Uniformly convex sets and extendability of Lipschitz quasiconvex functions

Joint work with **Carlo A. De Bernardi**

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"Convex" implies "quasiconvex", but not vice versa (e.g., $f(x) = \sqrt{\|x\|}$ on a normed space).

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A classical example of an application:
X a reflexive Banach space (e.g., a Hilbert space), $C \subset X$ a closed convex set, $f : C \to \mathbb{R}$ continuous quasiconvex function. If f is coercive (i.e., $f(x) \to \infty$ as $x \in C$, $\|x\| \to \infty$) then there exists min $f(C)$.

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Independently, [a particular form of] the same formula appears as a footnote in a paper by H. Whitney, 1934.
The convex case belongs essentially to the mathematical folklore.)

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Let's observe that *f can always be extended to an L-Lipschitz quasiconvex function to $\overline{A}$* (by continuity).

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Definition

An *$\Omega(A)$-family* is a family of sets $\{D_\alpha\}_{\alpha\in\mathbb{R}}$ such that:

$$\alpha < \beta \;\Rightarrow\; \overline{D}_\alpha^A \subset D_\beta\,; \qquad \bigcap_{\alpha\in\mathbb{R}} D_\alpha = \emptyset\,; \qquad \bigcup_{\alpha\in\mathbb{R}} D_\alpha = A\,.$$

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(c) [Easy exercise.] *For* $\{D_\alpha\}_\alpha$ *and* f *as in (a),* f *is L-Lipschitz (with* $L > 0$*) if and only if* $d(D_\alpha, A \setminus D_\beta) \ge (1/L)(\beta - \alpha)$ *for* $\alpha < \beta$*.*

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(c) [Easy exercise.] *For $\{D_\alpha\}_\alpha$ and f as in (a), f is L-Lipschitz (with $L > 0$) if and only if $d(D_\alpha, A \setminus D_\beta) \ge (1/L)(\beta - \alpha)$ for $\alpha < \beta$. More generally, f is uniformly continuous with an invertible modulus of continuity ω if and only if $d(D_\alpha, A \setminus D_\beta) \ge \omega^{-1}(\beta - \alpha)$ whenever $\alpha < \beta$.*

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has the following properties:

1. μ_C is a nonnegative, finite and sublinear (i.e., positively homogeneous and subadditive; hence convex) function;
2. μ_C is $(1/\delta)$-Lipschitz;
3. $[\mu_C < 1] = C$ and $[\mu_C \leq 1] = \overline{C}$.

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Then f is an L-Lipschitz quasiconvex function on A such that

$$[f < \alpha_n] = D_n \quad \textit{for each } n \in \mathbb{N}.$$

Counterexample 1

Simple counterexample

Let $A \subset \mathbb{R}^2$ *be an open convex set which is not strictly convex. Then there exists a Lipschitz quasiconvex function* f *on* A *such that* f *admits no continuous quasiconvex extension to* $\mathbb{R}^2$*.*
(Not even to any open convex set containing $A \cup \{x_0\}$ for a given $x_0 \notin \mathrm{ext}(A)$.)

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$(0, b_n)$ such that $b_n \nearrow -1$,
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By convexity, $F(0,b_n) \ge \alpha_n \ge \alpha_2$ for $n \ge 2$. So $F(0,-1) \ge \alpha_2$ by continuity. *Contradiction!*

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There exists a Lipschitz quasiconvex function on the open unit disc A in the Euclidean plane $\mathbb{R}^2$ that admits **no** *Lipschitz quasiconvex extension to $\mathbb{R}^2$.* (Not even to any open convex set containing $\overline{A}$.)

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For $n \in \mathbb{N}$ consider:

- the points $w_n := (\sqrt{1 - t_n^2}, t_n) \in \partial A$;
- the open half-planes $H_n := \left\{(x, y) : y - t_n - b_n(x - \sqrt{1 - t_n^2}) > 0\right\}$;
- the lines $L_n := \partial H_n \ni w_n$;
- the open convex sets $D_n := A \cap H_n$.

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By our *Construction Lemma*, there exist

- a sequence $1 = \alpha_1 < \alpha_2 < \ldots$ (depending on ν_k's), and
- an L-Lipschitz quasiconvex function f on A

such that $[f < \alpha_n] = D_n$ for each n.

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3. $A \cap \nu_{n+1} D_n \subset D_{n+1}$ for certain $\nu_{n+1} > 1$.

By our *Construction Lemma*, there exist

- a sequence $1 = \alpha_1 < \alpha_2 < \dots$ (depending on ν_k's), and
- an L-Lipschitz quasiconvex function f on A

such that $[f < \alpha_n] = D_n$ for each n.

The construction of f also gives

$$[f \le \alpha_n] = \overline{D}_n^A, \quad n \in \mathbb{N}.$$

Counterexample 2 – continuation 2

Proceeding by contradiction, assume that f admits an M-Lipschitz quasiconvex extension F, defined on $B = rA$ for some $r > 1$. And denote $E_n := [F \leq \alpha_n]_B$.

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Since F is M-Lipschitz on B, it easily follows that

$$d(E_n, B \setminus E_{n+1}) = d\big(\,[F \leq \alpha_n]_B\,,\,[F > \alpha_{n+1}]_B\,\big) \geq \tfrac{1}{M}(\alpha_{n+1} - \alpha_n).$$

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A precise calculation gives that $\Delta_n \to 0$, which is **a contradiction**.

