Optimality conditions for quadratic optimization problems with quadratic cone constraints

Giandomenico Mastroeni

Department of Computer Science, University of Pisa, Italy

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Outline

- The quadratic problem with cone quadratic constraints;
- Optimality conditions;
- Strong duality;
- The case of two quadratic equality constraints;
- Applications to quadratically constrained equilibrium problems.

The quadratic problem with cone quadratic constraints

We consider the quadratic problem

$$\mu := \inf \ f(x) \quad s.t. \ x \in K := \{x \in C : \ g(x) \in -P\}, \qquad \text{(QP)}$$

where

- $f(x) := \frac{1}{2}x^\top Ax + a^\top x + \alpha$, with A, real symmetric matrix; $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$;
- $g(x) := (g_1(x), \dots, g_m(x))$, $g_i(x) := \frac{1}{2}x^\top B_i x + b_i^\top x + \beta_i$, with B_i real symmetric matrices; $b_i \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$ for $i = 1, \dots, m$.
- P is a convex cone in $\mathbb{R}^m$, $C \subseteq \mathbb{R}^n$.

Applications

- Trust region problems;
- The standard quadratic problem;
- Robust optimization;
- Telecommunications;
- Merit functions for bimatrix games;
- Biology and Economics.

See, e.g.,

- Horst, R.,Pardalos, P., : Handbook of global optimization, nonconvex optimization and its applications, Kluwer, (1995).
- Ben-Tal, A., den Hertog, D.: Hidden conic quadratic representation of some nonconvex quadratic optimization problems, Math. Program. **143**, 1–29 (2014)

We associate with (QP) the Lagrangian function $L(\lambda, x) \doteq f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$ and its dual problem

$$v := \sup_{\lambda \in P^*} \inf_{x \in C} L(\lambda, x). \quad (1)$$

We say that strong duality holds for (QP), if there exists $\lambda^* \in P^*$ such that

$$\inf_{x \in K} f(x) = \inf_{x \in C} L(\lambda^*, x).$$

In case (QP) admits an optimal solution $\bar{x} \in K$, then the previous condition is equivalent to

- $L(\lambda^*, \bar{x}) \leq L(\lambda^*, x), \ \forall x \in C,$
- $\langle \lambda^*, g(\bar{x}) \rangle = 0,$
- $g(\bar{x}) \in -P, \quad \bar{x} \in C.$

The contingent cone $T(C;\bar{x})$ of C at $\bar{x} \in C$ is the set of all $v \in \mathbb{R}^n$ such that there exist sequences $(x_k, t_k) \in C \times \mathbb{R}_+$ with $x_k \to \bar{x}$ and $t_k(x_k - \bar{x}) \to v$.

Under suitable assumptions on $T(C;\bar{x})$, we first establish three general results.

- The first and the second consider the case where $\bar{x}$ is a KKT point and provide a sufficient optimality condition and a characterization of its optimality (in the case where $P = \{0\}^m$), respectively;
- the third one characterizes optimality under the assumption of strong duality.

Definition. Let $C \subseteq \mathbb{R}^n$.

- We say that a symmetric matrix B is positive semidefinite on the set C if $x^\top Bx \geq 0, \quad \forall x \in C$.
- $\mathrm{co}\ C$, $\mathrm{cl}\ C$, $\mathrm{ri}\ C$, denote the convex hull of C, the closure and the relative interior of C.
- $C^* := \{y^* \in \mathbb{R}^n : \langle y^*, x\rangle \geq 0,\ \forall x \in C\}$.

Proposition 1

Let f, $g_1, \ldots, g_m$ be quadratic functions as defined. Assume that $\bar{x} \in K$ is a KKT point for (QP), i.e., there exists $\lambda^* \in P^*$ such that

$$\nabla_x L(\lambda^*, \bar{x}) \in [T(C; \bar{x})]^*, \quad \langle \lambda^*, g(\bar{x}) \rangle = 0, \tag{2}$$

and, additionally, $(K - \bar{x}) \subseteq \mathrm{cl\ co}\ T(C; \bar{x})$. Then the following assertion holds.
If $\nabla_x^2 L(\lambda^*, \bar{x})$ is positive semidefinite on $K - \bar{x}$, then $\bar{x}$ is a (global) optimal solution for (QP).

