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On existence and approximations of solutions and bilevel Nash equilibria in multi-leader-multi-follower games

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From PARIS to NAPLES

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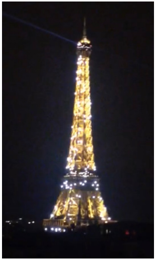

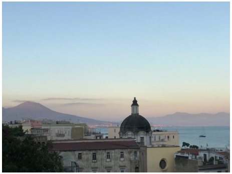

Zero-sum games

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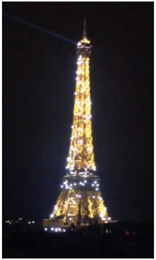

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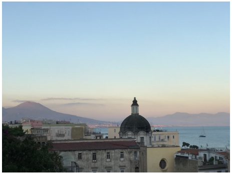

Non-zero sum (ratio-bounded) games

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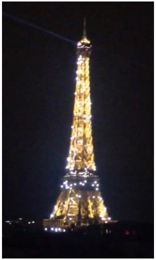

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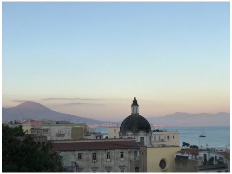

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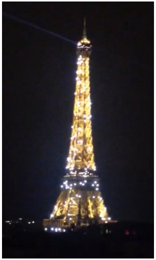

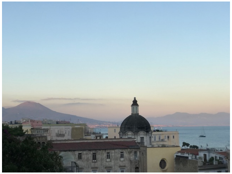

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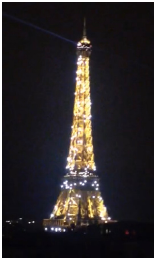

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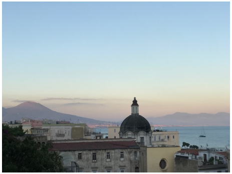

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Loridan-M.:MP'83 ...'00

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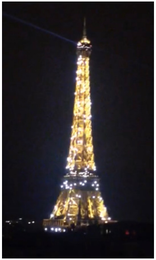

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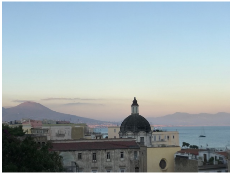

Non-zero sum (ratio-bounded) games

Caruso-Ceparano-M., SIOPT '20,'23

Lignola-M.:JOTA'90...'24

From NORMAL FORM Games and ONE-LEADER Stackelberg Games

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ONE-LEADER Stackelberg Games

in collaboration with:

Cavazzuti '83, Crisci '84, Loridan '85, Lignola '92, Delfour '92, Mallozzi '93, Raucci '97, Flam '02, DelPrete '03, Romaniello '03, Branzei, Tijs '03, Scalzo '03, Patrone '06, Bonnel '06, De Marco '07, Prieur '13, Ceparano '14, Caruso '18

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To MULTI-LEADER Stackelberg Games

in collaboration with:

Ceparano '17, Lignola '24, Caruso Ceparano '24

1. Multi-Leader-Common-Follower games (in short CF games)

- two-stage games with a k-players non-cooperative game at the first stage and a parametric one-player game at the second stage
- a particular case of the so-called Multi-Leader-Multi-Follower games

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Existence of solutions complicated by the presence of more leaders even if:

- the set of the optimal solutions in the second stage is a singleton
- the leaders payoffs are linear

A few number of papers concerns existence results for CF games:

- ✓ regarding a specific situation, derived from the real-world, which is solved by explicitly computing the solutions set:
  Aussel & Cervinka & Marechal '16, Aussel & Bendotti & Pištěk '17; Aussel&Svensson '20 (a review);
- ✓ considering the associated mathematical problems as special cases of equilibrium problems with equilibrium constraints:
  Leyffer & Munson '10, Hu & Fukushima '13,'15;
- ✓ considering classes of CF games satisfying specific conditions: Kulkarni & Shanbhag '14,'15.

