The Banach–Mazur distance between isomorphic spaces of continuous functions is not always an integer

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- The Banach-Mazur distance between two isomorphic Banach spaces X and Y is defined by

$$d(X,Y) = \inf\left\{\|\phi\|\,\|\phi^{-1}\| : \phi \text{ is an isomorphism from } X \text{ onto } Y\right\}.$$

- For a locally compact Hausdorff space K, $C_0(K)$ denotes the Banach space of all continuous real-valued functions vanishing at infinity on K, endowed with the supremum norm. We recall that a function $f : K \to \mathbb{R}$ is said to vanish at infinity on K if for every $\varepsilon > 0$ the set $\{x \in K : |f(x)| \geq \varepsilon\}$ is compact.
- As usual, if K is compact, we denote $C_0(K)$ by $C(K)$.
- If α is an ordinal number, $[1,\alpha]$ denotes the set of all ordinals λ such that $1 \leq \lambda \leq \alpha$, provided with the order topology.

Theorem (S. Mazurkiewicz, W. Sierpiński, Fund. Math. (1920))

Every compact countable metric space K is homeomorphic to the interval of ordinals $[1, \alpha]$ for some countable ordinal α.

Theorem (C. Bessaga, A. Pełczyński, Studia Math. (1960))

If $\omega \leq \alpha \leq \beta < \omega_1$, then $C([1, \alpha])$ is isomorphic to $C([1, \beta])$ if and only if $\beta < \alpha^\omega$, where ω denotes the first infinite ordinal and ω_1 denotes the first uncountable ordinal.

Theorem (A. A. Miljutin, Teor. Funkc. Anal. i Pril. (1966))

If K and L are uncountable compact metric spaces, then $C(K)$ and $C(L)$ are isomorphic.

Theorem (Banach–Stone)

If K and L are compact Hausdorff spaces, then $C(K)$ is isometrically isomorphic to $C(L)$ if and only if K and L are homeomorphic.

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Theorem (D. Amir, Israel J. Math. (1965))

Let K and L be compact Hausdorff spaces. If there exists an isomorphism T of $C(K)$ onto $C(L)$ with $\|T\| \, \|T^{-1}\| < 2$, then K and L are homeomorphic.

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Theorem (M. Cambern, Proc. Amer. Math. Soc. (1966))

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Observation

There are no two isomorphic $C_0(K)$ spaces for which the Banach–Mazur distance belongs to the interval $(1, 2)$.

In 1968, Aleksander Pełczyński posed the following question:

Problem 28 in Dissertationes Math. (1968)

Let K and L be compact Hausdorff spaces such that $C(K)$ is isomorphic to $C(L)$. Is it true that $d(C(K), C(L)) = n$ for some $n \in \mathbb{N}$?

Example (M. Cambern, Notices Amer. Math. Soc. (1969))

In 1969, Cambern gave an example of metric spaces K and L such that K is compact, L is locally compact but noncompact, and an isomorphism T of $C(K)$ onto $C_0(L)$ with $\|T\|\|T^{-1}\| = 2$. For this purpose, he considered the spaces $C([1, \omega])$ and $C_0([1, \omega 2))$.

Example (H. B. Cohen, Proc. Amer. Math. Soc. (1975))

In 1975, Cohen conctructed nonhomeomorphic uncountable compact Hausdorff spaces K and L and an isomorphism T of $C(K)$ onto $C(L)$ such that $\|T\|\|T^{-1}\| = 2$.

Theorem (M. Cambern, Studia Math. (1968))

$$d(c_0, c) = 3.$$

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Theorem (M. Cambern, Math. Ann. (1970))

Let Γ be an infinite set with the discrete topology.

1. *If K is any locally compact Hausdorff space which contains a point of accumulation, and if T is any isomorphism of $C_0(K)$ onto $C_0(\Gamma)$, then $\|T\|\,\|T^{-1}\| \geq 3$.*
2. *If Γ^* denotes the one-point compactification of Γ, then there exists an isomorphism T of $C(\Gamma^*)$ onto $C_0(\Gamma)$ satisfying $\|T\|\,\|T^{-1}\| = 3$.*

- For any ordinal number α, the αth derivative of K, $K^{(\alpha)}$, is defined by transfinite induction: $K^{(0)} = K$, $K^{(1)}$ is the set of non-isolated points of K, and

$$K^{(\alpha)} = \begin{cases} (K^{(\beta)})^{(1)} & \text{if } \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} K^{(\beta)} & \text{otherwise.} \end{cases}$$

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Theorem (Y. Gordon, Israel J. Math. (1970))

Let K, L be locally compact Hausdorff spaces and let T be an isomorphism of $C_0(K)$ into $C_0(L)$. If there is an ordinal α such that $\left|K^{(\alpha)}\right| > \left|L^{(\alpha)}\right|$, then $\|T\| \|T^{-1}\| \geq 3$.

Theorem (Y. Gordon, Israel J. Math. (1970))

$$d(C([0,1]), C_0([0,1) \cup \{2\})) = 3.$$

Remark

The same reasoning gives

$$d(C(\Delta), C_0(\Delta \setminus \{1\} \cup \{2\})) = 3,$$

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Theorem (Ł. Piasecki, J. Villada (2022))

$$d(C(\Delta), C_0(\Delta \setminus \{1\})) = 2.$$

Theorem (L. Candido, E. M. Galego, Fund. Math. (2012))

Suppose that $1 < n, k < \omega$. *Then*

$$d(c_0, C([1, \omega^n k])) = 2n + 1.$$

Theorem (Y. Gordon, Israel J. Math. (1970))

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Theorem (L. Candido, E. M. Galego, Studia Math. (2013))

Suppose that $1 < n, k < \omega$. *Then*

- $3 \leq d(c, C([1, \omega k])) \leq 2 + \sqrt{5}$,
- $2n - 1 \leq d(c, C([1, \omega^n])) \leq n + \sqrt{(n-1)(n+3)}$,
- $2n + 1 \leq d(c, C([1, \omega^n k])) \leq n + 1 + \sqrt{n(n+4)}$.

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Conjecture (L. Candido, E. M. Galego, Studia Math. (2013))

Let $n \geq 2$ *and* $k \geq 2$ *be integers. Then:*

- $d(c, C([1, \omega(k+1)]))$ *is equal to* 3, 4 *or* $2 + \sqrt{5}$,
- $d(c, C([1, \omega^n]))$ *is equal to* $2n - 1$, $2n$ *or* $n + \sqrt{(n-1)(n+3)}$,
- $d(c, C([1, \omega^n k]))$ *is equal to* $2n + 1$, $2n + 2$ *or* $n + 1 + \sqrt{n(n+4)}$.

Theorem (A. Gergont, Ł. Piasecki (2023))

Let $3 \leq k < \omega$. *Then*

$$d(c, C([1, \omega k])) \geq 3 + \frac{\sqrt{3k^2 - 2k + 1} - k - 1}{k}.$$

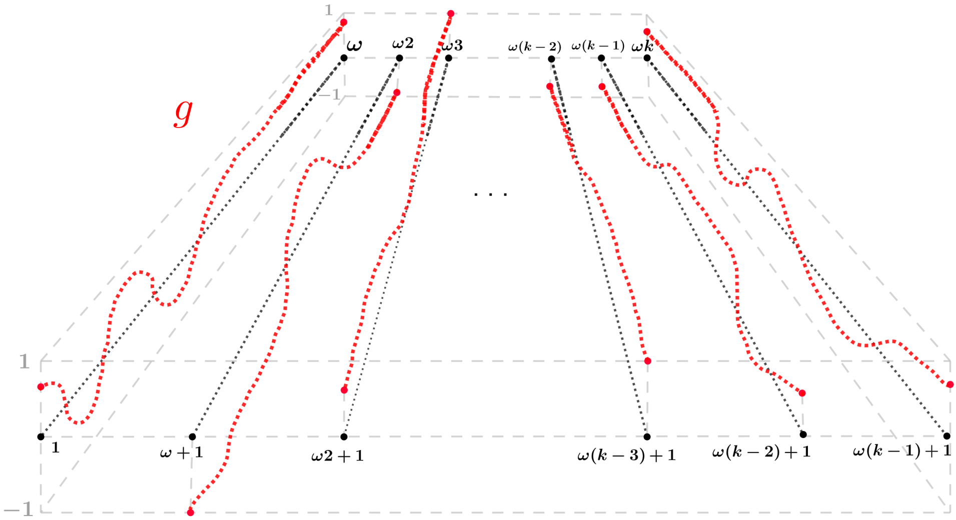
In particular, for $k = 3$ *we have*

$$d(c, C([1, \omega 3])) \geq 3 + \frac{\sqrt{22} - 4}{3} \approx 3.23.$$

Remark

Observe that if k *tends to infinity, then* $3 + \frac{\sqrt{3k^2-2k+1}-k-1}{k}$ *increases to* $2 + \sqrt{3} \approx 3.73$.

Proof.
Let T be an isomorphism from $C([1, \omega k])$ onto $C([1, \omega])$. Without loss of generality we may assume that $\|T^{-1}\| = 1$ (otherwise, we replace T by $S = \|T^{-1}\| T$). Then T is norm-increasing, that is, $\|T(f)\| \geq \|f\|$ for every $f \in C([1, \omega k])$. Since T is onto, there exists $g \in C([1, \omega k])$ such that $T(g) = \mathbf{1}$, where $\mathbf{1}$ denotes the constant function equal to 1 on $[1, \omega]$. Since $\|T^{-1}\| = 1$, we have $\|g\| \leq 1$.



