which will be often denoted by

$$x = (x_1, x_2, \ldots, x_i, \ldots, x_p) = (x_i, x_{-i}).$$

- Each player i has an objective function $\theta_i : C = \prod_{i=1}^{p} C_i \to \mathbb{R}$ that depends on all player's strategies, where $n = n_1 + \cdots + n_p$.
- Given the strategies $x_{-i} \in C_{-i}$ of the other players, the aim of player i is to choose a strategy $x_i \in C_i$ such that

$$\theta_i(x_i, x_{-i}) \leq \theta_i(y_i, x_{-i}) \text{ for all } \; y_i \in C_i. \qquad (\text{NEP}(i))$$

- A vector $\hat{x} \in C$ is a **Nash equilibrium** if for any i, $\hat{x}_i$ solves (NEP(i)) associated to $\hat{x}_{-i}$.
- We denote by NEP($\{\theta_i, C_i\}$) the set of Nash equilibria.

The Generalized Nash equilibrium problem (GNEP)

In **the generalized Nash equilibrium problem**

- Each player's strategy must belong to a set identified by the set-valued map $K_i : C \rightrightarrows C_i$ in the sense that the strategy space of player i is $K_i(x)$, which depends on all player's strategies.
- Given the strategy $x_{-i} \in C_{-i}$, player i chooses a strategy $x_i \in C_i$ such that $x_i \in K_i(x_i, x_{-i})$ and

$$\theta_i(x_i, x_{-i}) \leq \theta_i(y_i, x_{-i}) \text{ for all } y_i \in K_i(x_i, x_{-i}). \tag{GNEP(i)}$$

- Thus, a **generalized Nash equilibrium** [2] is a vector $\hat{x} \in C$ such that the strategy $\hat{x}_i$ is a solution of the problem (GNEP(i)) associated to $\hat{x}_{-i}$, for any i.
- We denote by GNEP($\{\theta_i, K_i, C_i\}$) the set of generalized Nash equilibria.

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- Thus, a **generalized Nash equilibrium** [2] is a vector $\hat{x} \in C$ such that the strategy $\hat{x}_i$ is a solution of the problem (GNEP(i)) associated to $\hat{x}_{-i}$, for any i.
- We denote by $\text{GNEP}(\{\theta_i, K_i, C_i\})$ the set of generalized Nash equilibria.

Remark

We notice that:

- Let $\hat{x} \in C$, then $\hat{x} \in \text{GNEP}(\{\theta_i, K_i, C_i\})$ if, and only if, $\hat{x} \in \text{NEP}(\{\theta_i, K_i(\hat{x})\})$.
- the map $K : C \rightrightarrows C$ defined as $K(x) = \prod K_i(x)$ is actually a self-map.

The Generalized Nash equilibrium problem (GNEP)

Theorem (♠)

For each i, $C_i \subset \mathbb{R}^{n_i}$ is compact, convex and non-empty. If for all i, the following hold:

1. *the objective function θ_i is quasiconvex in x_i ,*
2. *the objective function θ_i is continuous,*
3. *the set-valued map K_i is continuous with convex, closed and non-empty values;*

then the set $\mathrm{GNEP}(\{\theta_i, K_i, C_i\})$ has at least one element.

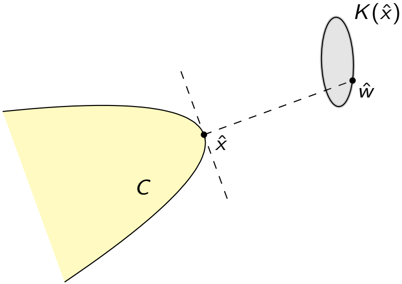
Projected solutions

Projected solutions

- For any i, let $K_i : C \rightrightarrows \mathbb{R}^{n_i}$ be a set-valued map.

- A vector $\hat{x}$ of C is said to be **projected solution** [3] of the generalized Nash equilibrium problem if there exists $\hat{w} \in \mathbb{R}^n$ such that:

  1. $\hat{x} \in P_C(\hat{w})$, that is $\hat{x}$ is a projection of $\hat{w}$ onto C;
  2. $\hat{w} \in \text{NEP}(\{\theta_i, K_i(\hat{x})\})$.



