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UNIVERSITÀ DEGLI STUDI DELL'AQUILA

A variational approach to weakly continuous preference relations

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D. Aussel - M. Milasi - D. Scopelliti

Setting

Let

- $\succ$ be a preference relation on a t.v.s. X

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$$\succ \qquad \Longrightarrow \qquad \succeq \qquad (x \succeq y \text{ if } y \not\succ x)$$

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$\succ$	$\Longrightarrow$	$\succeq$
asymmetric		*complete*
negatively transitive		*transitive*

- $\succeq$
  - *complete*: $x \succeq y$ or $y \succeq x$, for each $x, y \in X$
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$(x \succ y$ if $y \not\succeq x)$ $\quad \succ \quad \Longleftarrow \quad \succeq$

asymmetric / *negatively transitive* $\quad$ *complete* / *transitive*

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quasiconcave $u : X \to \mathbb{R}$ $\xrightarrow[\text{not surjective}]{x \succ y \Leftrightarrow u(x) > u(y)}$ convex preference relation $\succ$

Setting

$$U(x) = \{y \in X : y \succ x\} \qquad L(x) = \{y \in X : x \succ y\}$$

The preference relation $\succ$ is

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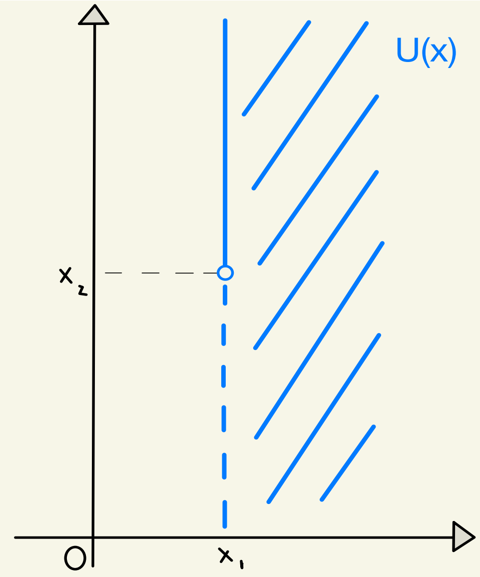
If $\succ$ has a numerical representation u

$\succ$ is upper (lower) semicontinuous $\overset{[1]}{\Longleftrightarrow}$ u is upper (lower) pseudocontinuous

[1] Morgan & Scalzo: Discontinuous but well-posed optimization problems. SIAM J. Optim. 17 (2006) 861–870

Lexicographic order



$$x \succ y \;\Leftrightarrow\; x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2$$

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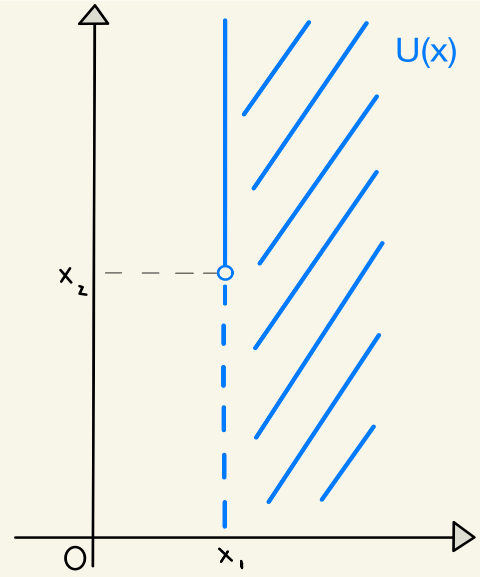

