Closed convex sets that are both Motzkin decomposable and generalized Minkowski sets

Juan Enrique Martínez-Legaz (UAB, Spain)
Cornel Pintea (Babeș-Bolyai University, Romania)

Variational Analysis and Optimization II

Milan, May 30, 2024

OUTLINE

- MOTZKIN DECOMPOSABLE SETS
- GENERALIZED MINKOWSKI SETS
- MdgM SETS AND THEIR CHARACTERIZATIONS
- FIXED POINT TYPE THEOREMS ON MdgM SETS
- SOME REFERENCES

MOTZKIN DECOMPOSABLE SETS

$\emptyset \neq F \subset \mathbb{R}^n$ is called *Motzkin decomposable*
if there exist a compact convex set C
and a closed convex cone D
such that $F = C + D$.

(C, D) is a *Motzkin decomposition* of F
with *compact and conic components* C and D,
respectively.

$$D = 0^+F := \{d \in \mathbb{R}^n : d + F \subset F\}$$

$$Q(F) := \operatorname{cl}\operatorname{conv}\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$$

If F contains no lines, then $Q(F) = \operatorname{cl}\operatorname{conv}\operatorname{extr} F$.

THEOREM

Let F be a nonempty closed convex set.

Then

(i) F is Motzkin decomposable

if and only if

$\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ is bounded.

In this case,

$Q(F)$ is a compact component of F.

(ii) If F is a Motzkin decomposable set without lines, then $Q(F)$ is the smallest compact component of F.

Example:

$F := \left\{(x, y) \in \mathbb{R}^2 : y \geq x^2\right\}$

$\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right) = \operatorname{bd} F$

COROLLARY

Every face of a Motzkin decomposable set is Motzkin decomposable, too.

THEOREM

Let F be a nonempty closed convex set.

Let $K := \left(0^+F\right) \cap (\operatorname{lin} F)^\perp$, and

$$M(F) := \left\{x \in F \cap (\operatorname{lin} F)^\perp : (x - K) \cap F = \{x\}\right\}.$$

Then the following statements hold:

(i) F is Motzkin decomposable

if and only if

$M(F)$ is bounded.

In this case,

$\operatorname{cl}\operatorname{conv} M(F)$ is a compact component of F.

(ii) If F is Motzkin decomposable

and contains no lines,

then $\operatorname{cl}\operatorname{conv} M(F) = Q(F)$.

THEOREM.

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set.

Then

C is Motzkin decomposable

if and only if

$C \cap (\mathrm{lin}\, F)^{\perp}$ is Motzkin decomposable.

In this case,

every compact component of $C \cap (\mathrm{lin}\, F)^{\perp}$

is a compact component of C, too.

A halfline L is an *asymptote* of F if
$F \cap L = \emptyset$ and $d(F, L) = 0$.

If $M \subseteq \mathbb{R}^2$ is Motzkin decomposable,
then M has no asymptote.

Example:

$$F := \left\{ (x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq \left(\sum_{i=1}^{n-1} x_i^2 \right)^{\frac{1}{2}} \right\}$$

F is Motzkin decomposable.
The intersection of F with $H : x_2 = \cdots = x_{n-1} = 1$
has two asymptotes:
the intersections of $x_n = x_1$ and $x_n = -x_1$ with H.
$F \cap H$ is not Motzkin decomposable.

PROPOSITION

Let A be a closed convex set
and B and $A+B$ be Motzkin decomposable sets
such that $0^{+}B \subset \operatorname{lin} A$.
Then A is Motzkin decomposable.

Example:

$$A := \left\{(x,y) \in \mathbb{R}^2 : y \geq x^2\right\} \qquad B := \left\{(x,y) \in \mathbb{R}^2 : y = 0\right\}$$

LEMMA

Let A and B be nonempty sets.
If A is convex, B is compact and $A+B$ is closed,
then A is closed.

COROLLARY

Let A and B be convex sets such that
B is bounded and $A+B$ is Motzkin decomposable.
Then A is Motzkin decomposable, too.

PROPOSITION.

Let $C_i \subseteq \mathbb{R}^{n_i}$ $(i = 1, ..., m)$ be nonempty closed convex sets.

Then the Cartesian product $\prod_{i=1}^{m} C_i$ is Motzkin decomposable

if and only if

C_i is *Motzkin decomposable* for every $i = 1, ..., m$.

GENERALIZED MINKOWSKI SETS

A nonempty closed convex set $C \subseteq \mathbb{R}^n$ is called a

generalized Minkowski set

if it is the convex hull of its minimal faces.