- We denote the set of projected solutions by $\text{PSGNEP}(\{\theta_i, K_i, C_i\})$.

Projected solutions

Such projected solutions depend on the chosen norm.

Example

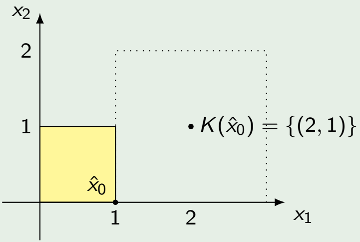
Consider for instance the strategy sets $C_1 = C_2 = [0, 1]$, functions θ_1 and θ_2 defined as

$$\theta_1(x_1, x_2) := (x_1 - x_2)^2 \text{ and } \theta_2(x_1, x_2) := (x_2)^2,$$

and constraint set-valued maps K_1 and K_2 defined as

$$K_1(x_1, x_2) := [2 - x_2, 2] \text{ and } K_2(x_1, x_2) := [1, 2 - x_1].$$

PSGNEP($\{\theta_i, K_i, C_i\}$)	
Euclidean norm	Maximum norm
$\{(1, 1)\}$	$\{(1, s) : \ s \in [0, 1]\}$



Existence results

Theorem

Assume the $\|\cdot\|$ *is a norm in* $\mathbb{R}^n$*, and for each player* i*:*

1. C_i *is convex, closed and non-empty subset of* $\mathbb{R}^{n_i}$*,*
2. K_i *is continuous with compact and non-empty values,*
3. K_i *is* ♠
4. θ_i *is* ♣
5. $\theta_i(\cdot, x_{-i})$ *is* ♦*, for all* x_{-i}*;*

then there exists a projected solution.

	[3] (2016)	[4] (2018)	[5] (2021)	[6] (2023)
C_i		Compactness	Compactness	
$\|\cdot\|$	Euclidean norm	Euclidean norm	any norm	Euclidean norm
K_i ♠	is single-valued or convex-valued with $int(K_i(x)) \neq \emptyset$, for all x	is convex-valued	convex-valued	is convex-valued
θ_i ♣	continuous differentiable	continuity	pseudo-continuity	continuity
♦	convexity	convexity	quasi-convexity	convexity

Pseudo-continuity

A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be **pseudocontinuous** [7] if, for each $x \in \mathbb{R}^n$ the following sets

$$\{y \in \mathbb{R}^n : \ h(y) \leq h(x)\} \text{ and } \{y \in \mathbb{R}^n : \ h(y) \geq h(x)\} \text{ are closed.}$$

Example

Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined as

$$h(x) = \begin{cases} x+1, & x > 0 \\ 0, & x = 0 \\ x-1, & x < 0 \end{cases}.$$

It is not difficult to verify that h is pseudocontinuous but it is not continuous.

The generalized Nash game proposed by Rosen [8]

Let C be a convex and non-empty subset of $\mathbb{R}^n$. For each i and each $x \in C$, we define

$$K_i(x) := \{y_i \in \mathbb{R}^{n_i} : \ (y_i, x_{-i}) \in C\}.$$

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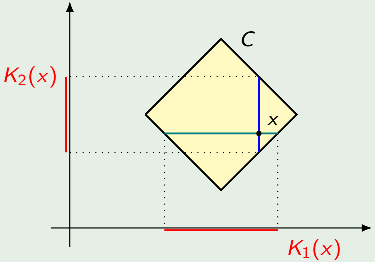
The following example shows that this kind of game could not be reduced to a classical Nash game.

Example

Consider $C \subset \mathbb{R}^2$ as in the following figure:



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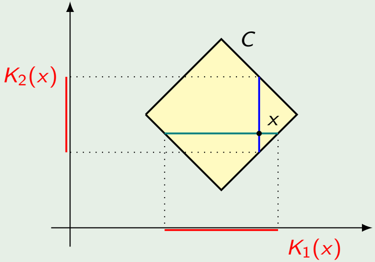

Remark

We observe that the map $K : C \rightrightarrows \mathbb{R}^n$ defined as $K(x) = \prod K_i(x)$ is not a self-map in general.

