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“In honor of our colleague and friend Nicolas Hadjisavvas”

Characterizing quasiconvexity of the pointwise infimum of a family of arbitrary translations of quasiconvex functions

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*Join work with:

- Fabián Flores, Departamento de Ingeniería Matemática, Universidad de Concepción.
- Nicolas Hadjisavvas, Department of Product and Systems Design Engineering, University of the Aegean.

Santiago-2014

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Arica-2024

This talk is based on:

- Characterizing quasiconvexity of the pointwise infimum of a family of arbitrary translations of quasiconvex functions, with applications to sums and quasiconvex optimization, *Mathematical Programming* **189** (2021) 315–337 (With F. Flores and N. Hadjisavvas)

- Closedness under addition for families of quasimonotone operators. Optimization, 73(4), (2022) 1267–1284. https://doi.org/10.1080/02331934.2022.2154127 (With F. Flores and N. Hadjisavvas)

The Framework:

X : real Banach space, X^* : continuous dual, $\langle \cdot, \cdot \rangle$ the pairing between X and X^*. For $f : X \to \mathbb{R} \cup \{+\infty\}$,

- $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ (domain).
- $\operatorname{Epi}(f) = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ (epigraph).
- For any $\lambda \in \mathbb{R}$

  $[f \leq \lambda] \doteq \{x \in X : f(x) \leq \lambda\}$ (*sublevel set at height* λ)

  $[f < \lambda] \doteq \{x \in X : f(x) < \lambda\}$ (*strict sublevel set at height* λ)

Motivation: convex functions vs quasiconvex functions

$$f \text{ convex} \Leftrightarrow \text{Epi}(f) \text{ is convex.}$$

$$f \text{ quasiconvex} \Leftrightarrow S_\lambda \text{ is convex}, \forall\, \lambda \in \mathbb{R}.$$

- If f, g convex $\Rightarrow f + g$ convex.
- If f, g quasiconvex $\Rightarrow f + g$ is not in general quasiconvex.

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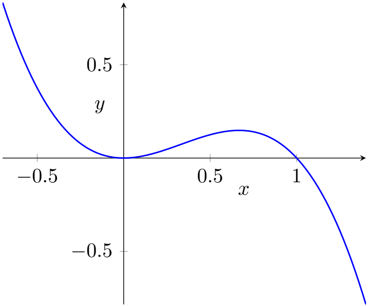
- If f, g quasiconvex $\Rightarrow f + g$ is not in general quasiconvex.

Example

$f(x) = x^2$, $g(x) = -x^3 \Rightarrow (f+g)(x) = x^2 - x^3$, is not a quasiconvex function.



Motivation: convex functions vs quasiconvex functions

Characterization subdifferential

- f convex, lsc: (classical) Subdifferential

f convex $\Leftrightarrow \partial f$ maximal monotone.

- f lsc: Clarke-Rockafellar subdifferential:

$$\partial f(x) = \{x^* \in X : f^{\uparrow}(x,u) \geq \langle x^*, u\rangle, \forall\, u \in X\},\ x \in \mathrm{dom} f. [7]$$

f quasiconvex $\Leftrightarrow \partial f$ quasimonotone. [3]

$$\partial f(x) = \left\{x^* \in X : f^{\uparrow}(x,u) \geq \langle x^*, u\rangle, \forall u \in X\right\},\ \text{if } x \in \mathrm{dom}\, f;$$

$$\partial f(x) = \emptyset, \text{ otherwise.}$$

$$f^{\uparrow}(x,u) = \sup_{\delta>0} \limsup_{\substack{y \to_f x \\ t \searrow 0^+}} \inf_{v \in B(u,\delta)} \frac{f(y+tv) - f(y)}{t},\ \text{for } x \in \mathrm{dom}\, f.$$

How do we preserve quasi-convexity under summation?

Easy example

$f : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex and $g : \mathbb{R} \to \mathbb{R}$ nondecreasing.

$$[g \circ f \leq \lambda] \, , \text{ is convex } \forall \lambda \in \mathbb{R}$$

so, $g \circ f$ quasiconvex and it is easy to show that

$$f + g \circ f \text{ is quasiconvex}$$

.

and in other cases?