THEOREM.
Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
Then, C is a generalized Minkowski set

if and only if

$C \cap U$ is a Minkowski set.
In particular, C is a generalized Minkowski set

if and only if

$C \cap (\text{lin } C)^\perp$ is a Minkowski set.

THEOREM.
Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set.
Then, C is a generalized Minkowski set

if and only if

there exist a Minkowski set $C_0 \subseteq \mathbb{R}^n$,
a supplementary subspace $V \subseteq \mathbb{R}^n$ to aff $C_0 -$ aff C_0
and a linear subspace $L \subseteq V$
such that

$$C = C_0 + L.$$

In particular, C is a generalized Minkowski set

if and only if

there exist a Minkowski set $C_0 \subseteq \mathbb{R}^n$
and a linear subspace $L \subseteq (\text{aff } C_0 - \text{aff } C_0)^\perp$
such that

$$C = C_0 + L.$$

Example:
$A := \left\{(x, y) \in \mathbb{R}^2 : y \geq x^2\right\}$ $\quad$ $B := \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\}$

V subspace of $\mathbb{R}^n$

S$\subseteq \mathbb{R}^n$ is said to be V-*invariant* if $S + V = S$.

PROPOSITION.

For a nonempty closed convex set $C \subseteq \mathbb{R}^n$,
the following statements are equivalent:

1. C is generalized Minkowski.
2. There exists a smallest (lin C)-invariant set $S \subseteq C$ such that conv $S = C$.
3. There exists a minimal (lin C)-invariant set $S \subseteq C$ such that conv $S = C$.

PROPOSITION.

Every face of a generalized *Minkowski* set
is a generalized *Minkowski* set, too.

PROPOSITION.

Let $C_i \subseteq \mathbb{R}^{n_i}$ $(i = 1, ..., m)$ be
nonempty closed convex sets.
Then the Cartesian product $\prod_{i=1}^{m} C_i$ is a generalized Minkowski set

if and only if

C_i is a *generalized Minkowski* set for every $i = 1, ..., m$.

MdgM SETS AND THEIR CHARACTERIZATIONS

A nonempty closed convex set $C \subseteq \mathbb{R}^n$ is
a *MdgM set*
if it is both Motzkin decomposable and
a generalized Minkowski set.

THEOREM.
Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
Then the following statements are equivalent:

a) C is *MdgM*.
b) $C \cap U$ is compact.
c) There exist a compact (convex) set $K \subseteq \mathbb{R}^n$ and a linear subspace $L \subseteq \mathbb{R}^n$ such that

$$C = K + L.$$

d) The total normal cone $N_C(\mathbb{R}^n)$ is a linear subspace.
e) The barrier cone $bar(C)$ is a linear subspace.
f) The recession cone $0^+(C)$ is a linear subspace.

COROLLARY.

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set.

Then the following statements are equivalent:

a) C is *MdgM*.

b) $N_C(\mathbb{R}^n) = (\text{lin } C)^{\perp}$.

c) $N_C(\mathbb{R}^n) = \left(0^+(C)\right)^{\perp}$.

d) $bar(C) = \left(0^+(C)\right)^{\perp}$.

e) $bar(C) = (\text{lin } C)^{\perp}$.

PROPOSITION.

If the sets $C_i \subseteq \mathbb{R}^n$ $(i = 1, ..., m)$ are *MdgM*,

then their sum $\sum\limits_{i=1}^{m} C_i$ is *MdgM*, too.

PROPOSITION.

Let $C_i \subseteq \mathbb{R}^{n_i}$ $(i = 1, ..., m)$ be

nonempty closed convex sets.

Then the Cartesian product $\prod\limits_{i=1}^{m} C_i$ is a *MdgM* set

if and only if

C_i is a *MdgM* set for every $i = 1, ..., m$.

FIXED POINT TYPE THEOREMS ON MdgM SETS

$C \subseteq \mathbb{R}^n$ nonempty closed convex set, $S \subseteq C$

$x \in S$ is a *lin-fixed point* of $F : S \rightrightarrows C$
(of $f : S \longrightarrow C$)
if $x \in F(x) + \text{lin } C$
(if $x \in f(x) + \text{lin } C$, respectively).

$x \in S$ is a lin-fixed point of $F : S \rightrightarrows C$
(of $f : S \longrightarrow C$)
if and only if
$[x] \in [F(x)]$
(if and only if $[x] = [f(x)]$, respectively),
with $[z] = z + \text{lin } C$
(the equivalence class of $z \in C$ in $C/\text{lin } C$)
and $[F(x)] := \{[y] : y \in F(x)\}$.