Remark

Proposition 1 is related to Theorem 2.1 in [Bomze (2015)] when applied to a quadratic problem. Indeed, in [Bomze (2015)], K is a convex set and $C := \mathbb{R}^n$, which guarantees that the condition $(K - \bar{x}) \subseteq \mathrm{cl\ co}\ T(C; \bar{x})$ is fulfilled.

Proposition 2

Let $f, g_1, \ldots, g_m$ be quadratic functions as above, let $P := \{0\}^m$ and $\bar{x} \in K$. Assume that

$$(K - \bar{x}) \subseteq \mathrm{cl\ co}\ T(C; \bar{x}) \subseteq -\mathrm{cl\ co}\ T(C; \bar{x}), \tag{3}$$

and that $\bar{x}$ is a KKT point for (QP), i.e., there exists $\lambda^* \in \mathbb{R}^m$ such that

$$\nabla_x L(\lambda^*, \bar{x}) \in [T(C; \bar{x})]^*. \tag{4}$$

Then the following conditions are equivalent:

(a) $\bar{x}$ is an optimal solution for the problem (QP);

(b) $\nabla_x^2 L(\lambda^*, \bar{x})$ is positive semidefinite on $K - \bar{x}$ and so on $\mathrm{cl\ cone}(K - \bar{x})$.

Remark

Note that the second inclusion in assumption (3) is not needed for proving that (b) implies (a), as shown by Proposition 1.

Strong duality

In the following proposition we characterize optimality under the strong duality property that can be considered as a regularity condition in view of the fulfillment of the KKT conditions.

Proposition 3

Let f, $g_1, \ldots, g_m$ be quadratic functions as above, let $\bar{x} \in K$, and assume that

$$(C - \bar{x}) \subseteq \mathrm{cl}\ \mathrm{co}\ T(C; \bar{x}) \subseteq -\mathrm{cl}\ \mathrm{co}\ T(C; \bar{x}). \tag{5}$$

Then the following assertions are equivalent:

(a) $\bar{x}$ is an optimal solution for the problem (QP) and strong duality holds;

(b) there exists $\lambda^* \in P^*$ such that (2) is fulfilled and $\nabla_x^2 L(\lambda^*, \bar{x})$ is positive semidefinite on $C - \bar{x}$.

Remark

We note that, for the implication $(b) \Rightarrow (a)$ in Proposition 3, the second inclusion in (5) is not needed.

Example 1

$$\inf\{\frac{1}{2}x^\top Ax + a^\top x : \ b^\top x = 0, \ x \in \mathbb{R}^2\}, \qquad \text{(QP)}$$

where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad a = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Here, $P = \{0\}$, $C = \mathbb{R}^2$. Notice that A is an indefinite matrix. The KKT conditions are given by

$$\begin{cases} x_1 + 2x_2 + 3 + \lambda = 0 \\ 2x_1 + x_2 - 1 - 2\lambda = 0 \\ x_1 - 2x_2 = 0, \ \lambda \geq 0 \end{cases} \quad \Rightarrow \begin{cases} \bar{x} = (-\frac{10}{13}, -\frac{5}{13}), \\ \lambda^* = -\frac{19}{13} \end{cases}$$

Consider the condition:

$$(x-\bar{x})^\top \nabla_x^2 L(\bar{x}, \lambda^*)(x-\bar{x}) = (x-\bar{x})^\top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} (x-\bar{x}) \geq 0, \quad \forall x \in K,$$