Quasiconvex and quasimonotone families

Definition

A family $\mathcal{A}$ of operators $T_i : X \rightrightarrows X^*$, $i \in I$, will be called a quasimonotone family, if the operator T with graph $\operatorname{Gr} T = \bigcup_{i \in I} \operatorname{Gr} T_i$ is quasimonotone.

Two operators T_1, T_2 will be called a quasimonotone pair if $\{T_1, T_2\}$ is a quasimonotone family.

Definition

A family of functions $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, is called a *quasiconvex family* if for every $i, j \in I$ and every $x, y \in X$, $z \in]x, y[$, the following implication holds:

$$f_i(x) < f_i(z) \Rightarrow f_j(z) \leq f_j(y).$$

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First result

Our first main result relates quasiconvexity of a pair of functions to quasi-monotonicity of the pair of subdifferentials.

Theorem

Let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be lsc functions. Then $\{f_1, f_2\}$ is a quasiconvex pair if and only if $\{\partial f_1, \partial f_2\}$ is a quasimonotone pair.

Tools:

[[2, Corollary 4.3]]

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function, and $a, b \in X$ with $f(a) < f(b)$. Then there exist $c \in [a, b[$ and sequences $x_n \to c$ and $x_n^* \in \partial f(x_n)$, such that $f(x_n) \to f(c)$ and $\langle x_n^*, x - x_n \rangle > 0$, for every $x = c + t(b - a)$ with $t > 0$.

[[1, Theorem 2.1]]

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. The following are equivalent:

(*i*) f is quasiconvex;

(*ii*) $\exists\, x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0 \Rightarrow f(z) \le f(y), \forall z \in [x, y]$.

How big is the class of quasiconvex pairs of lsc functions?

Candidates:

1. Type 1: f_1 and f_2 are quasiconvex, and there is a proportionality between the subdifferentials: $\partial f_i(x) \subseteq \mathbb{R}_+ \partial f_j(x)$,$\forall\, x \in X$, $i \neq j$,$i = 1$ or $i = 2$.

2. Type 2: f_1 and f_2 are nondecreasing transformations of a same quasiconvex function; that is, there exists a quasiconvex function g and nondecreasing functions h_1, h_2 such that $f_1 = h_1 \circ g$ and $f_2 = h_2 \circ g$.

3. Type 3: f_1,f_2 quasiconvex with $\operatorname{argmin} f_1 \cap \operatorname{argmin} f_2 \neq \emptyset$.

Examples:

Neither Type 1 nor Type 2

- $X = \mathbb{R}$;
- $f_1(x) = \begin{cases} x^2, & x < 1, \\ 1, & x \geq 1 \end{cases} \qquad f_2(x) = f_1(-x)$.
- $\partial f_1 \cup \partial f_2$ is quasimonotone $\Rightarrow f_1 + f_2$ is quasiconvex
- However, neither

$$\mathbb{R}_+ \partial f_1(x) \subseteq \mathbb{R}_+ \partial f_2(x), \ \forall\, x \in \operatorname{Dom} \partial f_2,$$

nor

$$\mathbb{R}_+ \partial f_2(x) \subseteq \mathbb{R}_+ \partial f_1(x), \ \forall x \in \operatorname{Dom} \partial f_1 \text{ hold.}$$

.

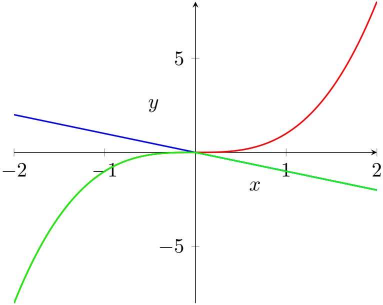
Type 3

- $X = \mathbb{R}$;
- $f_1(x) = \min\{0, x\}, \quad f_2(x) = \max\{0, x\}$.
- The subdifferentials are a quasimonotone pair, whereas one function has no minima, the other does have minima. Here, $\operatorname{argmin} f_1 \cap \operatorname{argmin} f_2 = \emptyset$.
  It is obvious that $f_1 + f_2$ is quasiconvex.