- D. Aussel, M. Cervinka and M. Marechal: Deregulated electricity markets with thermal losses and production bounds: models and optimality conditions, RAIRO-Oper. Res. 50(1), 19-38, (2016).
- D. Aussel, P. Bendotti and M. Pištěk: Nash equilibrium in a pay-as-bid electricity market: part 1 existence and characterization; part 2 best response of a producer, Optimization 66(6), 1013-1053, (2017).
- D. Aussel and A. Svensson: A Short State of the Art on Multi-Leader-Follower Games. In: Dempe S., Zemkoho A. (eds) *Bilevel Optimization*. Springer Optimization and its Applications 161. Springer, Cham. 53-76, (2020).
- M. Hu and M. Fukushima: Existence, uniqueness, and computation of robust Nash equilibria in a class of multi-leader-follower games. SIAM Journal on Optimization 23(2), 894-916, (2013).
- M. Hu and M. Fukushima: Multi-leader-follower games: models, methods and applications. J. Oper. Res. Soc. Jpn 58(1), 1-23, (2015).
- A. A. Kulkarni and U. V. Shanbhag: A shared-constraint approach to multi-leader multi-follower games, Set-Valued and Variational Anal. 22(4), 691-720, (2014).
- A.A. Kulkarni and U.V. Shanbhag: An Existence Result for Hierarchical Stackelberg v/s Stackelberg Games. IEEE Trans. Autom. Control 60(12), 3379-3384, (2015).
- S. Leyffer and T.S. Munson: Solving multi-leader-common-follower games. Optim. Methods Softw. 25(4), 601-623, (2010).

In *M.B. Lignola and J.M., Multi-Leader-Common-Follower games with pessimistic leaders: approximate and viscosity solutions, WpCSEFn.639, MTA '24*

We consider the following situation, also considered in:
Leyffer&Munson '10, Kulkarni&Shanbhag '14,'15, Aussel&Svensson '20

- possibly non-unique solution at the second stage
- pessimistic behavior of the leaders who cannot influence the common follower (that can choose any of the optimal responses in reaction to the choice of the leaders)

Key points

(i) A pessimistic solution concept for CF games

(ii) *Weighted Potential CF games* and their pecularities

(iii) *Approximate and viscosity* solutions for CF games and existence results

- k leaders, $k > 1$, 1 follower
- Finite-dimensional spaces (possible extension to infinite dimensional Banach spaces by appropriately balancing the use of strong and weak convergence in the hypotheses)
- Y nonempty subset of $\mathbb{R}^m$: actions of the common follower
- X_i nonempty closed subset of $\mathbb{R}^{l_i}$: actions of leader i, for $i = 1, ..., k$
- $X = \prod_{i=1,..,k} X_i$
- If $\bar{\boldsymbol{x}} = (\bar{x}_1, ..., \bar{x}_k) \Rightarrow \bar{\boldsymbol{x}}_{-i} = (\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_k)$ and $(x_i, \bar{\boldsymbol{x}}_{-i}) = (\bar{x}_1, , .., \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, ..., \bar{x}_k)$
- $K : \boldsymbol{x} \in X \rightrightarrows K(\boldsymbol{x}) \subseteq Y$ is a set-valued map with nonempty values

Follower: action set Y and payoff function F from $X \times Y$ to $\mathbb{R}$

Given $\boldsymbol{x} = (x_1, ..., x_k)$ an action profile of the leaders playing first non-cooperatively, the follower will choice y knowing the action profile $\boldsymbol{x}$ chosen by the leaders

Follower minimizes $F(\boldsymbol{x}, y)$ **with respect to** $y \in K(\boldsymbol{x})$

$$P(\boldsymbol{x}) \qquad \text{find } \bar{y} \in K(\boldsymbol{x}) \text{ such that } F(\boldsymbol{x}, \bar{y}) \leq F(\boldsymbol{x}, y) \ \ \forall\, y \in K(\boldsymbol{x})$$

The solution map (argmin map)

$$\mathcal{M} : \boldsymbol{x} \in X \rightrightarrows \mathcal{M}(\boldsymbol{x}) = \{y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq F(\boldsymbol{x}, z)\ \forall\, z \in K(\boldsymbol{x})\}$$

also called Best Response Map

Leader i: action set X_i and payoff function L_i from $X \times Y$ to $\mathbb{R}$

All leaders, when prepared for the worst and having in mind to minimize their payoffs, consider the functions $\mathcal{P}_i$, called pessimistic payoffs:

$$\mathcal{P}_i : \boldsymbol{x} \in X \to \mathcal{P}_i(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} L_i(\boldsymbol{x}, y), \text{ for } i = 1, ..., k$$

Leaders play a k-player non-cooperative normal form game:
$\boldsymbol{\Gamma} = \{k, X_1, .., X_k, \mathcal{P}_1, .., \mathcal{P}_k\}$