1
−1
1
−1
g
ω
ω2
ω3
ω(k − 2)
ω(k − 1)
ωk
1
ω + 1
ω2 + 1
ω(k − 3) + 1
ω(k − 2) + 1
ω(k − 1) + 1

Step 1. We claim that there exist $i_1, i_2, \ldots, i_{k-2} \in \{0, 1, \ldots, k-1\}$ such that $i_m \neq i_n$ for $m \neq n$ and

$$T\left(\sum_{j=1}^{k-2} g\chi_{[\omega i_j+1,\omega(i_j+1)]}\right)(\omega) \geq \frac{k-2}{k},$$

where χ_A denotes the characteristic function of $A \subset [1, \omega k]$.
Suppose it is not true. Then, for any $i_1, i_2, \ldots, i_{k-2} \in \{0, 1, \ldots, k-1\}$ such that $i_m \neq i_n$ for $m \neq n$, we have

$$T\left(\sum_{j=1}^{k-2} g\chi_{[\omega i_j+1,\omega(i_j+1)]}\right)(\omega) < \frac{k-2}{k}.$$

Therefore, for any $i, j \in \{0, 1, \ldots, k-1\}$ such that $i \neq j$, we have

$$T\left(g\chi_{[\omega i+1,\omega(i+1)]} + g\chi_{[\omega j+1,\omega(j+1)]}\right)(\omega) > \frac{2}{k}.$$

Hence, if k is an even number, we get a contradiction,

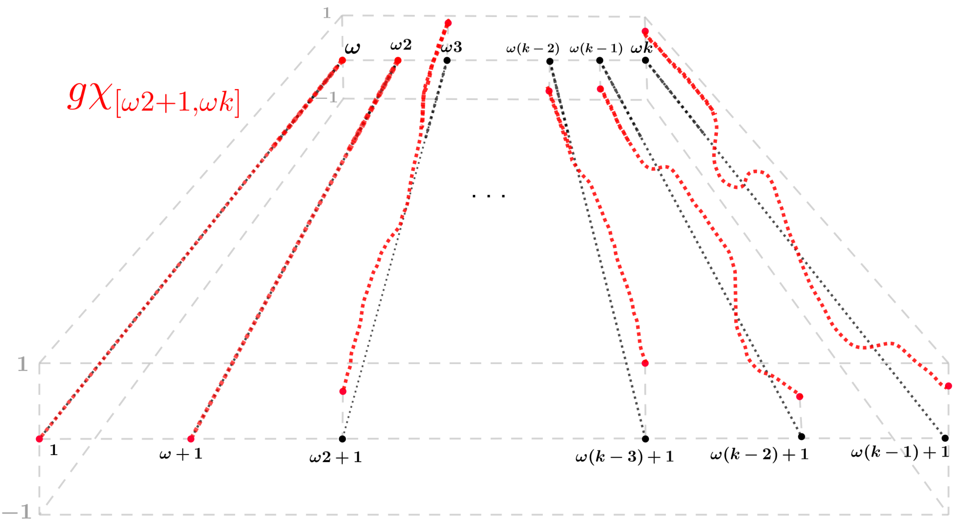
$$\begin{aligned} 1 &= \mathbf{1}(\omega) = T\left(\sum_{i=0}^{k-1} g\chi_{[\omega i+1,\omega(i+1)]}\right)(\omega) = \sum_{i=0}^{k-1} T\left(g\chi_{[\omega i+1,\omega(i+1)]}\right)(\omega) \\ &= \sum_{i=0}^{\frac{k}{2}-1} T\left(g\chi_{[\omega(2i)+1,\omega(2i+1)]} + g\chi_{[\omega(2i+1)+1,\omega(2i+2)]}\right)(\omega) > \frac{k}{2}\cdot\frac{2}{k} = 1. \end{aligned}$$

If, on the other hand, k is an odd number,

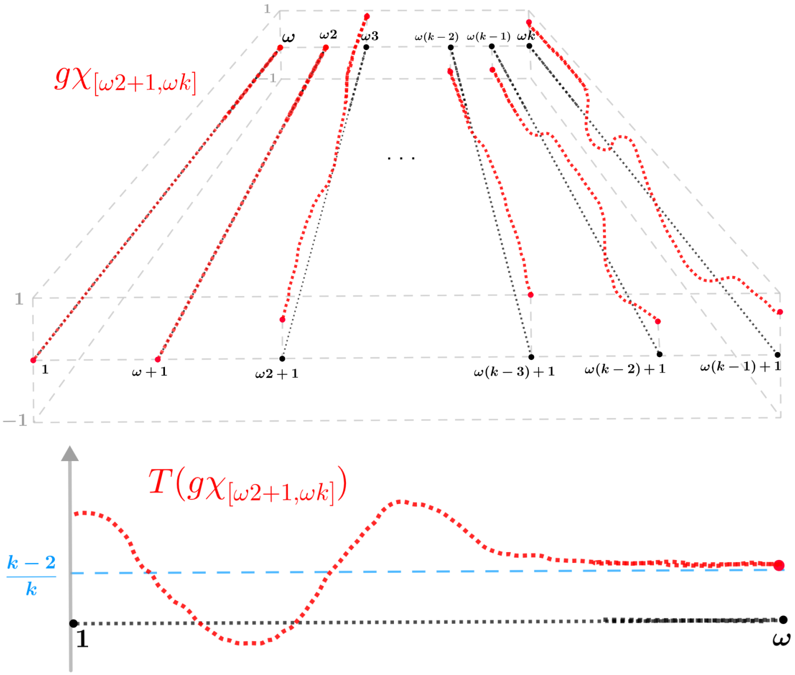
$$\begin{aligned} 2 &= T\left(\sum_{i=0}^{k-1} 2g\chi_{[\omega i+1,\omega(i+1)]}\right)(\omega) = T(g\chi_{[1,\omega]} + g\chi_{[\omega(k-1)+1,\omega k]})(\omega) \\ &\quad + \sum_{i=0}^{k-2} T(g\chi_{[\omega i+1,\omega(i+1)]} + g\chi_{[\omega(i+1)+1,\omega(i+2)]})(\omega) > k\cdot\frac{2}{k} = 2, \end{aligned}$$

a contradiction. The proof of our claim is completed. In what follows, without loss of generality we can assume that

$$T(g\chi_{[\omega 2+1,\omega k]})(\omega) \geq \frac{k-2}{k}.$$



$g\chi_{[\omega 2+1,\omega k]}$
ω
$\omega 2$
$\omega 3$
$\omega(k-2)$
$\omega(k-1)$
ωk
$\cdots$
1
-1
1
-1
1
$\omega+1$
$\omega 2+1$
$\omega(k-3)+1$
$\omega(k-2)+1$
$\omega(k-1)+1$



$gX_{[\omega 2+1,\omega k]}$
ω
$\omega 2$
$\omega 3$
$\omega(k-2)$
$\omega(k-1)$
ωk
$\cdots$
1
-1
1
$\omega+1$
$\omega 2+1$
$\omega(k-3)+1$
$\omega(k-2)+1$
$\omega(k-1)+1$
$T(gX_{[\omega 2+1,\omega k]})$
$\frac{k-2}{k}$
1
ω

Step 2. Let X be a linear subspace of $C([1, \omega k])$ spanned by $\chi_{[1,\omega]}$ and $\chi_{[\omega+1,\omega 2]}$,

$$X = \{\alpha_1 \chi_{[1,\omega]} + \alpha_2 \chi_{[\omega+1,\omega 2]} : \alpha_1, \alpha_2 \in \mathbb{R}\}.$$

Consider a function $x^* : X \to \mathbb{R}$ defined by

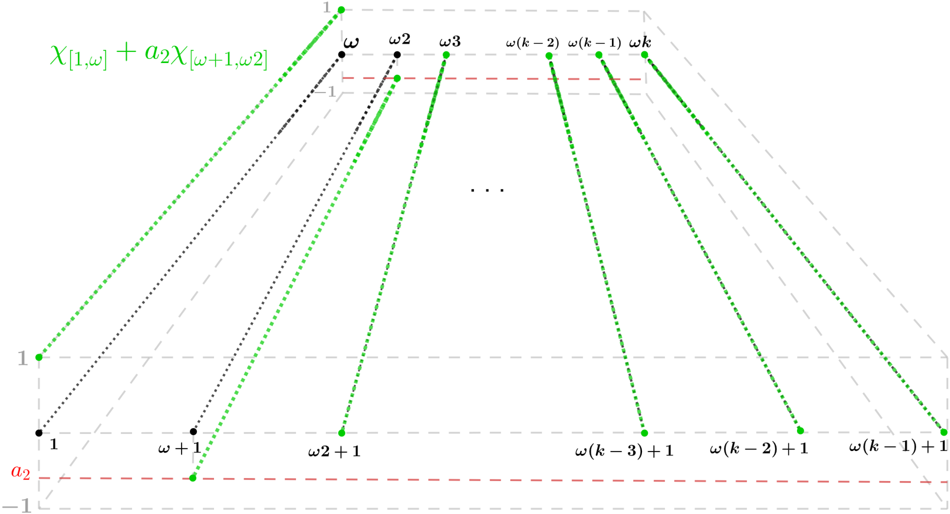
$$x^*(\alpha_1 \chi_{[1,\omega]} + \alpha_2 \chi_{[\omega+1,\omega 2]}) = T(\alpha_1 \chi_{[1,\omega]} + \alpha_2 \chi_{[\omega+1,\omega 2]})(\omega).$$