Noticing that $K = \{(x_1, x_2) : x_1 = 2x_2\}$, this amount to check that

$$(x_1 + \frac{10}{13}, x_2 + \frac{5}{13}) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} (x_1 + \frac{10}{13}, x_2 + \frac{5}{13})^T \geq 0, \quad \forall (x_1, x_2) : x_1 = 2x_2,$$

or, equivalently,

$$2(2x_2+\frac{10}{13})^2+5(x_2+\frac{5}{13})^2 \geq 0, \quad \forall x_2 \in \mathbb{R},$$

which obviously holds. Therefore, the point $\bar{x}$ is optimal for QP.
For strong duality, we have to consider the condition:

$$(x-\bar{x})^\top \nabla_x^2 L(\bar{x},\lambda^*)(x-\bar{x}) = (x-\bar{x})^\top \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} (x-\bar{x}) \geq 0, \quad \forall x \in C = \mathbb{R}^2,$$

which cannot hold being the matrix A indefinite.
Therefore, strong duality does not hold for (QP).

Remark

Condition (5), i.e.,

$$(C-\bar{x}) \subseteq \mathrm{cl\ co}\ T(C;\bar{x}) \subseteq -\mathrm{cl\ co}\ T(C;\bar{x})$$

is fulfilled under the following circumstances:

(*i*) $\bar{x} \in \mathrm{int}\, C$;

(*ii*) C is defined by linear equalities, i.e., $C := \{x \in \mathbb{R}^n : Hx = d\}$, $H \in \mathbb{R}^{p \times n}$, $d \in \mathbb{R}^p$;

(iii) $C := \{x \in \mathbb{R}^n : h(x) = 0\}$, where h is a quadratic function with $\nabla h(\bar{x}) = 0$.

In fact, in case (iii), it can be proved that

$$T(C;\bar{x}) = C - \bar{x} = \{v \in \mathbb{R}^n : \; v^\top H v = 0\},$$

and, since $T(C;\bar{x}) = -T(C;\bar{x})$, then (5) is fulfilled.

Case (iii) will be of interest when we will consider a quadratic problem with two quadratic equality constraints.

Example 2

$$\inf\{\frac{1}{2}x^\top Ax : \ \frac{1}{2}x^\top Bx \le 0, \ x \in \mathbb{R}^2\}, \qquad \text{(QP)}$$

where

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

Here, $P = \mathbb{R}_+$, $C = \mathbb{R}^2$. Notice that A and B are indefinite matrices. The KKT conditions are given by

$$\begin{cases} -x_1 + 2x_2 + \lambda(x_1 - x_2) = 0 \\ 2x_1 - \lambda x_1 = 0 \\ \lambda(\frac{1}{2}x_1^2 - x_1x_2) = 0 \\ \frac{1}{2}x_1^2 - x_1x_2 \le 0, \ \lambda \ge 0 \end{cases} \Rightarrow \begin{cases} \bar{x} = (0, x_2), \quad x_2 \in \mathbb{R}, \\ \lambda^* = 2 \end{cases}$$

$$\nabla_x^2 L(\bar{x}, \lambda^*) = A + \lambda^* B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which is positive semidefinite. Therefore, the points $(0, x_2)$, $x_2 \in \mathbb{R}$, are optimal and strong duality holds for (QP).

Applications to particular cases

The previous propositions generalize optimality conditions for classical quadratic programming to a quadratic problem with cone constraints and a geometric constraint set.

We now present some particular cases:

We first consider the quadratic programming problem with bivalent constraints (QP1) defined by

$$\inf_{x \in K} \; f(x) := x^\top Ax + 2a^\top x + \alpha,$$

where

- $K := \{x \in C : g_i(x) := x^\top B_i x + 2b_i^\top x + \beta_i = 0,\; i = 1, .., m,$ $g_{m+j}(x) := x^\top E_{m+j} x - 1 = 0,\; j = 1, \ldots, n\},$
- $E_{m+j} = \mathrm{diag}(e_j)$ and e_j is a vector in $\mathbb{R}^n$ whose *j-th* element is equal to 1 and all the other entries are equal to 0.

Let $L(\lambda, \gamma, x) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{n} \gamma_j g_{m+j}(x)$, be the Lagrangian function associated with (QP1).

By Proposition 2 we recover Lemma 3.1 of [Li G. (2012)] which can be stated as follows.