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usc	lsc
no	no

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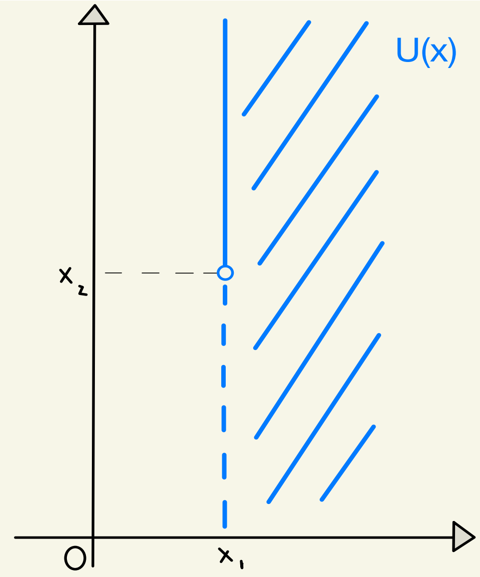

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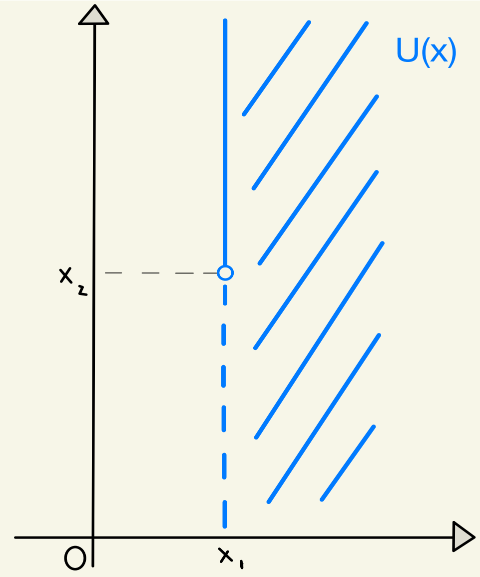

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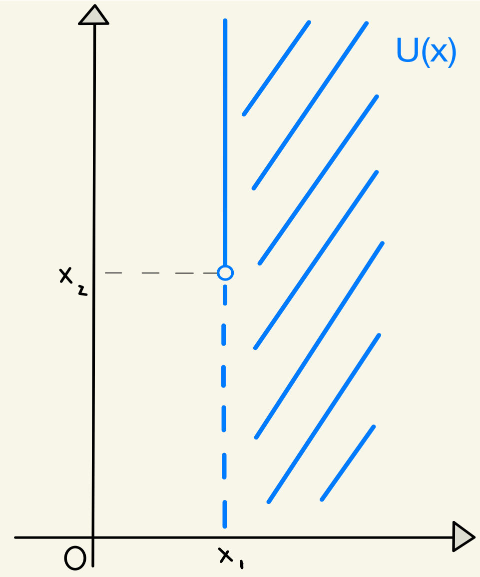

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usc	lsc	wusc
no	no	yes

- *weak upper semicontinuous* at x if
  $y \in \operatorname{int} U(x) \Rightarrow y \in U(z),\ \forall z \in V_x$
- *solid* at x if
  $U(x) \neq \emptyset \Rightarrow \operatorname{int} U(x) \neq \emptyset$

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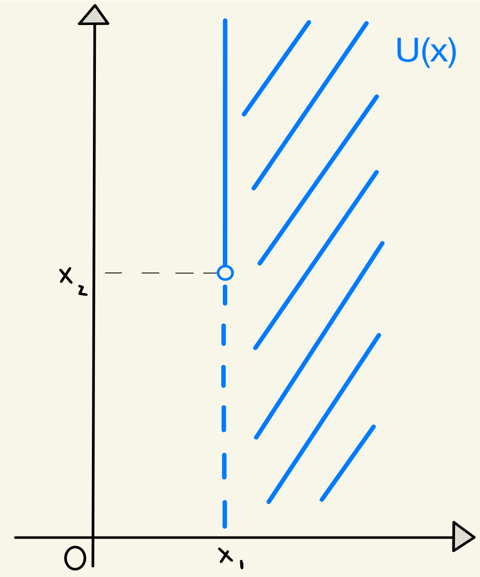

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$\succ$ is weakly lower continuous $\overset{[2]}{\Longleftrightarrow}$ $y \succ x \Rightarrow y \succeq z, \forall z \in V_x$