Clearly, x^* is a linear functional. Since $\dim X = 2 > \dim x^*(X)$, there exists $a = (a_1, a_2)$ with $\max\{|a_1|, |a_2|\} = 1$ such that

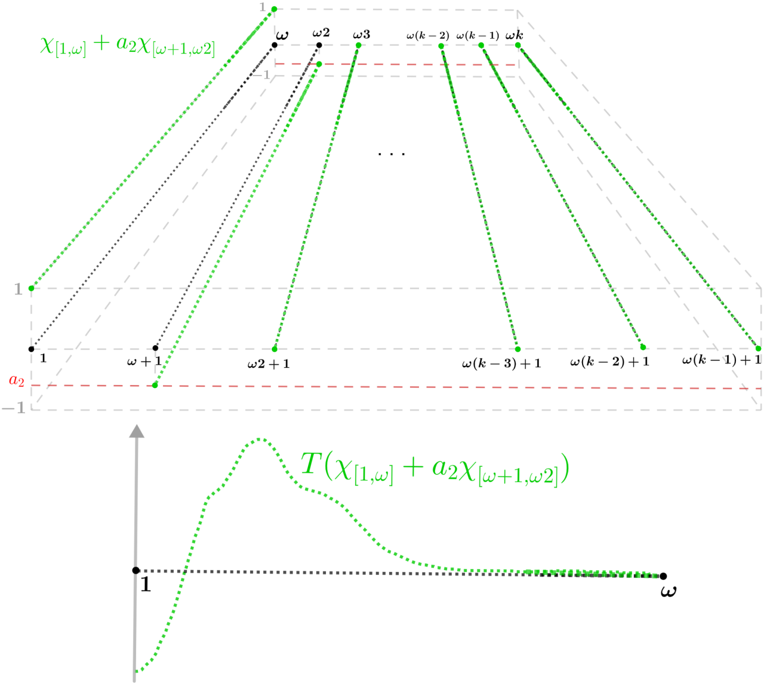
$$x^*(a_1 \chi_{[1,\omega]} + a_2 \chi_{[\omega+1,\omega 2]}) = 0.$$

In what follows, w.l.o.g. we can assume that $a_1 = 1$, that is,

$$x^*(\chi_{[1,\omega]} + a_2 \chi_{[\omega+1,\omega 2]}) = T(\chi_{[1,\omega]} + a_2 \chi_{[\omega+1,\omega 2]})(\omega) = 0.$$



$\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]}$
1
-1
ω
$\omega2$
$\omega3$
$\omega(k-2)$
$\omega(k-1)$
ωk
$\cdots$
1
a_2
-1
1
$\omega+1$
$\omega2+1$
$\omega(k-3)+1$
$\omega(k-2)+1$
$\omega(k-1)+1$



$\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]}$
1
-1
ω
$\omega2$
$\omega3$
$\omega(k-2)$
$\omega(k-1)$
ωk
. . .
1
a_2
-1
1
$\omega+1$
$\omega2+1$
$\omega(k-3)+1$
$\omega(k-2)+1$
$\omega(k-1)+1$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]})$
1
ω

Step 3. We claim that for every $\varepsilon > 0$ and every $M \in \mathbb{N}$ there is only a finite number of elements of $\{\chi_{\{l\}} : l \in \mathbb{N}\}$ for which the following condition holds: for at least one $n \in \{1, 2, \ldots, M\}$

$$\left|T(\chi_{\{l\}})(n)\right| \geq \varepsilon.$$

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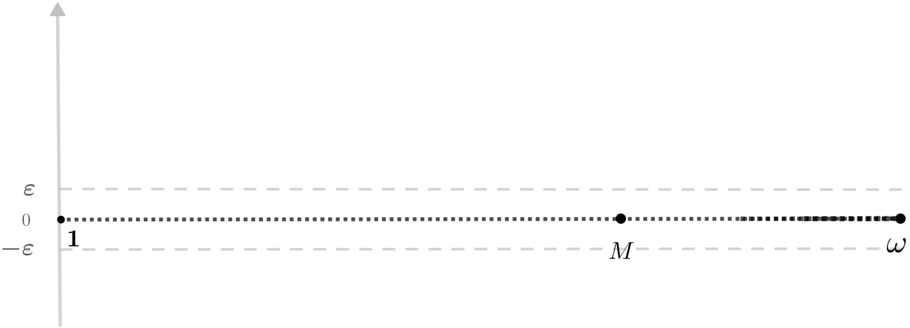

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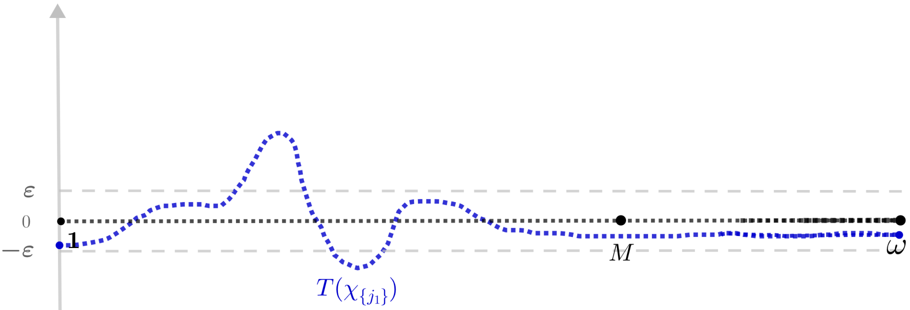

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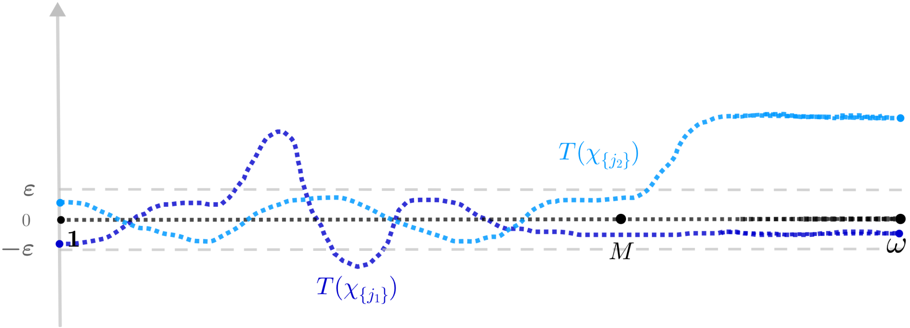

Suppose it is not true. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ s.t.

$$\left|T(\chi_{\{l\}})(n_0)\right| \geq \varepsilon \text{ for infinitely many } l \in \mathbb{N}.$$

For $N \in \mathbb{N}$ we define $f_N = \sum_{l=1}^{N} \operatorname{sign}\left(T(\chi_{\{l\}})(n_0)\right) \cdot \chi_{\{l\}}$. Then $\|f_N\| = 1$ and

$$\|T(f_N)\| \geq |T(f_N)(n_0)| = \sum_{l=1}^{N} \left|T(\chi_{\{l\}})(n_0)\right|.$$

Letting $N \to \infty$, we get a contradiction with boundedness of T.

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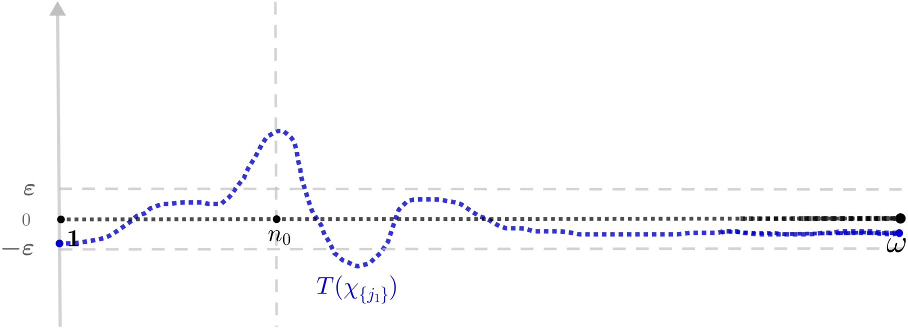

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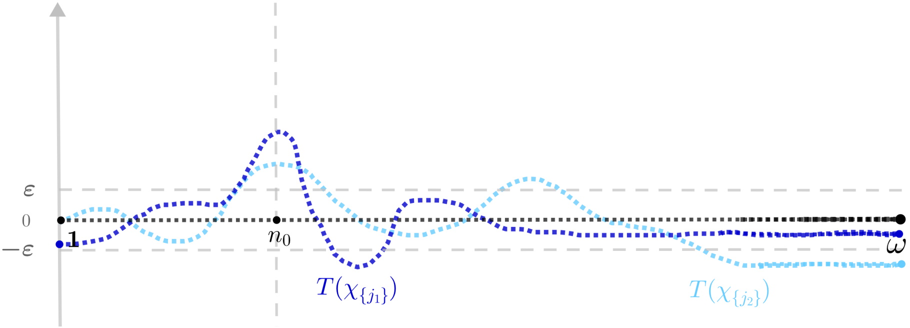