Proposition 4

Let $C := \mathbb{R}^n$ and $\bar{x} \in K$. Assume that there exist $\lambda \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ such that $\nabla_x L(\lambda, \gamma, \bar{x}) = 0$. Then $\bar{x}$ is an optimal solution for (QP1) if and only if $\nabla_x^2 L(\lambda, \gamma, \bar{x})$ is positive semidefinite on $Z(\bar{x})$ defined by

$$Z(\bar{x}) := \bigcap_{i=1}^{m+n} Z_i(\bar{x}). \tag{6}$$

where, $Z_i(\bar{x}) := \{v \in \mathbb{R}^n : \nabla g_i(\bar{x})^\top v + \frac{1}{2} v^\top B_i v = 0\}$, for $i = 1, \ldots, m+n$.

Remark

In fact, it is possible to show that

$$Z(\bar{x}) = K - \bar{x}. \tag{7}$$

where $K := \{x \in \mathbb{R}^n : g_i(x) = 0, i = 1, \ldots, m+n\}$.

Next result follows from Proposition 3. It is inspired by Theorem 3.1 of [Li G. (2012)] and provides a characterization and a sufficient condition for strong duality for (QP1).

Proposition 5

Let $\bar{x} \in K$ with $C := \mathbb{R}^n$. Consider the following assertions:

(a) $\bar{x}$ is an optimal solution for (QP1) and strong duality holds;

(b) there exist $\lambda \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ such that $\nabla_x L(\lambda, \gamma, \bar{x}) = 0$ and $\nabla_x^2 L(\lambda, \gamma, \bar{x})$ is positive semidefinite;

(c) $A - \mathrm{diag}(\bar{X} A \bar{x} + \bar{X} a)$ is positive semidefinite, where $\bar{X} := \mathrm{diag}(\bar{x}_1, \ldots, \bar{x}_n)$.

Then $(c) \Rightarrow (b) \Leftrightarrow (a)$.

Conditions (3) and (5), i.e.,

$$(K-\bar{x}) \subseteq \text{cl co } T(C;\bar{x}) \subseteq -\text{cl co } T(C;\bar{x}),$$

$$(C-\bar{x}) \subseteq \text{cl co } T(C;\bar{x}) \subseteq -\text{cl co } T(C;\bar{x}),$$

in general are not fulfilled for a quadratic problem with bivalent constraints.

Example 3

Let $C := \{x \in \mathbb{R}^2 : x_1^2 = 1\}$, $K := \{x \in \mathbb{R}^2 : x_1^2 = 1, x_2^2 = 1\}$, $\bar{x} = (1,1) \in K$. Then,

$$T(C,\bar{x}) = \{x \in \mathbb{R}^2 : x_1 = 0\} = \text{cl co } T(C;\bar{x}),$$

$$K - \bar{x} = \{(0,0),(0,-2),(-2,-2),(-2,0)\} \not\subseteq \text{cl co } T(C;\bar{x}).$$

This also implies that $C - \bar{x} \not\subseteq \text{cl co } T(C;\bar{x})$ so that Propositions 2 and 3 in general cannot be applied to problem (QP1).

Consider the problem

$$\mu := \inf\{f(x) : \ g_1(x) \leq 0, \dots, g_m(x) \leq 0, \ x \in C\}, \tag{8}$$

where $C := \{x \in \mathbb{R}^n : Hx = d\}$, H is a $(p \times n)$ matrix and $f, g_i, \ i = 1, \dots, m$, are quadratic functions defined as in the beginning.

Recalling that for the above C, condition (5) is fulfilled, the following results are all particular cases of Proposition 3.

Corollary ([Jeyakumar- Li (2015)(Theorem 2.1)] and [Zheng, Sun, Li, Xu (2011) (Theorem 1)]

Let $\bar{x}$ be feasible for (8). The following assertions are equivalent:

(a) $\bar{x}$ is an optimal solution and strong duality holds for (8);

(b) there exists $\lambda^* \in \mathbb{R}^m_+$ such that $\nabla_x L(\bar{x}, \lambda^*) \in H^\top(\mathbb{R}^p)$, $\lambda_i^* g_i(\bar{x}) = 0, \ i = 1, \dots, m$, and $\nabla_x^2 L(\bar{x}, \lambda^*)$ is positive semidefinite on Ker H.