[2] Campbell & Walker: Maximal elements of weakly continuous relations. J. Econom. Theory 50 (1990) 459–464

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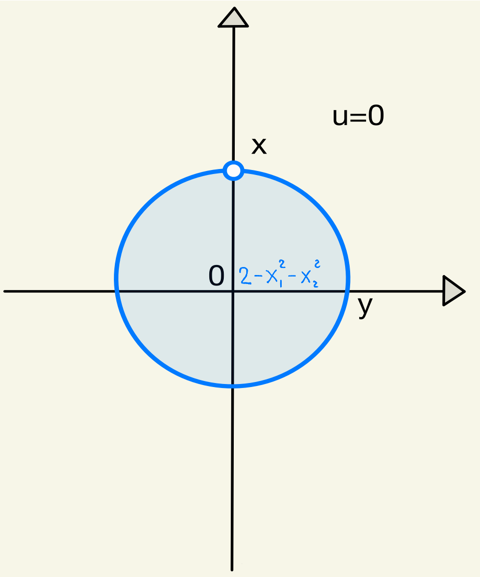

$$u(x_1, x_2) = \begin{cases} 2 - x_1^2 - x_2^2 & (x_1, x_2) \in B \setminus (0,1) \\ 0 & \text{otherwise} \end{cases}$$

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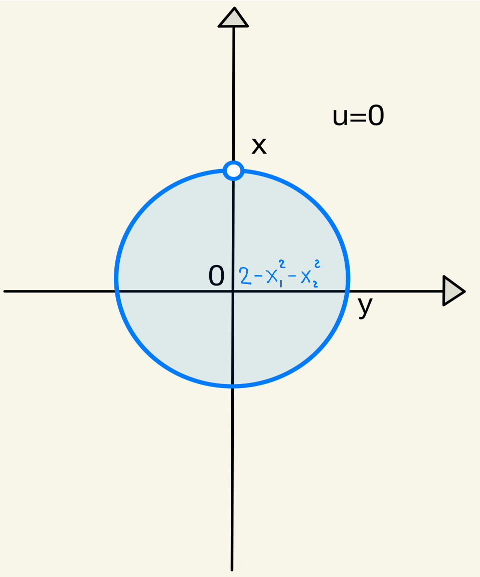

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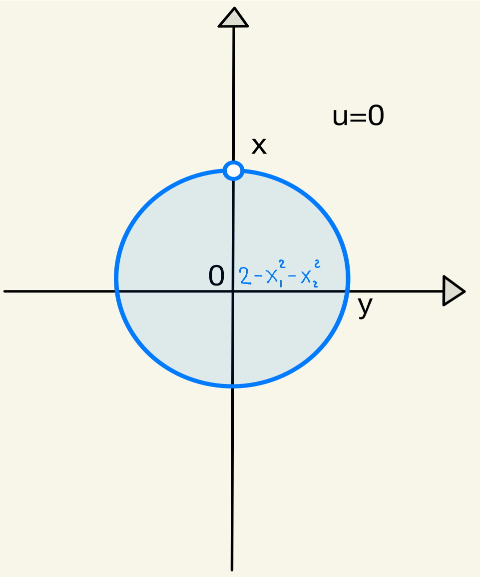

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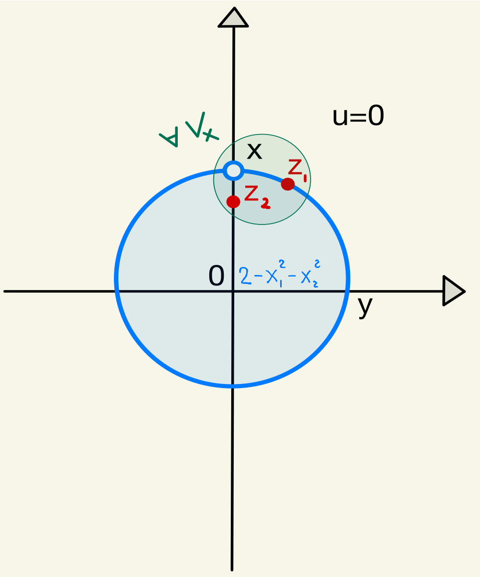