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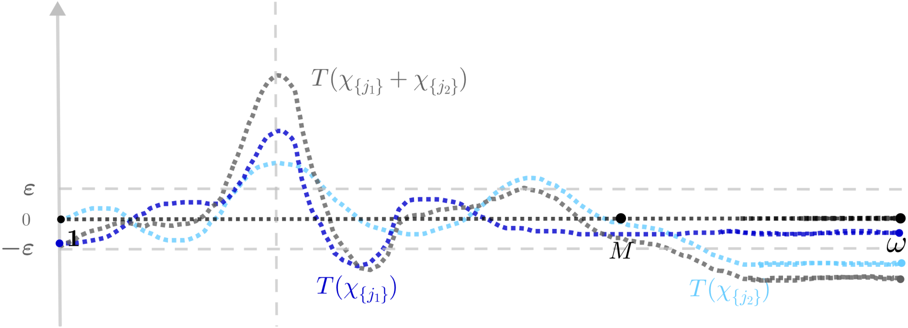

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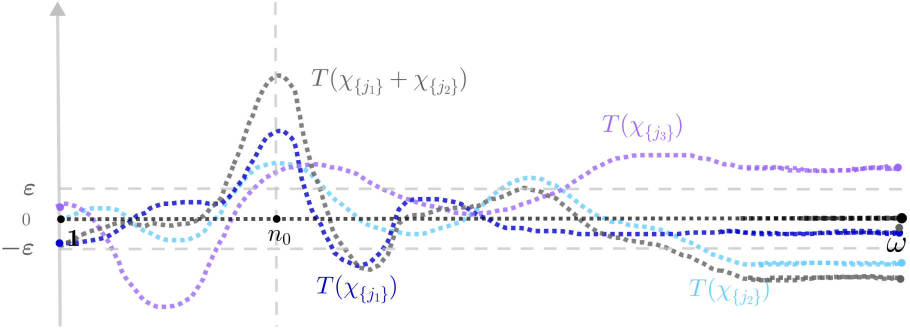


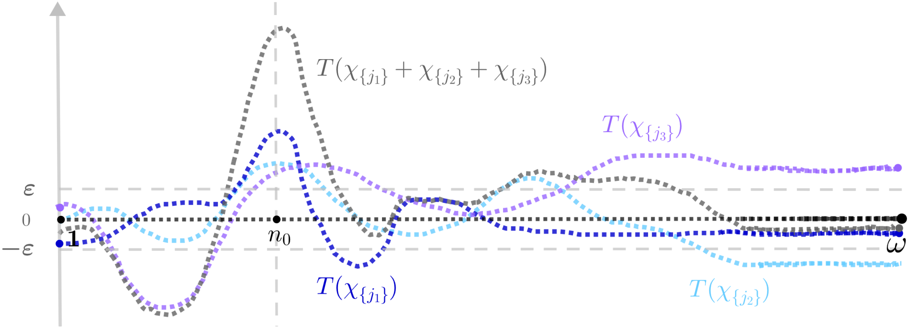
Suppose it is not true. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ s.t.

$$\left|T(\chi_{\{l\}})(n_0)\right| \geq \varepsilon \text{ for infinitely many } l \in \mathbb{N}.$$

For $N \in \mathbb{N}$ we define $f_N = \sum_{l=1}^{N} \operatorname{sign}\left(T(\chi_{\{l\}})(n_0)\right) \cdot \chi_{\{l\}}$. Then $\|f_N\| = 1$ and

$$\|T(f_N)\| \geq |T(f_N)(n_0)| = \sum_{l=1}^{N} \left|T(\chi_{\{l\}})(n_0)\right|.$$

Letting $N \to \infty$, we get a contradiction with boundedness of T.

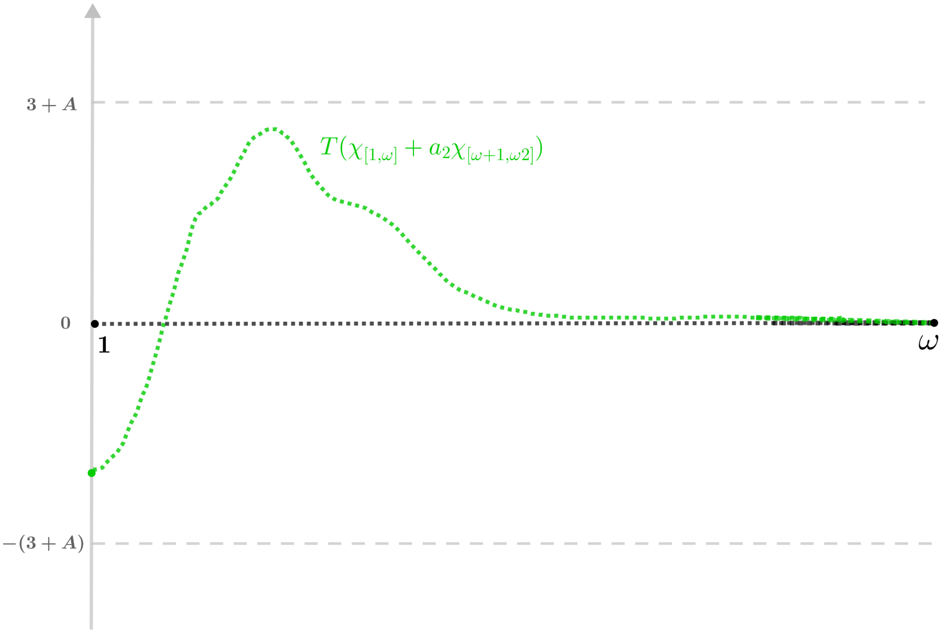


Step 4. Suppose now that

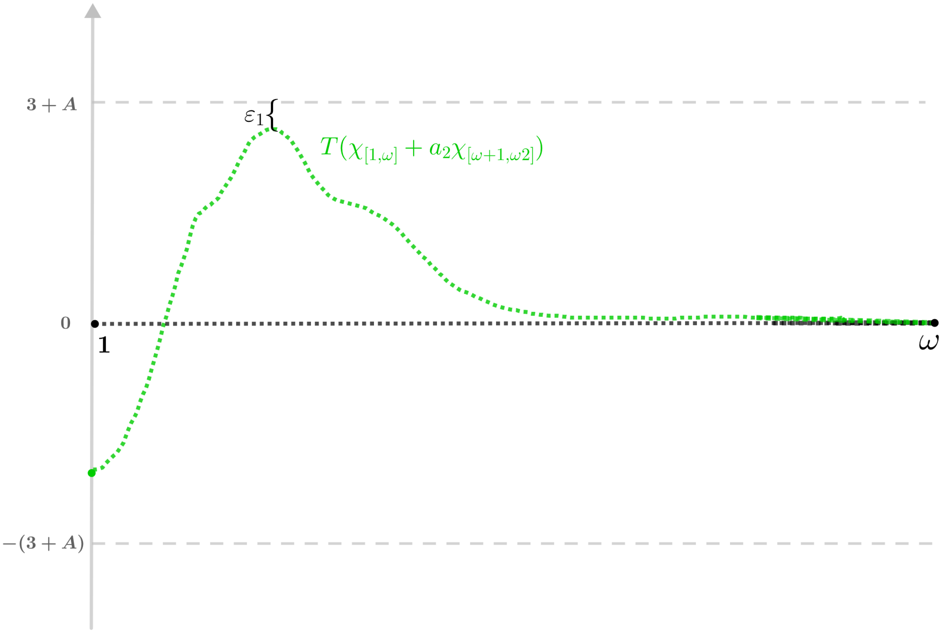
$$\|T\| < 3 + \frac{\sqrt{3k^2 - 2k + 1} - k - 1}{k}.$$

We show that this assumption leads to a contradiction. Put

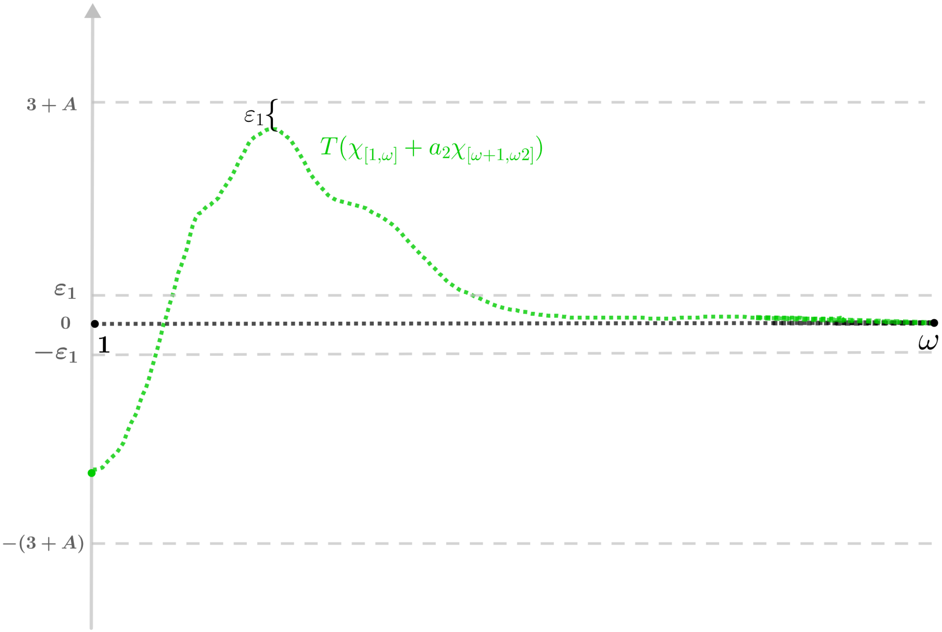
$$A = \frac{\sqrt{3k^2 - 2k + 1} - k - 1}{k}.$$