- $X = \mathbb{R}^2$;
- $f_1(x_1, x_2) = \min\left\{100x_1^2 + x_2^2, 1\right\}$
  $f_2(x_1, x_2) = \min\left\{x_1^2 + 100x_2^2, 1\right\}$.
- $\operatorname{argmin} f_1 = \operatorname{argmin} f_2 = \{0\}$.
- For $f = f_1 + f_2$: $f(0.8, 0) = f(0, 0.8) = 1.64$; $f(0.4, 0.4) = 2$
  $\Rightarrow f$ is not quasiconvex.

Other problem that in apparent is not relared to the problem of the sum

$f(x) = -x$, $g(x) = x^3 \Rightarrow \min\{f, g\}$ is not a quasiconvex function.



Characterizations of quasiconvexity of the minimum of any vertical translation of two quasiconvex functions.

Theorem

Assume that the functions f_1, f_2 are quasiconvex. Then, the following assertions are equivalent:

(a) $\{f_1, f_2\}$ *is a quasiconvex pair.*

(b) *for every* $\lambda_1, \lambda_2 \in \mathbb{R}$, $[f_1 \leq \lambda_1] \cup [f_2 \leq \lambda_2]$ *is convex.*

(c) *for every* $\lambda_1, \lambda_2 \in \mathbb{R}$, $[f_1 < \lambda_1] \cup [f_2 < \lambda_2]$ *is convex.*

(d) *for every* $x \in X$, $[f_1 < f_1(x)] \cup [f_2 < f_2(x)]$ *is convex.*

(e) *for every* $\alpha \in \mathbb{R}$, *the function* h_α *defined by* $h_\alpha(x) \doteq \min\{f_1(x) + \alpha, f_2(x)\}$ *is quasiconvex.*

* *Equivalences* $(a) - (e)$ *are true in any vector space*

Example

Consider the functions f_1, f_2 defined on $\mathbb{R}^2$ by

$$f_1(x_1, x_2) = \begin{cases} \max\{\arctan x_1, 0\} & \text{if} \quad -1 \leq x_2 \leq +1 \\ \frac{\pi}{2} & \text{if} \quad x_2 < -1 \\ x_2 - 1 + \frac{\pi}{2} & \text{if} \quad x_2 > 1 \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} \max\{-\arctan x_1, 0\} & \text{if} \quad -1 \leq x_2 \leq +1 \\ -x_2 - 1 + \frac{\pi}{2} & \text{if} \quad x_2 < -1 \\ \frac{\pi}{2} & \text{if} \quad x_2 > 1 \end{cases}$$

One may check that the union of any two sublevel sets is convex, so the functions are a quasiconvex pair $\Rightarrow \min\{f_1, f_2\}$ is quasiconvex.

Some consequences for the sum of quasiconvex functions

Let $J \doteq \{1, 2, \ldots, m\}$.

Theorem

Let $\{f_i : i \in J\}$ *be a quasiconvex family. Then* $f_1 + f_2 + \cdots + f_m$ *and* $\min\{f_1, f_2, \cdots, f_n\}$ *are quasiconvex.*

Another characterization:

Proposition

Let f_1, f_2 *be functions on* X*. Then* $\{f_1, f_2\}$ *is a quasiconvex pair, iff for every pair of nondecreasing functions* h_1, h_2*, the function* $h_1 \circ f_1 + h_2 \circ f_2$ *is quasiconvex.*

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Remark

The result that the sum of two convex functions is convex, and so quasiconvex, cannot be re-obtained with our results ☹.
But

- We show that our class of functions, for which the sum of quasiconvex functions is quasiconvex, contains not trivial functions and
- ...

Other properties that are preserved under summation in a quasiconvex family

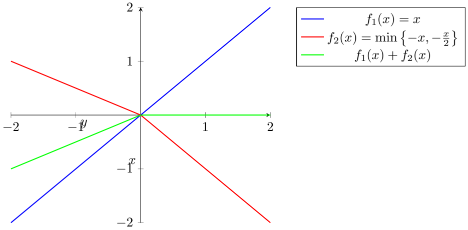
f is semistrictly quasiconvex if f is quasiconvex and

$$[x, y \in \mathrm{dom}\, f, f(x) \neq f(y)] \Rightarrow \left[f(tx + (1-t)y) < \max\{f(x), f(y)\}, \forall\, t \in]0,1[.\right]$$

The sum of two semistrictly quasiconvex functions is not necessarily semistrictly quasiconvex.