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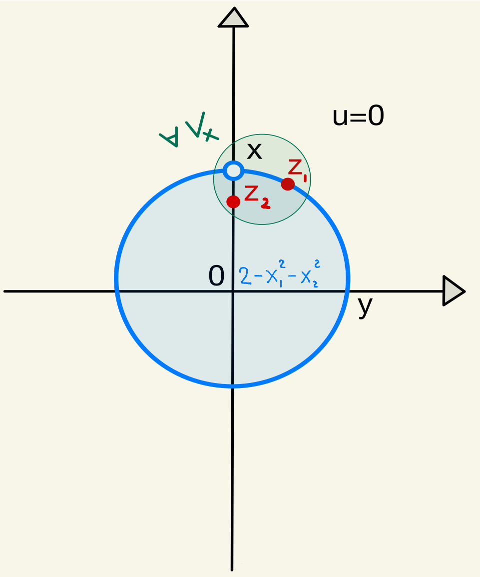

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upc	wlc
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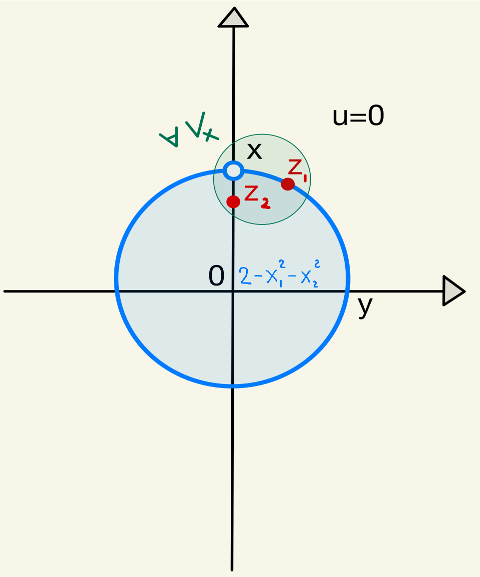

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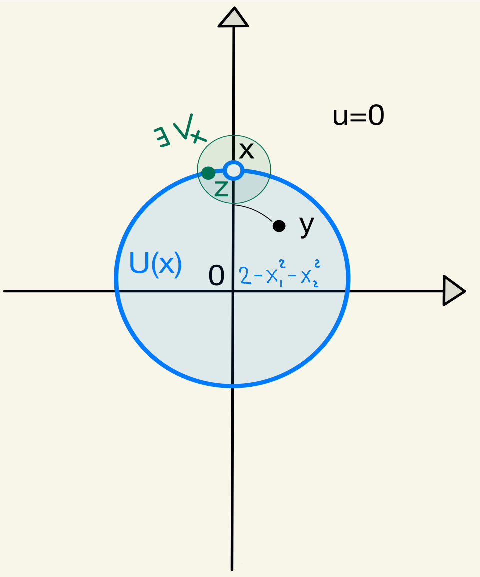

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upc	wlc	lpc	wusc
no	no	no	yes

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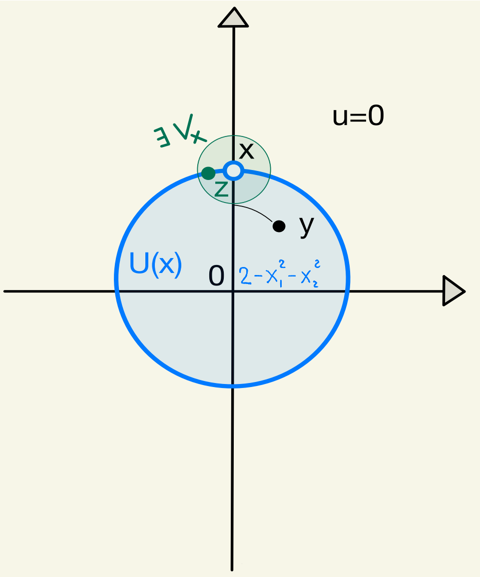