When $C := \mathbb{R}^n$, then (b) reduces to the following:

(b′) there exists $\lambda^* \in \mathbb{R}^m_+$ such that $\nabla_x L(\bar{x}, \lambda^*) = 0$, $\lambda_i^* g_i(\bar{x}) = 0, \ i = 1, \dots, m$ and $\nabla_x^2 L(\bar{x}, \lambda^*)$ is positive semidefinite.

The Case with Two Quadratic Equality Constraints

Consider a quadratic problem with two quadratic equality constraints:

$$\mu := \inf\{f(x) : \ g_1(x) = 0, \ g_2(x) = 0\}, \tag{9}$$

where $f, g_i, \ i = 1,2$ are quadratic functions as previously defined.
Let $K := \{x \in \mathbb{R}^n : \ g_1(x) = 0, \ g_2(x) = 0\}$.

The standard Lagrangian associated with (9) $L_S : \mathbb{R}^2 \times \mathbb{R}^n \longmapsto \mathbb{R}$ is:

$$L_S(\lambda_1, \lambda_2, x) := f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x).$$

The following result is a consequence of Proposition 2.

Proposition 6

Let f, g_1, g_2 be defined as above, let $\bar{x} \in K$ be a KKT point for (9), i.e., there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) = 0$.
Then the following conditions are equivalent:

(a) $\bar{x}$ is an optimal solution for (9);

(b) $A + \lambda_1 B_1 + \lambda_2 B_2$ is positive semidefinite on $K - \bar{x}$.

If, additionally, $\nabla g_2(\bar{x}) = 0$ then (b) is equivalent to:

$(b1)$ $A + \lambda_1 B_1$ is positive semidefinite on $K - \bar{x}$.

Duality

In the following we set $C := \{x \in \mathbb{R}^n : \ g_2(x) = 0\}$, so that $K = \{x \in C : \ g_1(x) = 0\}$.
The dual problem and the standard dual problem associated with (9) are, respectively, defined by:

$$\nu := \sup_{\lambda_1 \in \mathbb{R}} \inf_{x \in C} \{L(\lambda_1, x)\}; \tag{10}$$

$$\nu_S := \sup_{\lambda_1, \lambda_2 \in \mathbb{R}} \inf_{x \in \mathbb{R}^n} \{L_S(\lambda_1, \lambda_2, x)\}. \tag{11}$$

We say that standard strong duality (SSD) holds for problem (9) if $\mu = \nu_S$ and problem (11) admits solution.

Remark

It easy to check that $\nu_S \leq \nu \leq \mu$.

Next theorem provides a characterization of strong duality for a quadratic problem with two quadratic equality constraints.

Theorem 1

Let $\bar{x} \in K$ be feasible for (9) and suppose that $\mu \in \mathbb{R}$.

(*a*) Assume that $\nabla g_2(\bar{x}) \neq 0$. Then the following assertions are equivalent

- (*a*1) $\bar{x}$ is an optimal solution and strong duality holds for problem (9);
- (*a*2) $\exists\ \lambda_1,\ \lambda_2 \in \mathbb{R}$ such that $\nabla_x L_S(\lambda_1, \lambda_2, \bar{x}) = 0$ and $A + \lambda_1 B_1 + \lambda_2 B_2$ is positive semidefinite on $C - \bar{x}$ (and so on $\mathrm{cl\,cone}(C - \bar{x})$).

(*b*) Assume that $\nabla g_2(\bar{x}) = 0$, and B_2 positive (or negative) semidefinite. Then, (*a*1) is equivalent to

- (*b*1) $\exists\ \lambda_1 \in \mathbb{R}$ and $\exists y \in \mathbb{R}^n$ s.t. $\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + B_2 y = 0$ and $A + \lambda_1 B_1$ is positive semidefinite on $\ker B_2 = C - \bar{x}$.