The generalized Nash game proposed by Rosen

A solution of this Rosen game is a vector $\hat{x} \in C$ such that

$$\hat{x} \in \mathrm{NEP}(\{\theta_i, K_i(\hat{x})\}).$$

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Theorem

Assume that C is a convex, compact and non-empty subset of $\mathbb{R}^n$. If for each i the objective function θ_i is

1. *continuous and*
2. ♠ *with respect to its player's variable,*

then there exists at least a generalized Nash equilibrium.

where ♠ means	Convex	Semi Strictly quasi-convex	Quasi-convex
	Rosen [8] (1965)	Aussel-Dutta [9] (2008)	Bueno-Calderón-C [10] (2023)

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A vector $\hat{x} \in C$ is a projected solution, if there exists $\hat{y}$ such that

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Proposition ([11])

By considering the **Euclidean** *norm, any projected solution is a classical solution.*

Since $\|\hat{y} - \hat{x}\|^2 \leq \|\hat{y} - x\|^2$, for all $x \in C$, and the fact that $x = (\hat{y}_{i_0}, \hat{x}_{-i_0}) \in C$, for all i_0, we deduce

$$\sum \|\hat{y}_i - \hat{x}_i\|^2 \leq \sum_{i \neq i_0} \|\hat{y}_i - \hat{x}_i\|^2.$$

This implies $\|\hat{y}_{i_0} - \hat{x}_{i_0}\|^2 \leq 0$ and consequently $\hat{y}_{i_0} = \hat{x}_{i_0}$. Hence $\hat{y} = \hat{x}$ and the result follows.

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Remark

A natural question: is it possible to consider any norm in the previous result?

Reformulation

The problem of finding projected solutions for GNEPs can be associated to a particular GNEP by adding a new player.

Reformulation

The problem of finding projected solutions for GNEPs can be associated to a particular GNEP by adding a new player.

- For each $i \in M = \{1, 2, \dots, p, p+1\}$, we consider the sets

$$\hat{C}_i = \begin{cases} \mathrm{co}(C_i \cup K_i(C)), & \text{if } i \leq p; \\ C, & \text{if } i = p+1 \end{cases}$$

- As usual $\mathbf{x} = (\mathbf{x}_i, \mathbf{x}_{-i}) \in \hat{C} = \prod \hat{C}_i$. We also write $\mathbf{x}^0$ instead $\mathbf{x}_{-(p+1)}$.
- For each $i \in M$, $\hat{K}_i : \hat{C} \rightrightarrows \hat{C}_i$ and $\hat{\theta}_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are defined as

$$\hat{K}_i(\mathbf{x}) = \begin{cases} K_i(\mathbf{x}_{p+1}), & \text{if } i \leq p \\ C, & \text{if } i = p+1 \end{cases} \text{ and } \hat{\theta}_i(\mathbf{x}) = \begin{cases} \theta_i(\mathbf{x}^0), & \text{if } i \leq p \\ \|\mathbf{x}^0 - \mathbf{x}_{p+1}\|, & \text{if } i = p+1. \end{cases}$$

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Proposition ([11])

The following implications hold:

1. *If* $\hat{\mathbf{x}} \in \mathrm{GNEP}(\{\hat{\theta}_i, \hat{K}_i\})$, *then* $\hat{\mathbf{x}}_{p+1} \in \mathrm{PSGNEP}(\{\theta_i, K_i\})$.
2. *If* $\hat{x} \in \mathrm{PSGNEP}(\{\theta_i, K_i\})$, *then there is* $\hat{y} \in \mathbb{R}^n$ *such that* $\hat{\mathbf{x}} = (\hat{y}, \hat{x}) \in \mathrm{GNEP}(\{\hat{\theta}_i, \hat{K}_i\})$.

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Thank you!!