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From now on

- X and X^* will be equipped by the norm topology and the weak* topology, respectively

Adjusted contour set

Let $\succ$ be a preference relation and $x \in X$

$$S^a_\succ(x) = \begin{cases} B(U(x), \rho_x) \cap L^c(x) & \text{if } U(x) \neq \emptyset \\ L^c(x) & \text{if } U(x) = \emptyset \end{cases}$$

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Basic facts

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- $U(x) \neq \emptyset \quad \Rightarrow \quad x \notin \mathrm{int}\, S^a_\succ(x)$
- $U(x) \subseteq S^a_\succ(x) \subseteq L^c(x)$
- $\succ$ convex $\quad \Rightarrow \quad S^a_\succ(x)$ convex

Adjusted contour set

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$$S^a_{\succ}(x) = \begin{cases} B(U(x), \rho_x) \cap L^c(x) & \text{if } U(x) \neq \emptyset \iff x \notin \text{argmax} \\ L^c(x) & \text{if } U(x) = \emptyset \overset{\text{def}}{\iff} x \in \text{argmax} \end{cases}$$

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- $U(x) \neq \emptyset \quad \Rightarrow \quad x \notin \text{int}\, S^a_{\succ}(x)$
- $U(x) \subseteq S^a_{\succ}(x) \subseteq L^c(x)$
- $\succ$ convex $\quad \Rightarrow \quad S^a_{\succ}(x)$ convex
- $\succ$ wusc solid $\quad \Rightarrow \quad$ argmax closed

Adjusted normal cone operator

To the preference relation $\succ$ we associate the map $N^a_{\succ} : X \rightrightarrows X^*$ defined by

$$N^a_{\succ}(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall y \in S^a_{\succ}(x)\}$$

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- u represents $\succ$ $\quad\Rightarrow\quad$ $N^a_\succ(x) = N^a_u(x)$ as defined in [3]

[3] Aussel & Hadjisavvas: Adjusted sublevel sets, normal operator, and quasi-convex programming. SIAM J. Optim. 16 (2005) 358–367

Continuity of set-valued maps

Let $\Phi : X \rightrightarrows Y$ be a set-valued map with Y Hausdorff

- Φ is closed if

$$\operatorname{gph}\Phi = \{(x,y) \in X \times Y : y \in \Phi(x)\} \quad \text{is closed in } X \times Y$$

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- if Φ is compact, that is maps X into a compact subset of Y then

$$\Phi \text{ is closed} \iff \Phi \text{ is upper semicontinuous and closed-valued}$$

Upper semicontinuity of N^a

A subset K of X^* is a cone if

- $tx^* \in K$, for any $x^* \in K$ and $t \geq 0$

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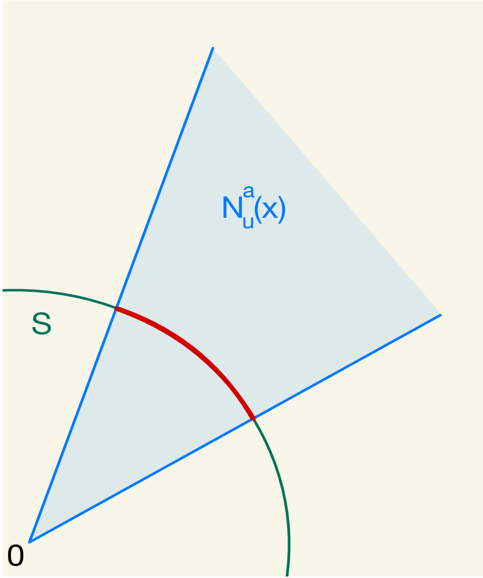
Problem

Find a map $A : X \rightrightarrows X^*$ *such that*

- $A(x)$ *is a weak***-compact base of* $N^a_{\succ}(x)$ *for each* $x \notin \operatorname{argmax}$
- A *is norm-to-weak** *upper semicontinuous*