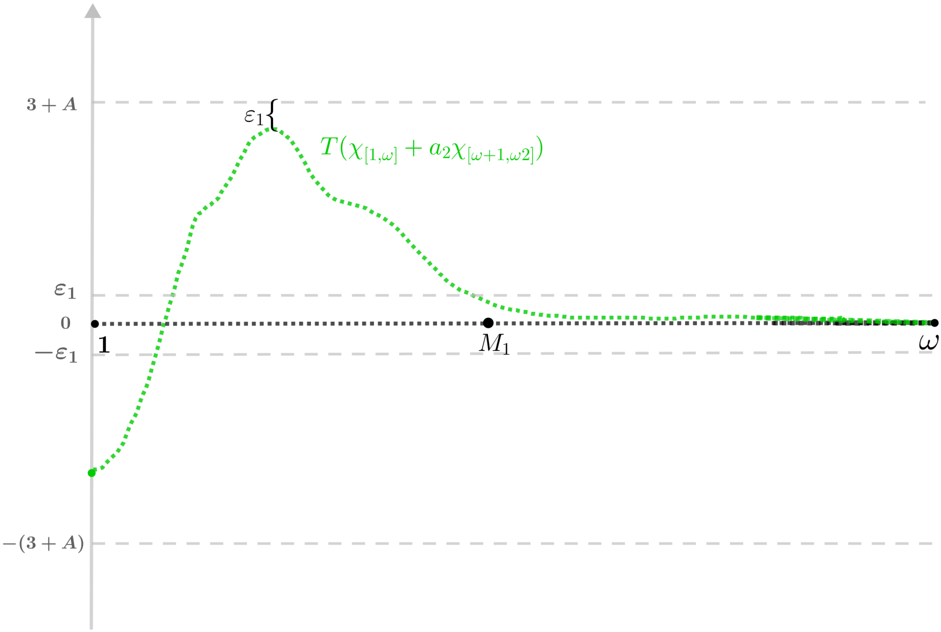
$3+A$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
0
1
ω
$-(3+A)$



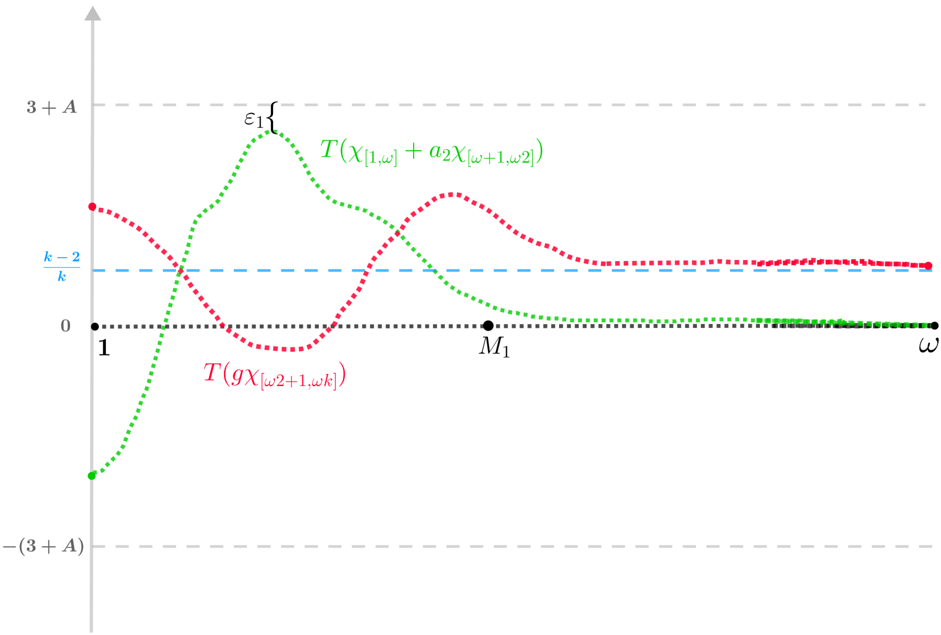
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]})$
0
1
ω
$-(3+A)$



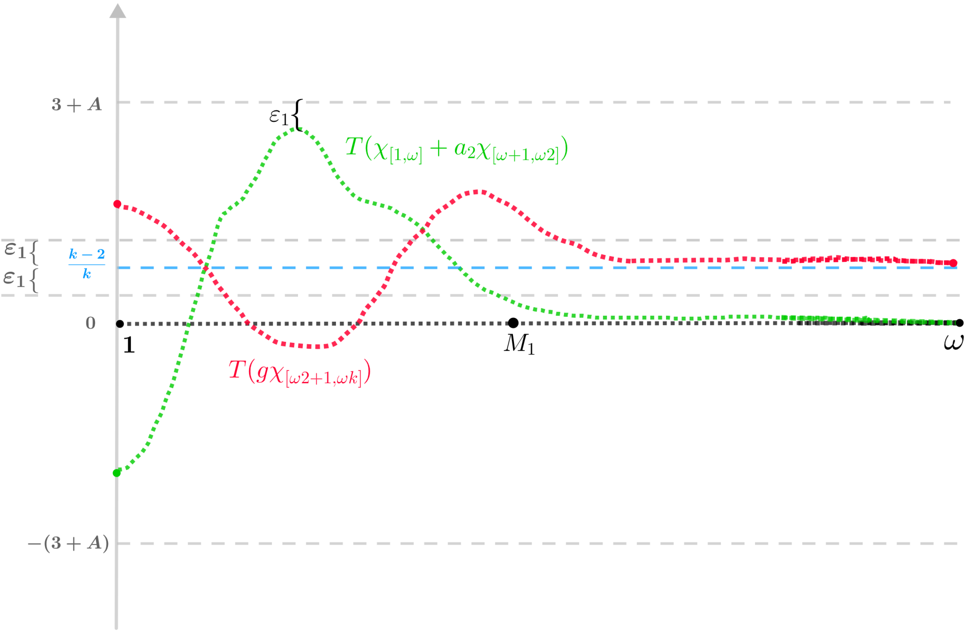
$3+A$
$\varepsilon_1\{$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]})$
ε_1
0
$\mathbf{1}$
ω
$-\varepsilon_1$
$-(3+A)$



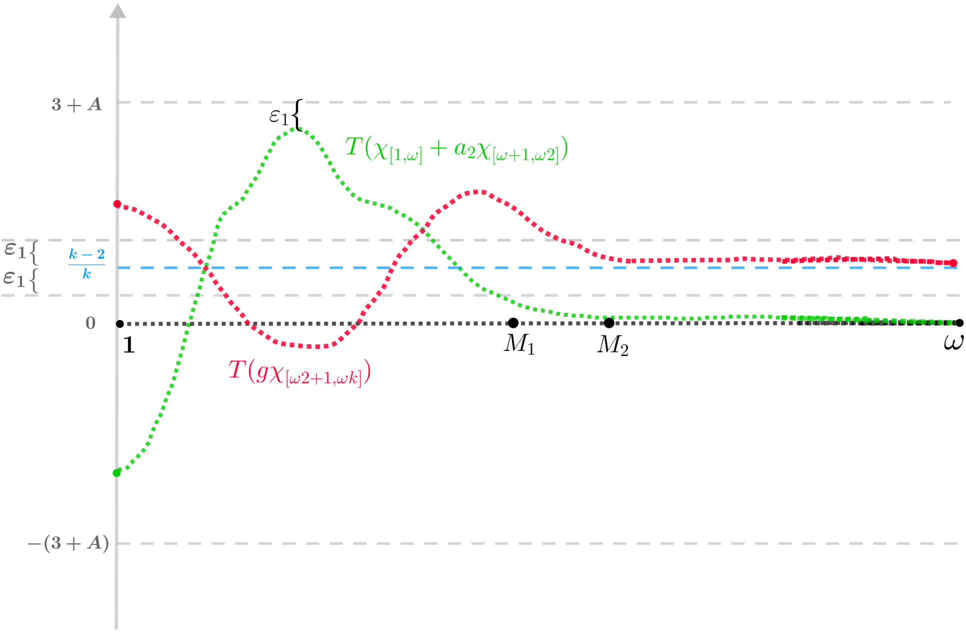
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]})$
ε_1
0
1
M_1
ω
$-\varepsilon_1$
$-(3+A)$