(*c*) Assume that $\nabla g_2(\bar{x}) = 0$, and B_2 indefinite. Then, (*a*1) is equivalent to

- (*c*1) $\exists\ \lambda_1 \in \mathbb{R}$ s.t. $\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) = 0$ and $A + \lambda_1 B_1$ is positive semidefinite on $C - \bar{x}$ (and so on $\mathrm{cl\,cone}(C - \bar{x})$).

Further developments

- Local optimality conditions
- Applications to quadratically constrained equilibrium problems (in particular, variational inequalities)
- Extensions to multiobjective quadratic optimization problems

Applications to equilibrium problems

Consider the following equilibrium problem (EP), which consists in finding

$$\bar{x} \in K \quad s.t. \quad f(\bar{x}, y) \geq 0, \quad \forall y \in K,$$

where $f : K \times K \to \mathbb{R}$, $K \subseteq \mathbb{R}^n$ and $f(x, x) = 0$, $\forall x \in K$.

Lemma

$\bar{x}$ is a solution of (EP) if and only if $\bar{x}$ is an optimal solution of the following constrained extremum problem:

$$\min_{y \in K} f(\bar{x}, y) \qquad (P(\bar{x}))$$

Remark

If we assume that $f(x, \cdot)$ is a quadratic function for every $x \in K$ and K is defined by means of quadratic constraints plus a geometric one ($x \in C$), then we can apply the results obtained for problem (QP).

The equilibrium problem with cone quadratic constraints

We consider the quadratic equilibrium problem which consists in finding

$$\bar{x} \in K := \{x \in C : \ g(x) \in -P\} \quad s.t. \ f(\bar{x}, y) \geq 0, \ \forall y \in K, \qquad \text{(EP)}$$

where

- $f(x,y) := \frac{1}{2} y^\top A(x) y + a(x)^\top y + \alpha(x)$, with $A(x)$ real symmetric matrix, $a(x) \in \mathbb{R}^n$ and $\alpha(x) \in \mathbb{R}$, for every $x \in K$;
- $g(x) := (g_1(x), \ldots, g_m(x))$, $g_i(x) := \frac{1}{2} x^\top B_i x + b_i^\top x + \beta_i$, with B_i real symmetric matrices; $b_i \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$ for $i = 1, \ldots, m$.
- P is a convex cone in $\mathbb{R}^m$, $C \subseteq \mathbb{R}^n$.

Remark

If $A(x) \equiv 0$, $\alpha(x) := -a(x)^T x$, then (EP) collapses to the *VI*:

$$a(\bar{x})^T (y - \bar{x}) \geq 0, \quad \forall y \in K.$$

Are there other interesting cases to be considered?

The condition to be fulfilled is

$$f(x,x) = \frac{1}{2}x^\top A(x)x + a(x)^\top x + \alpha(x) = 0, \quad \forall x \in K.$$

Let us go back to optimality conditions for (EP). To this aim, consider problem $P(\bar{x})$ under the given assumptions. By the previous Lemma, we have that

Proposition

$\bar{x}$ is a solution of (EP) if and only if $\bar{x}$ is an optimal solution of

$$\min_{y \in K}\{\frac{1}{2}y^\top A(\bar{x})y + a(\bar{x})^\top y + \alpha(\bar{x})\} \qquad (P(\bar{x}))$$

The Lagrangian function associated with $P(\bar{x})$ is defined by:

$$L_{\bar{x}}(\lambda, y) := f(\bar{x}, y) + \sum_{i=1}^{m} \lambda_i g_i(y) =$$

$$= \frac{1}{2}y^\top A(\bar{x})y + a(\bar{x})^\top y + \alpha(\bar{x}) + \sum_{i=1}^{m} \lambda_i(\frac{1}{2}y^\top B_i y + b_i^\top y + \beta_i)$$

Proposition 1 applied to problem $P(\bar{x})$ becomes:

Proposition 1a

Let $f(x,\cdot)$, $g_1,\dots,g_m$ be quadratic functions as defined, for every $x \in K$. Assume that $\bar{x} \in K$ is a KKT point for $(P(\bar{x}))$, i.e., there exists $\lambda^* \in P^*$ such that

$$\nabla_y L_{\bar{x}}(\lambda^*,\bar{x}) \in [T(C;\bar{x})]^*, \quad \langle \lambda^*, g(\bar{x})\rangle = 0,$$

and, additionally, $(K-\bar{x}) \subseteq \mathrm{cl\ co}\ T(C;\bar{x})$. Then, the following assertion holds:
If $\nabla_y^2 L_{\bar{x}}(\lambda^*,\bar{x})$ is positive semidefinite on $K-\bar{x}$, then $\bar{x}$ is a solution of (EP).

Remark

In the present case

- $\nabla_y L_{\bar{x}}(\lambda^*,\bar{x}) = A(\bar{x})\bar{x} + a(\bar{x}) + \sum_{i=1}^{m} \lambda_i^*(B_i\bar{x} + b_i)$
- $\nabla_y^2 L_{\bar{x}}(\lambda^*,\bar{x}) = A(\bar{x}) + \sum_{i=1}^{m} \lambda_i^* B_i$

Similarly, Proposition 2, applied to $P(\bar{x})$, becomes:

Proposition 2a

Let $f(x,\cdot)$, $g_1,\ldots,g_m$ be quadratic functions as defined above.
Let $P := \{0\}^m$, $\bar{x} \in K$ and assume that

$$(K - \bar{x}) \subseteq \mathrm{cl\ co}\ T(C;\bar{x}) \subseteq -\mathrm{cl\ co}\ T(C;\bar{x}), \tag{12}$$

and that $\bar{x}$ is a KKT point for (P($\bar{x}$)), i.e., there exists $\lambda^* \in \mathbb{R}^m$ such that

$$\nabla_y L_{\bar{x}}(\lambda^*, \bar{x}) \in [T(C;\bar{x})]^*. \tag{13}$$

Then the following conditions are equivalent:

- (a) $\bar{x}$ is a solution of (EP);
- (b) $\nabla_y^2 L_{\bar{x}}(\lambda^*, \bar{x})$ is positive semidefinite on $K - \bar{x}$ (and so on $\mathrm{cl\,cone}(K - \bar{x})$).

Remark

Note that the second inclusion in assumption (3) is not needed for proving that (b) implies (a), as shown by Proposition 1a.

An equilibrium problem with two quadratic equality constraints

Consider an equilibrium problem with two quadratic equality constraints:

$$f(\bar{x},y) \geq 0, \quad \forall y \in K := \{x \in \mathbb{R}^n : \; g_1(x) = 0, \; g_2(x) = 0\} \tag{14}$$

where $f(x,\cdot), g_i, \; i = 1,2$ are quadratic functions as previously defined.

The standard Lagrangian associated with $P(\bar{x})$ is:

$$L_{\bar{x}}(\lambda_1, \lambda_2, x) := f(\bar{x},y) + \lambda_1 g_1(y) + \lambda_2 g_2(y).$$

The following result is the analogous of Proposition 6.

Proposition 6a

Let $f(x,\cdot), g_1, g_2$ be quadratic functions as above, let $\bar{x} \in K$ be a KKT point for (14), i.e., there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\nabla_y f(\bar{x},\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) = 0$.

Then the following conditions are equivalent:

(a) $\bar{x}$ is a solution of (EP) defined by (14);

(b) $A(\bar{x}) + \lambda_1 B_1 + \lambda_2 B_2$ is positive semidefinite on $K - \bar{x}$.

If, additionally, $\nabla g_2(\bar{x}) = 0$ then (b) is equivalent to:

$(b1)$ $A(\bar{x}) + \lambda_1 B_1$ is positive semidefinite on $K - \bar{x}$.

Further remarks

It would be of interest to

- Investigate the possible applications of Proposition 3 and Theorem 1 in the context of the analysis of duality for a quadratically constrained quadratic equilibrium problem;
- Analyse the applications to vector quadratic equilibrium problems.

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