Finite dimensional case



u is a quasiconcave representation of $\succ$

$$N_u^a : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$$

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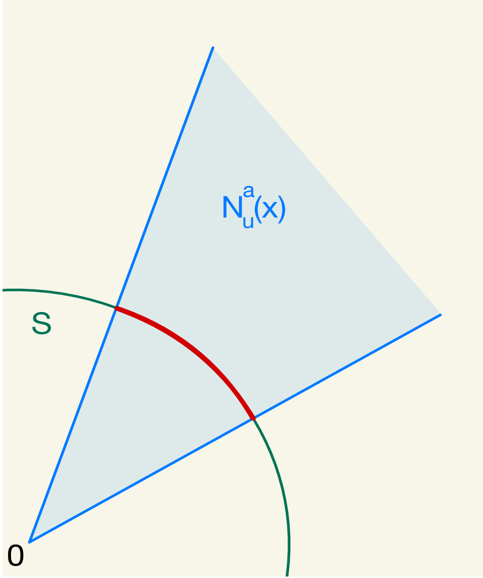

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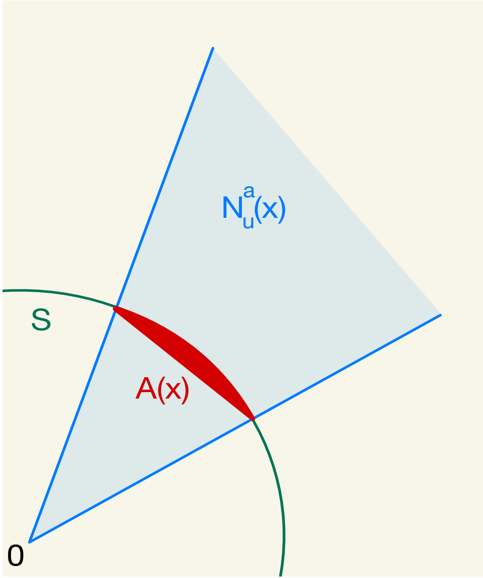

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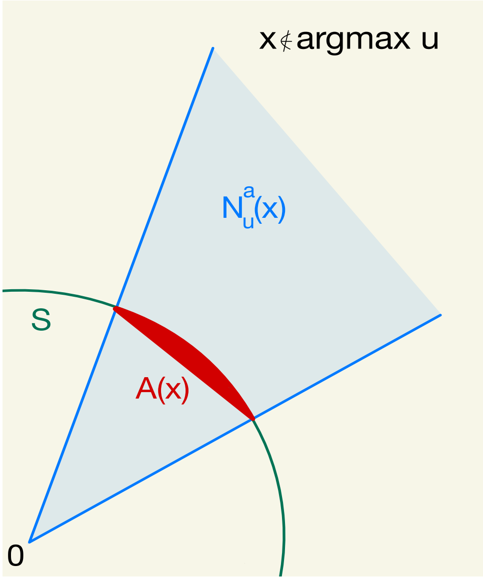

$$A(x) = \begin{cases} \mathrm{co}\,(N_u^a(x) \cap S) & \text{if } x \notin \operatorname{argmax} u \\ B & \text{if } x \in \operatorname{argmax} u \end{cases}$$

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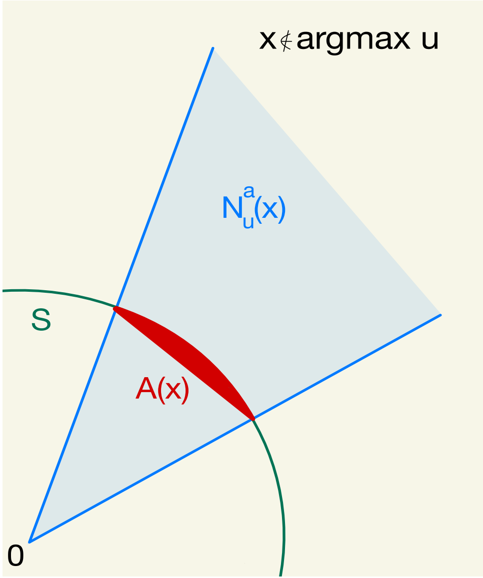