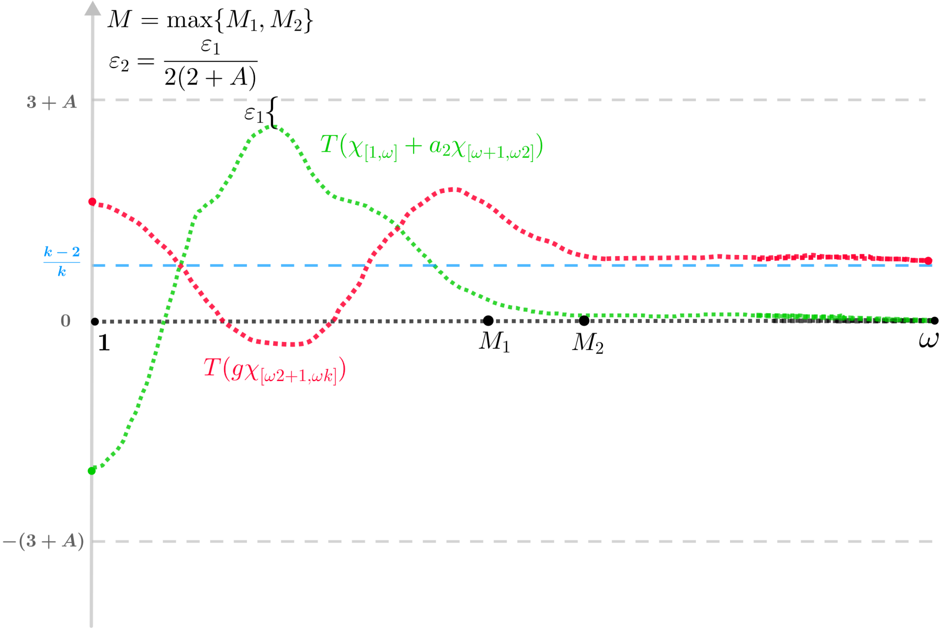
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$\frac{k-2}{k}$
0
1
M_1
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$

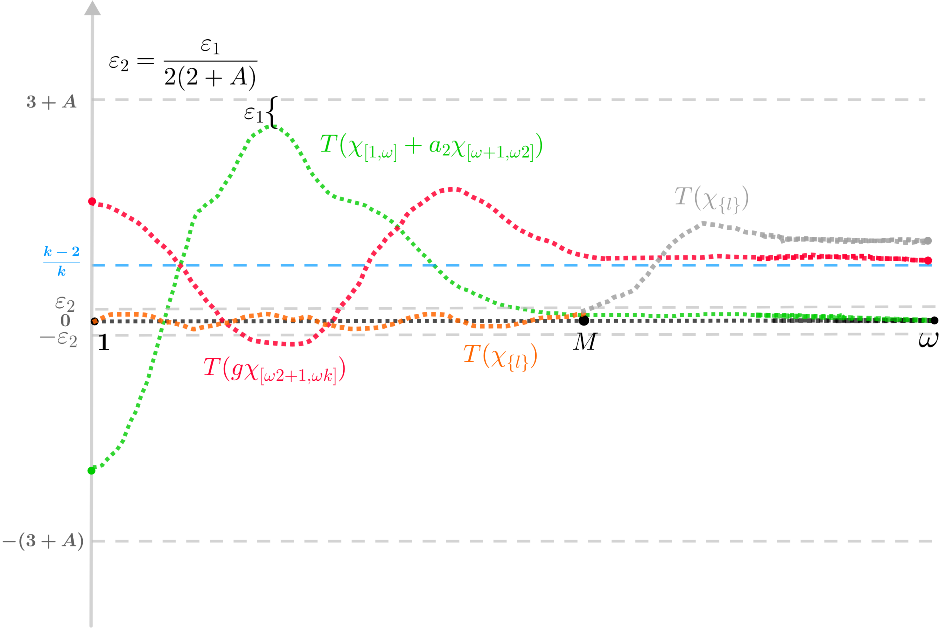


$3+A$
$\varepsilon_1\{$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$\varepsilon_1\{$
$\frac{k-2}{k}$
$\varepsilon_1\{$
0
1
M_1
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$

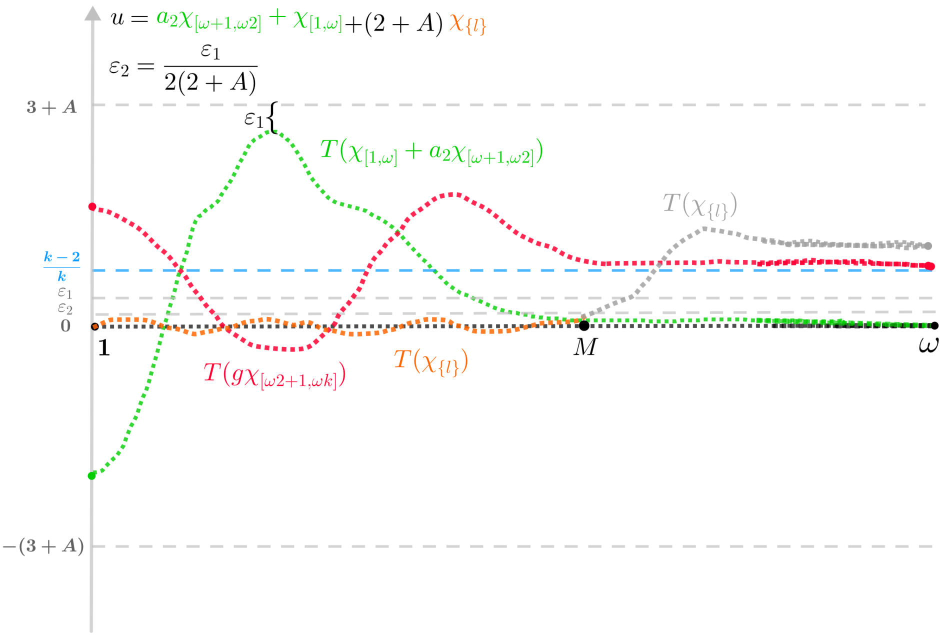


$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
ε_1
$\frac{k-2}{k}$
ε_1
0
1
M_1
M_2
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$

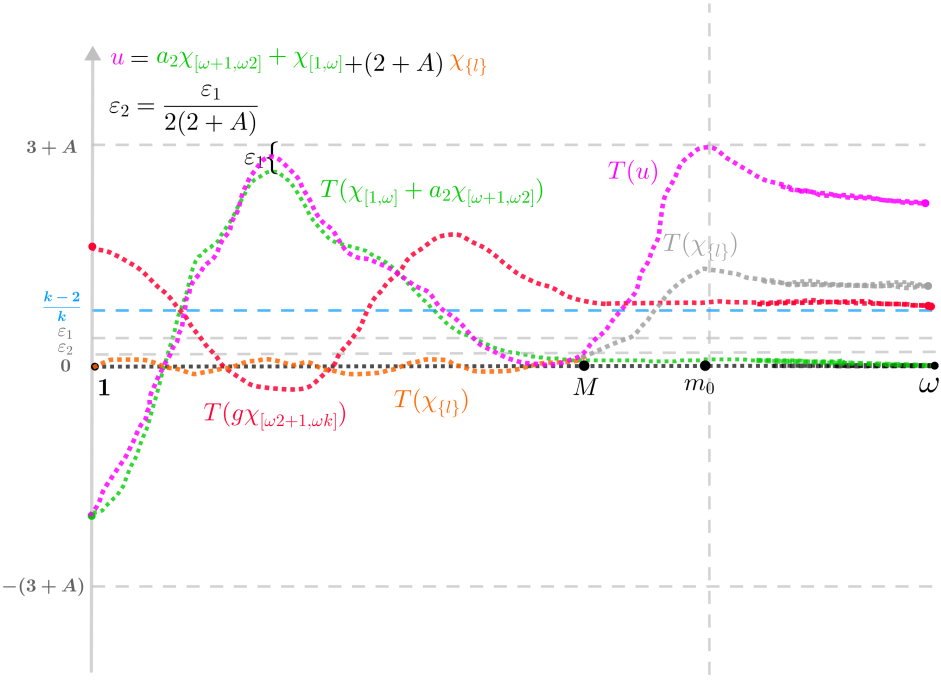

$M = \max\{M_1, M_2\}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$\frac{k-2}{k}$
0
1
M_1
M_2
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$