Example

$f_1\,(x) = x$ and $f_2(x) = \min\left\{-x, -\frac{x}{2}\right\}$.
f_1, f_2 : semistrictly quasiconvex, but their sum is not.



Proposition

Let $f_1, f_2, \dots, f_m$ be lsc and semistrictly quasiconvex $($and so quasiconvex$)$ functions. If $\{f_i : i \in J\}$ is a quasiconvex family, then the sum $f_1 + f_2 + \cdots + f_m$ is also semistrictly quasiconvex.

The $\mathcal{C}$-family

$C \subseteq \mathbb{R}^n$

$C^\infty \doteq \{v \in \mathbb{R}^n : \exists\, t_k \to +\infty, \exists\, x_k \in C, \dfrac{x_k}{t_k} \to v\}$. (asymptotic cone).

Definition [9]

It is said that f belongs to $\mathcal{C}$ if for all $x \in \operatorname{dom} f$ and all $v \in (\operatorname{dom} f)^\infty$, $v \neq 0$, one has either

(i) $0 \leq t \mapsto f(x+tv)$ is nonincreasing, or

(ii) $\lim_{t\to+\infty} f(x+tv) = +\infty$.

Remark: f convex or coercive then $f \in \mathcal{C}$;

Proposition

Let $f_1, f_2, \ldots, f_m$ be lsc functions on X, such that

$$\{f_i : \; i \in J\} \;\; \textit{is a quasiconvex family}.$$

If $f_i \in \mathcal{C}$ for all $i \in J$, then $f_1 + f_2 + \cdots + f_m \in \mathcal{C}$.

Q-subdifferential ([13])

Geven $f : \mathbb{R}^n \to \mathbb{R}$,
$\partial^Q f(x) = \left\{(v,t) \in \mathbb{R}^{n+1} : \langle v, x\rangle \geq t \text{ and } f(y) \geq f(x) \text{ if } \langle v, y\rangle \geq t\right\}.$

Theorem ([13])

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ For each $x \in \mathbb{R}^n$, assume that at least one of the following conditions is satisfied:

(i) $[f < f(x)] \subseteq [g < g(x)]$ *(or $[g < g(x)] \subseteq [f < f(x)]$)*

(ii) $\partial^Q f(x) \subseteq \partial^Q g(x)$ *(or $\partial^Q g(x) \subseteq \partial^Q f(x)$.)*

Then $f + g$ is quasiconvex.

Proposition

Let f, g be as in Theorem 7. Then $\{f, g\}$ is a quasiconvex pair, and so $f + g$ and $\min\{f, g\}$ are quasiconvex.

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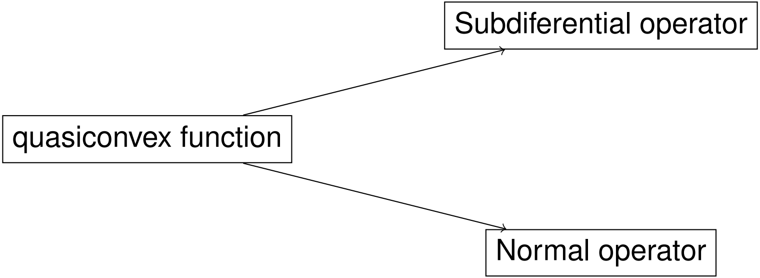
Operators associated to quasiconvex functions

- via subdifferentials:

  D.-T. Luc -'93 ; Aussel & Corvellec & Lassonde-'94; Daniilidis & Hadjisavvas-'02
- via normal cones to the sublevel sets :

  Borde & Crouzeix-'90; Aussel & Danilidis-' 20.