Good news: extending from a **subspace** works!

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Theorem

Let Y be a subspace of a normed space X. Then each L-Lipschitz qc. function on Y admits an L-lipschitz (the same constant!) qc. extension to the whole X.

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Rough idea. $C_\alpha := [f < \alpha]$, $\alpha \in \mathbb{R}$, forms an $\Omega(Y)$-family of open convex subsets of Y s.t. $d(c_\alpha, Y \setminus C_\beta) \geq \omega^{-1}(\beta - \alpha)$, $\alpha < \beta$. Consider the sets

$$D_\alpha := \bigcup_{y \in Y} U_X\left[y, d(y, Y \setminus C_\alpha)\right].$$

They extend C_α's in an appropriate way. Define $F(x) := \sup\{\alpha : x \notin D_\alpha\}$.

The result, *not as good as for subspaces*, is as follows. The definitions and a more precise statement will be given later.

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Let $A \subsetneq X$ be a nonempty open convex set. If A is uniformly convex, then every Lipschitz quasiconvex function f on A admits a uniformly continuous (but not necessarily Lipschitz) quasiconvex extension F to the whole X.

Uniformly convex sets

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Let $C \subset X$ be a convex set whose interior and boudary are nonempty. We say that C is *uniformly convex* if

$$x_n, y_n \in \partial C \ (n \in \mathbb{N}),\ d(\tfrac{x_n+y_n}{2}, \partial C) \to 0 \quad \Rightarrow \quad \|x_n - y_n\| \to 0.$$

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The above equalities (known for $C = B_X$ [Day, 1944]), except for the the last one, are *not* trivial for a general C.

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Easy: $\delta_C(0) = 0$, and $\delta_C(\varepsilon) \leq \frac{1}{2}\varepsilon$ for $0 \leq \varepsilon < \operatorname{diam}(C)$.

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Let C be a convex set in a normed space X, $\mathrm{int}(C) \neq \emptyset \neq \partial C$.

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Consequently, if C is uniformly convex then δ_C can be extended to a homeomorphism of $[0, \infty)$ onto itself.

The result

Theorem

Let $A \subsetneq X$ be a nonempty open convex set. If A is uniformly convex, then every Lipschitz quasiconvex function f on A admits a uniformly continuous (but not necessarily Lipschitz) quasiconvex extension F to the whole X.

More precisely, if f is L-Lipschitz and $\delta_A\colon [0,\infty) \to [0,\infty)$ is a homeomorphism that extends the corresponding modulus of convexity δ_A, then the qc. extension F is uniformly continuous with the invertible modulus of continuity $\tilde{\omega}(t) = L \cdot \delta_A^{-1}(t)$.

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More general statement: if f is uniformly continuous with an invertible modulus of continuity ω, then F is uniformly continuous with the invertible modulus of continuity $\tilde{\omega}(t) = \omega\big(\delta_A^{-1}(t)\big)$.

Moreover $F(X) \subset \overline{f(A)}$.

Idea of the proof

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The strict sublevel sets

$$C_\alpha := [f < \alpha], \quad \alpha \in \mathbb{R},$$

are extended to $D_\alpha := \emptyset$ if $C_\alpha = \emptyset$; $D_\alpha := X$ if $C_\alpha = A$; otherwise

$$D_\alpha := \bigcap_{y \in \overline{A \cap \partial C_\alpha}} K(y, C_\alpha)\,.$$

Here:

$$K(y, C_\alpha) = \bigcap [f \leq f(y)]$$

where the intersection is w.r.t. all nonzero supporting functionals to C_α at y.

A general **non**-extendability result

Theorem

Let $A \subsetneq X$ be a nonempty open convex set. If A is not LUR (locally uniformly convex/rotund), then there exists a Lipschitz quasiconvex function $f\colon A \to \mathbb{R}$ that does not admit any continuous quasiconvex extension to X.

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Remains open:

if every Lipschitz qc. function on A admits a uniformly continuous qc. extension to the whole X, is then A necessarily uniformly convex?

LUR points and the local modulus of convexity

Definition

Let $C \subset X$ be a convex set whose interior and boudary are nonempty, $x \in \partial C$. We say that C is LUR (locally uniformly convex/rotund) at x if

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For $C = B_X$, the first complete proof is due to J. Daneš, 1976.

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We say that C is LUR if C is LUR at each point $x \in \partial C$.

Papers

- C.A. De Bernardi and L. V.,
  Rotundity properties, and non-extendability of Lipschitz quasiconvex functions,
  J. Conv. Anal. **30** (2023), 329–342.
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  On extension of uniformly continuous quasiconvex functions,
  Proc. Amer. Math. Soc. **151** (2023), 1705–1716.

Papers

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  J. Conv. Anal. **30** (2023), 329–342.

- C.A. De Bernardi and L. V.,
  On extension of uniformly continuous quasiconvex functions,
  Proc. Amer. Math. Soc. **151** (2023), 1705–1716.

- C.A. De Bernardi and L. V.,
  Global moduli of convexity of convex sets, preprint in preparation.

- C.A. De Bernardi and L. V.,
  Local moduli of convexity of convex sets, preprint in preparation.

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Thank you for your attention!