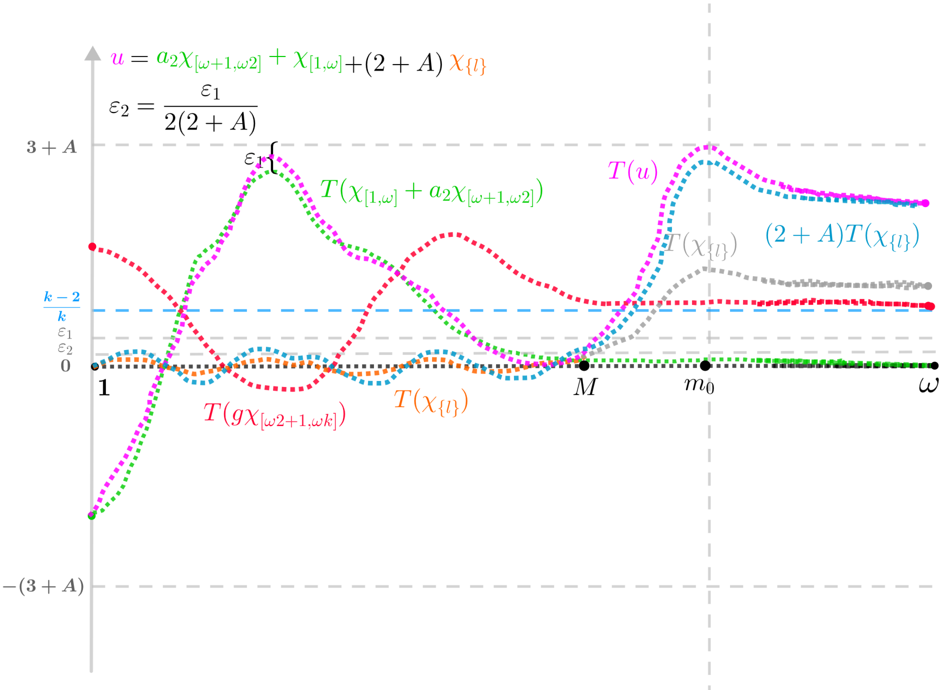
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega2]})$
$T(\chi_{\{l\}})$
$\frac{k-2}{k}$
ε_2
0
$-\varepsilon_2$
1
M
ω
$T(\chi_{\{l\}})$
$T(g\chi_{[\omega2+1,\omega k]})$
$-(3+A)$

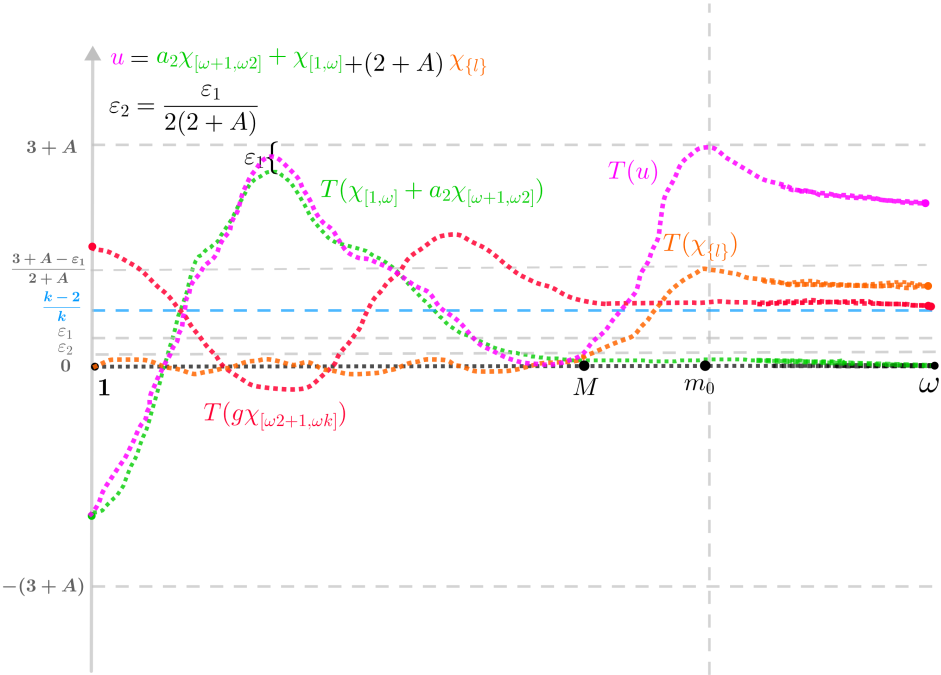


$u = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} + (2+A)\,\chi_{\{l\}}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(\chi_{\{l\}})$
$\frac{k-2}{k}$
ε_1
ε_2
0
1
M
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$T(\chi_{\{l\}})$
$-(3+A)$

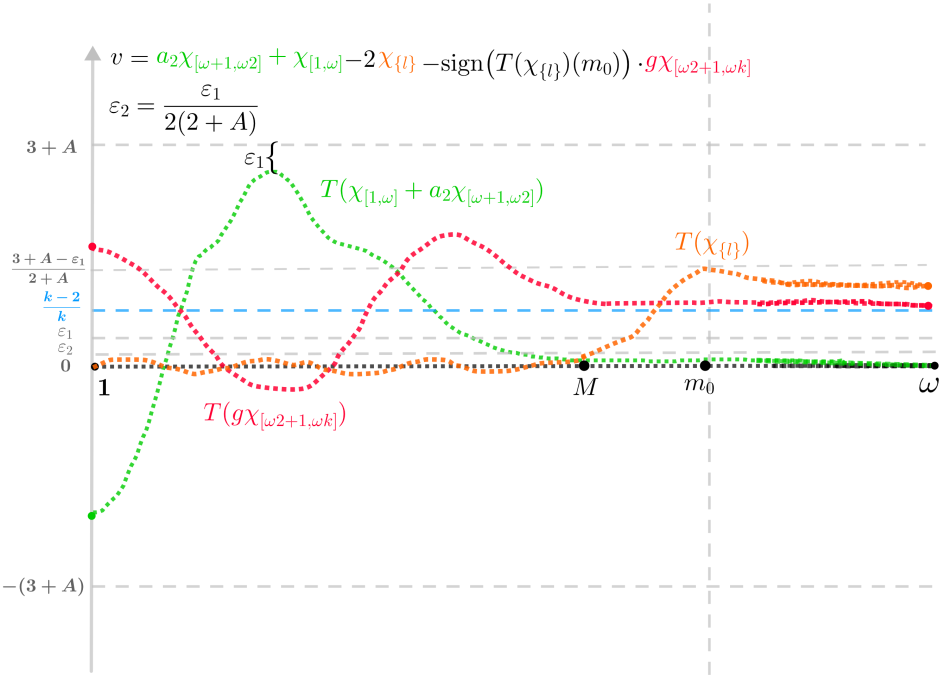


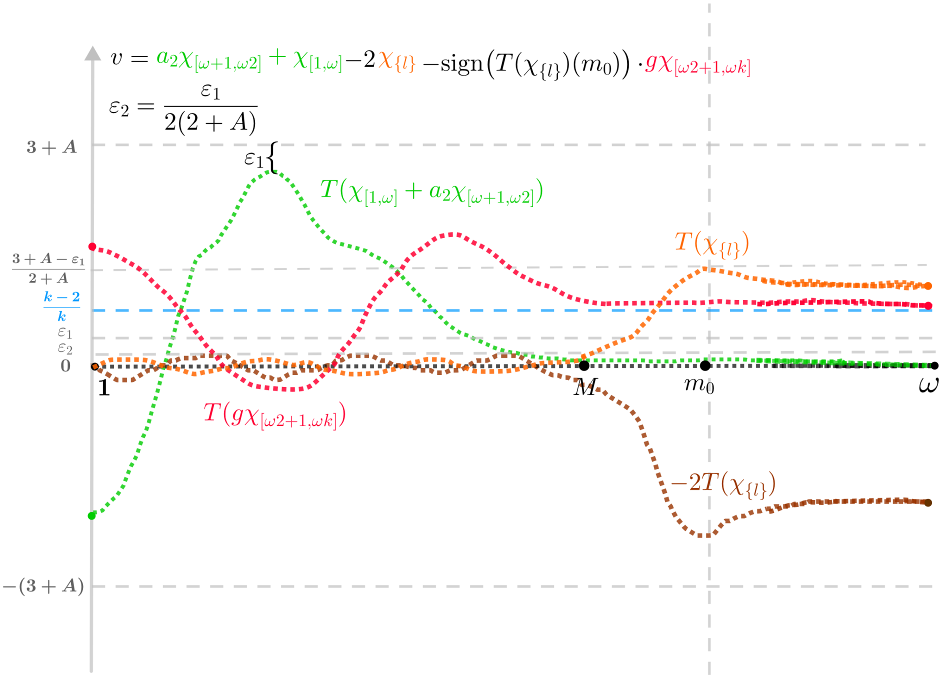
$u = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} + (2+A)\,\chi_{\{l\}}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(u)$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(\chi_{\{l\}})$
$\frac{k-2}{k}$
ε_1
ε_2
0
1
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$T(\chi_{\{l\}})$
$-(3+A)$