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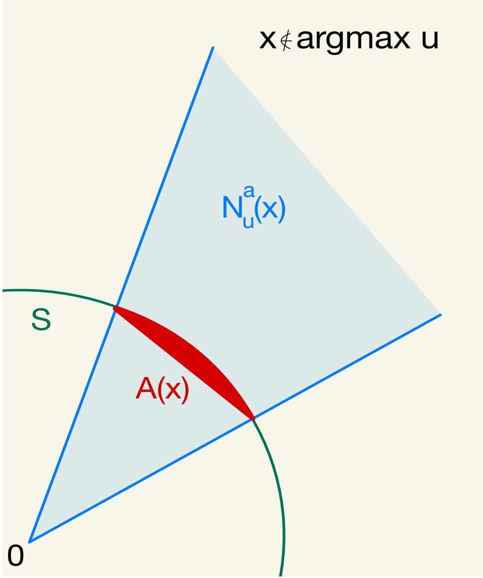

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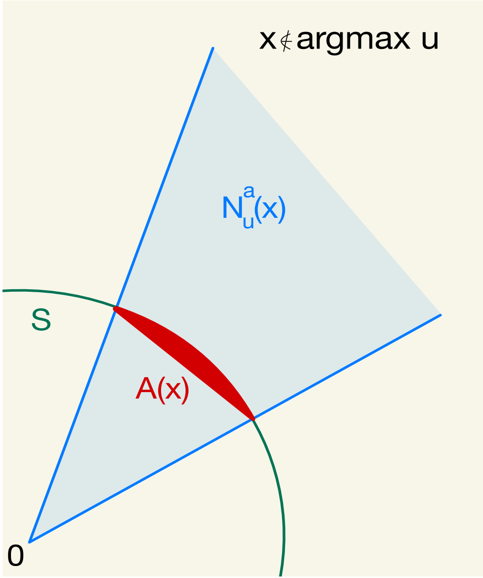

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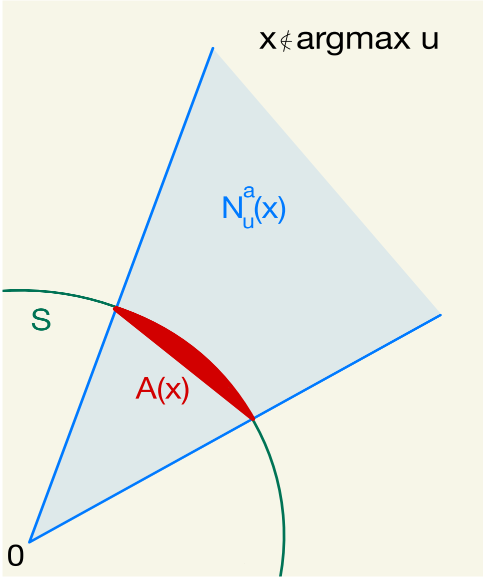

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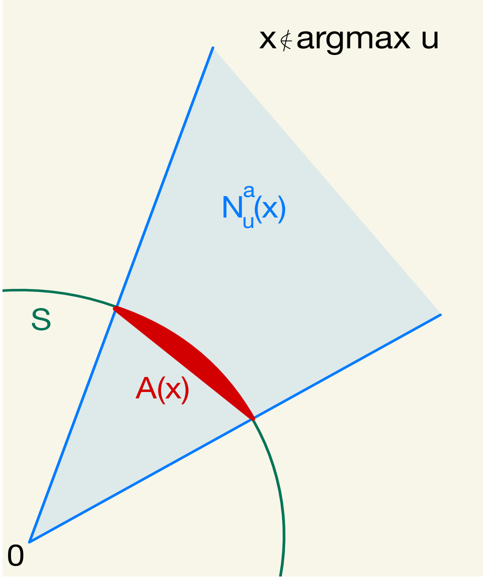