Operators

For $T : X \rightrightarrows X^*$, we denote by

$$\mathrm{Gr}\ T = \big\{(x, x^*) \in X \times X^* : x^* \in T(x)\big\}$$

its *graph* , and for

$$\mathrm{Dom}\ T = \big\{x \in X : \exists x^* \in X, (x, x^*) \in \mathrm{Gr}\ T\big\}$$

its *Domain* .
An operator $T : X \rightrightarrows X^*$ is called *quasimonotone*, if $\forall\ (x, x^*) \in \mathrm{Gr}\ T$ and $(y, y^*) \in \mathrm{Gr}\ T$,

$$\langle x^*, y - x\rangle > 0 \ \Rightarrow\ \langle y^*, y - x\rangle \geq 0;$$

Quasimonotone family (recall)

Definition

A family $\mathcal{A}$ of operators $T_i : X \rightrightarrows X^*$, $i \in I$, is called a *quasimonotone family*, if the operator T with graph $\mathrm{Gr}\ T = \bigcup_{i \in I} \mathrm{Gr}\ T_i$ is quasimonotone.

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Remark

A famiy of monotone operators not necessarily satisfies this definition: $T_1, T_2 : \mathbb{R} \to \mathbb{R}$, $T_1(x) = x$, $T_2(x) = 1$ *(or* $T_2(x) = e^x$*) are monotone operators but* $\{T_1, T_2\}$ *is not a quasimonotone pair. It is enough to take* $x = -1$ *and* $y = 1$.

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More about quasimonotone families

Proposition

Let $T_1, T_2 : X \rightrightarrows X^*$ *be two operators.The following are equivalent:*

(a) *The family* $\{T_1, T_2\}$ *is a quasimonotone pair on* $\mathrm{Dom}\ T_1 \cap \mathrm{Dom}\ T_2$*.*

(b) *For every pair of functions* $f_1, f_2 : X \to \mathbb{R}_+$*, the operator* $f_1T_1 + f_2T_2$ *is quasimonotone.*

Proposition

Let $J = \{1, 2, \cdots m\}$*. Assume that* T_i*,* $i \in J$ *(or their restrictions on* $\bigcap_{i \in J} \mathrm{Dom}\ T_i$*) constitute a quasimonotone family of operators. Then,*

$$T_1 + T_2 + \cdots + T_m \text{ is quasimonotone.}$$

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A particular case of quasimonotone family

Proposition

Let T_i, $i \in J$ *be quasimonotone operators such that*

$$\mathbb{R}_+ T_i(x) \subseteq \mathbb{R}_+ T_{i+1}(x) \;\; \forall\, x \in X,\; i = 1, 2, \ldots, m-1.$$

Then $\{T_i : i \in J\}$*, is a quasimonotone family.*

Corollary

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Then $T_1 + T_2 + \cdots + T_m$ *is quasimonotone.*

Strictly quasimonotone operators

- $T : X \rightrightarrows X^*$ is said to be *strictly quasimonotone* if it is quasimonotone and for all $x, y \in \mathrm{dom}T$, $x \neq y$

$$\exists z \in]x, y[,\ \exists z^* \in T(z) : \langle z^*, y - x\rangle \neq 0.$$

Example

$T_1 : \mathbb{R} \to \mathbb{R}$, $T_1(x) = x^2$ and $T_2 : \mathbb{R} \to \mathbb{R}$, $T_2(x) = -x^2$ are strictly quasimonotone operators.
Obviously $T_1 + T_2$ is not strictly quasimonotone.

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Obviously $T_1 + T_2$ is not strictly quasimonotone.

Proposition

Assume that T_i, $i \in J$, *are strictly quasimonotone, and constitute a quasimonotone family. If the operator* T *with graph* $\mathrm{Gr}\ T = \bigcup_{i \in J} \mathrm{Gr}\ T_i$ *satisfies*

$$\forall\, x, y \in \mathrm{Dom}\ T,\ x \neq y,\ ]x, y[\cap \mathrm{Dom}\ T \neq \emptyset. \tag{1}$$

then T *is strictly quasimonotone.*

Remark:

Equation (1) is satisfied if $\mathrm{Dom}\ T$ is convex.