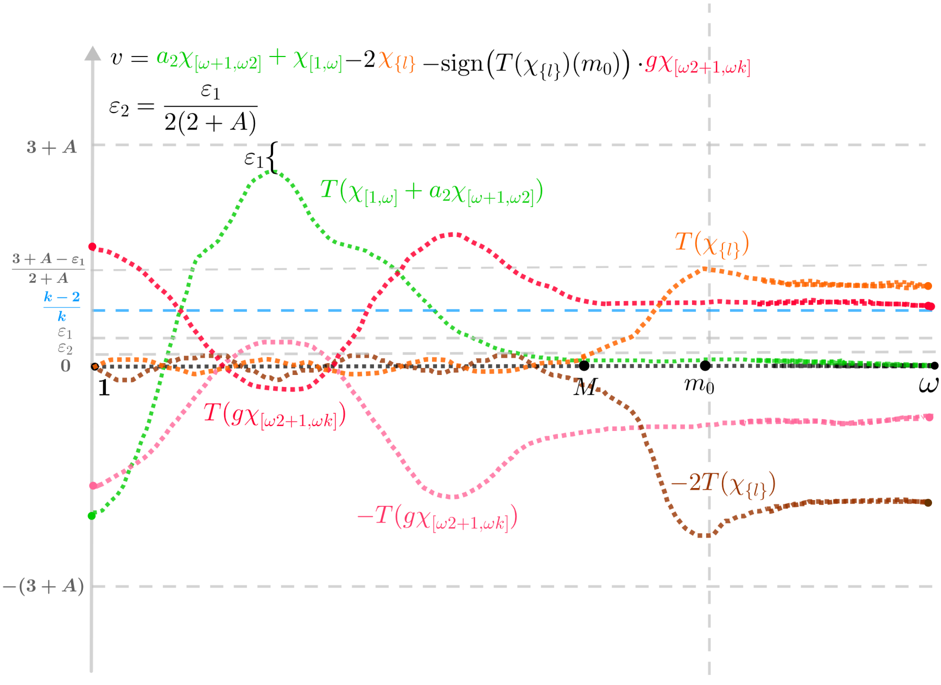

$u = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} + (2+A)\,\chi_{\{l\}}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(u)$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(\chi_{\{l\}})$
$(2+A)T(\chi_{\{l\}})$
$\frac{k-2}{k}$
ε_1
ε_2
0
1
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$T(\chi_{\{l\}})$
$-(3+A)$



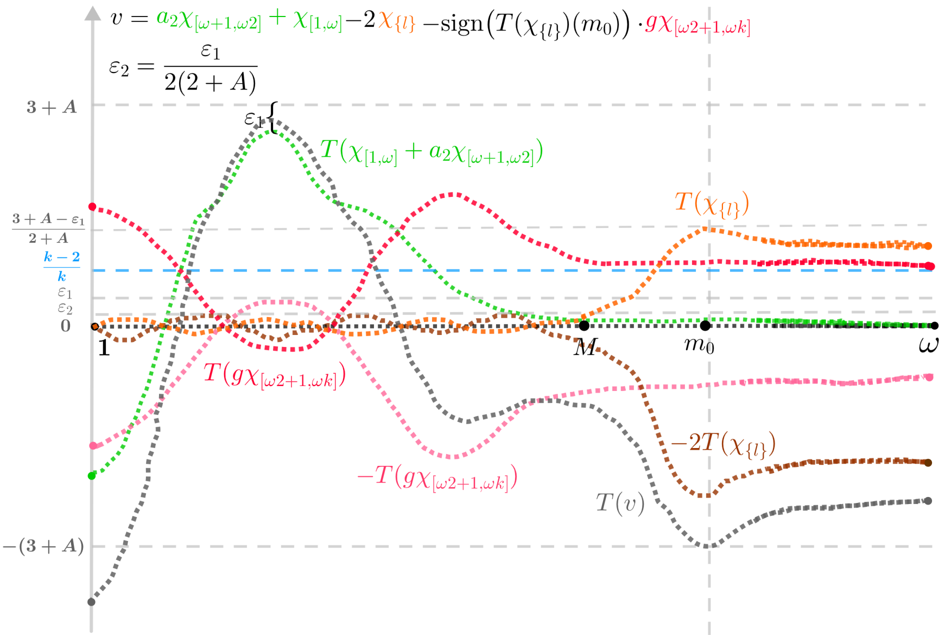
$u = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} + (2+A)\,\chi_{\{l\}}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
$\varepsilon_1\{$
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(u)$
$T(\chi_{\{l\}})$
$\dfrac{3+A-\varepsilon_1}{2+A}$
$\dfrac{k-2}{k}$
ε_1
ε_2
0
1
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$


$v = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} - 2\chi_{\{l\}} - \mathrm{sign}\big(T(\chi_{\{l\}})(m_0)\big) \cdot g\chi_{[\omega 2+1,\omega k]}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(\chi_{\{l\}})$
$\frac{3+A-\varepsilon_1}{2+A}$
$\frac{k-2}{k}$
ε_1
ε_2
0
1
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$


$v = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} - 2\chi_{\{l\}} - \mathrm{sign}\big(T(\chi_{\{l\}})(m_0)\big)\cdot g\chi_{[\omega 2+1,\omega k]}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(\chi_{\{l\}})$
$\frac{3+A-\varepsilon_1}{2+A}$
$\frac{k-2}{k}$
ε_1
ε_2
0
1
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-2T(\chi_{\{l\}})$
$-(3+A)$



$v = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} - 2\chi_{\{l\}} - \mathrm{sign}\big(T(\chi_{\{l\}})(m_0)\big) \cdot g\chi_{[\omega 2+1,\omega k]}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$T(\chi_{\{l\}})$
$\dfrac{3+A-\varepsilon_1}{2+A}$
$\dfrac{k-2}{k}$
ε_1
ε_2
0
1
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-2T(\chi_{\{l\}})$
$-T(g\chi_{[\omega 2+1,\omega k]})$
$-(3+A)$



$v = a_2\chi_{[\omega+1,\omega 2]} + \chi_{[1,\omega]} - 2\chi_{\{l\}} - \mathrm{sign}\big(T(\chi_{\{l\}})(m_0)\big) \cdot g\chi_{[\omega 2+1,\omega k]}$
$\varepsilon_2 = \dfrac{\varepsilon_1}{2(2+A)}$
$3+A$
ε_1
$T(\chi_{[1,\omega]} + a_2\chi_{[\omega+1,\omega 2]})$
$\frac{3+A-\varepsilon_1}{2+A}$
$\frac{k-2}{k}$
ε_1
ε_2
0
1
$T(\chi_{\{l\}})$
M
m_0
ω
$T(g\chi_{[\omega 2+1,\omega k]})$
$-2T(\chi_{\{l\}})$
$-T(g\chi_{[\omega 2+1,\omega k]})$
$T(v)$
$-(3+A)$

Example 1 (A. Gergont, Ł. Piasecki (2023))

Consider a mapping $T : C([1, \omega 3]) \to C([1, \omega])$ defined for every $f \in C([1, \omega 3])$ as follows:

$$
\begin{aligned}
T(f)(1) &= -f(\omega 2) - f(\omega 3), \\
T(f)(2) &= f(\omega) + \frac{15}{30} f(\omega 2) - \frac{14}{30} f(\omega 3), \\
T(f)(\omega) &= -\frac{8}{30} f(\omega) + \frac{9}{30} f(\omega 2) - \frac{9}{30} f(\omega 3), \\
T(f)(3m) &= -\frac{25}{30} f(m) + \frac{17}{30} f(\omega) + \frac{9}{30} f(\omega 2) - \frac{9}{30} f(\omega 3), \\
T(f)(3m+1) &= \frac{26}{30} f(\omega + m) - \frac{8}{30} f(\omega) - \frac{17}{30} f(\omega 2) - \frac{9}{30} f(\omega 3), \\
T(f)(3m+2) &= -\frac{26}{30} f(\omega 2 + m) - \frac{8}{30} f(\omega) + \frac{9}{30} f(\omega 2) + \frac{17}{30} f(\omega 3),
\end{aligned}
$$

where $1 \leq m < \omega$. Standard calculations show that T is an isomorphism and $\|T\|\|T^{-1}\| = \frac{7659}{1930}$. Therefore,

$$
d(C([1, \omega]), C([1, \omega 3])) \leq \frac{7659}{1930} < 4.
$$

Remark

Isomorphism in Example 1 is a particular case of a mapping $T : C([1,\omega 3]) \longrightarrow C([1,\omega])$ defined for every $f \in C([1,\omega 3])$ as follows:

$$\begin{aligned}
T(f)(1) &= a_1 f(\omega) + b_1 f(\omega 2) + c_1 f(\omega 3),\\
T(f)(2) &= a_2 f(\omega) + b_2 f(\omega 2) + c_2 f(\omega 3),\\
T(f)(\omega) &= a_3 f(\omega) + b_3 f(\omega 2) + c_3 f(\omega 3),\\
T(f)(3m) &= a_4 f(m) + b_4 f(\omega + m) + c_4 f(\omega 2 + m) + (a_3 - a_4) f(\omega)\\
&\quad + (b_3 - b_4) f(\omega 2) + (c_3 - c_4) f(\omega 3),\\
T(f)(3m+1) &= a_5 f(m) + b_5 f(\omega + m) + c_5 f(\omega 2 + m) + (a_3 - a_5) f(\omega)\\
&\quad + (b_3 - b_5) f(\omega 2) + (c_3 - c_5) f(\omega 3),\\
T(f)(3m+2) &= a_6 f(m) + b_6 f(\omega + m) + c_6 f(\omega 2 + m) + (a_3 - a_6) f(\omega)\\
&\quad + (b_3 - b_6) f(\omega 2) + (c_3 - c_6) f(\omega 3),
\end{aligned}$$

where $1 \leq m < \omega$. It is possible to find coefficients a_i, b_i, c_i, where $i = 1, \ldots, 6$, such that $\|T\|\|T^{-1}\| < 3.88$. For this purpose, the following code can be used:

https://github.com/agnieszkagergont/Isomorphism-model-no.1

$$3.23 < d(C([1, \omega]), C([1, \omega 3])) < 3.88.$$

Thank You