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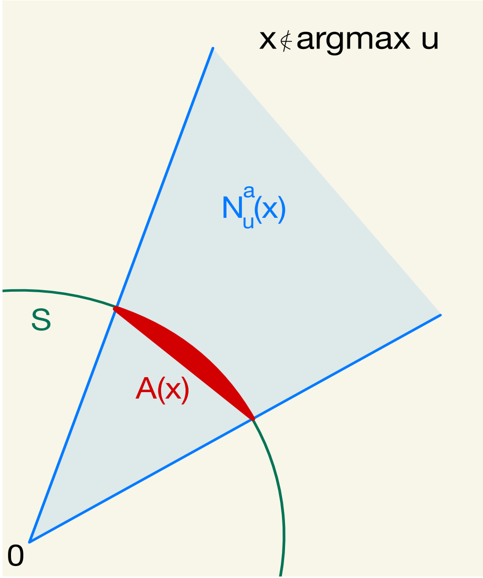

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[4] Aussel & Cotrina: Quasimonotone quasivariational inequalities: existence results and applications. J. Optim. Theory Appl. 158 (2013) 637–652

Upper semicontinuity of N^a

Theorem 3 in [5]

Let $u : X \to \mathbb{R}$ be quasiconcave upper semicontinuous and solid. Then

(i) N_u^a *is norm-to-weak* closed at any* $x \notin \operatorname{argmax} u$

(ii) *there exists a norm-to-weak* upper semicontinuous set-valued map* $A : X \rightrightarrows B^*$ *such that* $A(x)$ *is a weak*-compact base of* $N_u^a(x)$*, for all* $x \notin \operatorname{argmax} u$

[5] Castellani & Giuli: A continuity result for the adjusted normal cone operator. J. Optim. Theory Appl. 200 (2024) 858–873

Upper semicontinuity of N^a

Corollary 4.11 in [6]

Let $\succ$ be a weak upper semicontinuous, solid, and convex preference relation on X. Then there exists a norm-to-weak* upper semicontinuous set-valued map $A : X \rightrightarrows B^*$ such that $A(x)$ is a weak*-compact base of $N^a_{\succ}(x)$, for all $x \notin \operatorname{argmax}$

[6] Aussel, Giuli, Milasi, Scopelliti: A variational approach to weakly continuous relations in Banach spaces. Submitted

An application

- N finite family of players

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For each $i \in N$

- X_i Banach strategy space

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Preference Equilibrium Problem

Find $x \in C$ *such that* $x \in K(x)$ *and* $U_i(x_i) \cap K_i(x) = \emptyset, \forall i \in N$

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Quasi Variational Inequality Problem

Find $x \in K(x)$ *such that* $\exists x^* \in \prod(N^a_{\succ_i}(x_i) \setminus \{0_i\})$ *with* $\sum\langle x_i^*, y_i - x_i\rangle \geq 0, \forall y \in K(x)$

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Proposition 5.6 in [6]

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- A preference relation $\succ$ on X is *sub-boundarily constant* on $C \subseteq X$ if, for any $x, y \in C$

$$y \in U(x) \quad \Rightarrow \quad [y, x) \cap \operatorname{int} S^a_{\succ}(x) \neq \emptyset$$

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$$y \in U(x) \quad \Rightarrow \quad [y, x) \cap \operatorname{int} S^a_{\succ}(x) \neq \emptyset$$

- Any lower semicontinuous preference relation $\succ$ is *sub-boundarily constant*

Existence result

Theorem 5.4 in [6]

For any $i \in N$, let

- $C_i \subseteq X_i$ *nonempty and convex*
- $\succ_i$ *weak upper semicontinuous solid convex preference relation on* X_i
- $\succ_i$ *sub-boundarily constant on* C_i
- K_i *lower semicontinuous compact with nonempty values in* $\mathcal{D}(X_i)$ *and* fix K *closed*

Then there exists a preference equilibrium

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