Proposition

Suppose that $T_i,\, i \in J$ *is a quasimonotone family of operators on* $\bigcap\limits_{i\in j} \operatorname{Dom} T_i$. *If* T_1 *is a strictly quasimonotone operator such that either*

(a) $\bigcap\limits_{i\in J} \operatorname{Dom}\ T_i$ *is convex, or*

(b) $\operatorname{Dom} T_1 \subseteq \operatorname{Dom} T_i$, *for* $i \in J,\ i \neq 1$,

then

$$T_1 + \cdots + T_m$$

is strictly quasimonotone.

Semistrictly quasimonotone operators

$T : X \rightrightarrows X^*$ is said to be

- *semistrictly quasimonotone* if it is quasimonotone and for all $(x, x^*) \in \mathrm{Gr}\, T$ and $y \in \mathrm{dom} T$

$$\langle x^*, y - x\rangle > 0 \Rightarrow \exists z \in]\frac{x+y}{2}, y[,\ \exists z^* \in T(z) : \langle z^*, z - x\rangle > 0.$$

Example

Let $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$T_1(x) = \begin{cases} [0, +\infty[, & \text{if } x = 0; \\ \{0\}, & \text{if } x < 0; \\ \emptyset, & \text{otherwise.} \end{cases} \quad T_2(x) = \begin{cases}]-\infty, 0], & \text{if } x = 0; \\ \{0\}, & \text{if } x > 0; \\ \emptyset, & \text{otherwise.} \end{cases},$$

Let T be the operator whose graph is $\mathrm{Gr}\, T_1 \cup \mathrm{Gr}\, T_2$.
• T_1 and T_2 are semistrictly quasimonotone and constitute a quasimonotone pair.
• T is quasimonotone but not semistrictly quasimonotone.

Proposition

Assume that T_i, $i \in J$ *are semistrictly quasimonotone and* $\bigcap\limits_{i \in j} \operatorname{Dom} T_i$ *is convex. If* T_i, $i \in J$ *constitute a quasimonotone family of operators on* $\bigcap\limits_{i \in j} \operatorname{Dom} T_i$, *then*

$$T_1 + T_2 + \cdots + T_m,$$

is semistrictly quasimonotone.

Normal operators associated to a quasiconvex functions

Given a lsc function $f : X \to \mathbb{R}$, the set-valued operator $N_f : X \rightrightarrows X^*$ defined by

$$N_f(x) = \begin{cases} N_{[f \le f(x)]}(x), & \text{if } x \in \operatorname{dom} f \\ \emptyset, & \text{otherwise,} \end{cases}$$

is the *Normal Operator* associated to f.

Fact:

- (Aussel & Daniilidis) Let X be a reflexive Banach space, and $f : X \to \mathbb{R}$ a lsc function. Then, the following statements are equivalent:
  - (i) f is quasiconvex.
  - (ii) $x, y \in \operatorname{dom} f$, $x^* \in N_f(x)$ and $\langle x^*, y - x \rangle > 0 \Rightarrow f(x) < f(y)$.
  - (iii) N_f is quasimonotone.

Normal operators associated to a quasiconvex functions

Proposition

Let $f_i : X \to \mathbb{R} \cup \{+\infty\}, i \in J$ *be a family of lsc functions. Consider the following assertions:*

(a) $\{f_1, \dots, f_m\}$ *is a quasiconvex family.*

(b) $\{N_{f_1}, \dots, N_{f_m}\}$ *is a quasimonotone family.*

Then $(a) \Rightarrow (b)$*. If, in addition* $f_i, i \in J$ *are continuous or,* X *admits a Gâteaux-smooth renorm, then* $(b) \Rightarrow (a)$*.*

(Semi)strictly quasiconvex case

Proposition

Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in J$, *be lsc functions that constitute a quasiconvex family, where each function is continuous on its respective domain. The following hold:*

(a) *If at least one* f_i, $i \in J$ *is strictly quasiconvex, then* $N_{f_1} + \cdots + N_{f_m}$ *is strictly quasimonotone.*

(b) *If every function* f_i, $i \in J$ *is semistrictly quasiconvex, then* $N_{f_1} + \cdots + N_{f_m}$ *is semistrictly quasimonotone.*

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Thank you!


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