This is **not** equivalent to saying that $[x]$ is a fixed point for a certain self map on the quotient space $\mathbb{R}^n/\text{lin } C$.

THEOREM.
Let $C \subseteq \mathbb{R}^n$ be a *MdgM* set and
$U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
If $F : C \cap U \rightrightarrows C$ has a closed graph and
has nonempty, convex and compact images,
then it has a lin-fixed point.

COROLLARY.
Let $C \subseteq \mathbb{R}^n$ be a *MdgM* set and
$U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
If $f : C \cap U \longrightarrow C$ is continuous,
then it has a lin-fixed point.

n it has a lin-fixed point.

COROLLARY.
Let $C \subseteq \mathbb{R}^n$ be a non compact *MdgM* set and
$U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
If $F : C \cap U \rightrightarrows C$ has a closed graph and
has nonempty, convex and compact images,
then it has infinitely many lin-fixed points.

COROLLARY.

Let $C \subseteq \mathbb{R}^n$ be a non compact *MdgM* set and $U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
If $f : C \cap U \longrightarrow C$ is continuous,
then it has infinitely many lin-fixed points.

PROPOSITION.

Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $f : C \longrightarrow C$.
For $x \in C$, the following statements are equivalent:

a) $f^k(x) \in x + \text{lin } C$ for every $k \geq 1$

b) $f^k(x)$ is a lin-fixed point of f for every $k \geq 0$.
(with the convention that f^0 is the identity).

$C \subseteq \mathbb{R}^n$ nonempty closed convex set.
$x \in C$ is a *strongly lin-fixed point* of $f : C \longrightarrow C$
if it satisfies the equivalent conditions a) and b) of the preceding proposition.

PROPOSITION.
Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $f : C \longrightarrow C$ be such that

$$f(x + \text{lin } C) \subseteq f(x) + \text{lin } C, \ \forall x \in C.$$

Then every lin-fixed point of f
is a strongly lin-fixed point of f.

$$U \text{ supplementary subspace to lin } C$$

$$g : C \cap U \to C, \qquad \tau : C \to \mathbb{R}, \qquad h : \text{lin } C \to \text{lin } C$$

$$\text{f=g} \circ (p_U)_{|C} + \tau \cdot \left(h \circ (p_{\text{lin } C})_{|C} \right)$$

PROPOSITION.
Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set,
$f : C \longrightarrow C$, and
$U \subseteq \mathbb{R}^n$ be a supplementary subspace to lin C.
Then

$$f(x + \text{lin } C) \subseteq f(x) + \text{lin } C, \ \forall x \in C$$

if and only if

$$p_U \circ f \circ (p_U)_{|C} = p_U \circ f.$$

THEOREM.

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $f : C \longrightarrow C$ be such that $p_{\text{lin } C} \circ f$ is a contraction. Then, for every strongly lin-fixed point x of f, $\left(f^k(x)\right)_k$ converges to a fixed point of f.

COROLLARY.

Let $C \subseteq \mathbb{R}^n$ be a *MdgM* set and $f : C \longrightarrow C$ be a continuous mapping satisfying

$$f(x + \text{lin } C) \subseteq f(x) + \text{lin } C, \ \forall x \in C.$$

and such that $p_{\text{lin } C} \circ f$ is a contraction.
Then f has a fixed point.

SOME REFERENCES

Goberna, M.A.; González, E.; Martínez-Legaz, J.E.; Todorov, M.I.
Motzkin decomposition of closed convex sets.
J. Math. Anal. Appl. 364, No. 1, 209-221 (2010).

Goberna, M.A.; Martínez-Legaz, J.E.; Todorov, M.I.
On Motzkin decomposable sets and functions.
J. Math. Anal. Appl. 372 (2010), No. 2, 525–537.

Martínez-Legaz, J.E., Pintea, C.
Closed convex sets of Minkowski type.
J. Math. Anal. Appl. 444, No. 2, 1195-1202 (2016).

Martínez-Legaz, J.E.; Pintea, C.
Closed convex sets of Motzkin, generalized Minkowski, and Pareto bordered types.
Commun. Optim. Theory 2024 (2024) 2, 17 pp.

Martínez-Legaz, J.E.; Pintea, C.
Closed convex sets that are both Motzkin decomposable and generalized Minkowski sets.
J. Nonlinear Var. Anal. 8, No. 4, 571-579 (2024).