Associated classical Nash Equilibrium Problem:

$$\text{Find } \bar{\boldsymbol{x}} \in X \text{ such that } \mathcal{P}_i(\bar{\boldsymbol{x}}) = \inf_{x_i \in X_i} \mathcal{P}_i(x_i, \bar{\boldsymbol{x}}_{-i}) \text{ for } i = 1, .., k$$

called Pessimistic Multi-Leader-Common-Follower problem (PCF):

(PCF) find $\bar{\boldsymbol{x}} \in X$ such that:

$$\sup_{y \in \mathcal{M}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}}, y) = \inf_{x_i \in X_i} \sup_{y \in \mathcal{M}(x_i, \bar{\boldsymbol{x}}_{-i})} L_i(x_i, \bar{\boldsymbol{x}}_{-i}, y) \ \forall\, i = 1, .., k$$

A solution to (PCF) is called a pessimistic solution to the CF game

CF games may fail to have pessimistic solutions even if the best response map $\mathcal{M}$ is single-valued and the leaders' payoffs are linear in all the variables.

Example 1 (Pang and Fukushima, Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games: Comput Manag Sci 2, 2005)

$$X_1 = X_2 = [0,1], Y = [0,+\infty[, F(x_1,x_2,y) = y(x_1+x_2-1) + \frac{1}{2}y^2,$$

$$L_1(x_1,x_2,y) = \frac{1}{2}\,x_1 + y, \qquad L_2(x_1,x_2,y) = -\left(\frac{1}{2}\,x_2 + y\right)$$

The argmin map $\mathcal{M}$ of the follower is single-valued:

$$\mathcal{M} : \boldsymbol{x} = (x_1,x_2) \in X \rightrightarrows \mathcal{M}(\boldsymbol{x}) = \{\max(0, 1-x_1-x_2)\} \subseteq Y$$

and one can see that the normal form game $(X_1, X_2, \mathcal{P}_1, \mathcal{P}_2)$, where

$$\mathcal{P}_1(x_1,x_2) = \max\left(\frac{1}{2}\,x_1, 1-\frac{1}{2}\,x_1 - x_2\right) \quad \mathcal{P}_2(x_1,x_2) = \min\left(-\frac{1}{2}\,x_2, -1+x_1+\frac{1}{2}\,x_2\right),$$

does not have any Nash equilibrium and the CF game does not have any pessimistic solution.

Formulation of problem (PCF) and existence theorem of Nash equilibria for normal form games

$\Downarrow \; \Downarrow$

Require semicontinuity and *quasi-convexity* properties of the pessimistic payoffs of the leaders (*marginal functions* of the sup-type)

Let g be a real-valued function defined in $U \times W \subseteq \mathbb{R}^m \times \mathbb{R}^h$, where U and W are nonempty closed sets, and let T be a set-valued map from U to W.
Let $s(u) = \sup_{w \in T(u)} g(u, w)$
1. (Berge '59, Aubin & Frankowska '90)
- the set W is compact and the set-valued map T is closed over U;
- the function g is upper semicontinuous over $U \times W$;
$\Rightarrow$ $s(u)$ is upper semicontinuous over U.

In order to apply this results to CF games we take $T = argminmap = \mathcal{M}$: closedness of $\mathcal{M}$ can be easily guaranteed

2. (Berge '59, Aubin & Frankowska '90)
- the set-valued map T is lower semicontinuous over U;
- the function g is lower semicontinuous over $U \times W$;
$\Rightarrow$ $s(u)$ is lower semicontinuous over U.

3. (Fiacco & Kyparisis '86)
- the sets U and W are convex and the map T is concave over U;
- the function g is quasi-convex over $U \times W$;
$\Rightarrow$ $s(u)$ is quasi-convex over U.

- The lower semicontinuity and the concavity CANNOT be easily guaranteed for map $\mathcal{M}$ even for nice data.
- So, we consider particularly tractable CF games, inspired by the class of weighted potential normal form games introduced by Monderer and Shapley: Potential games, Games Econom. Behav. 14(1),(1996).

In fact, the search of a Nash equilibrium for a normal form game $(H_1, ..., H_k, G_1, ..., G_k)$ can be reduced to the search of a minimum point whenever we deal with a *weighted potential game*.

A normal form game $(H_1, ..., H_k, G_1, ..., G_k)$ is a *weighted potential game* whenever there exist a real-valued function Φ defined on H and a vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k)$ such that $\alpha_i > 0 \; \forall \; i = 1, ..., k$ and, for any $i = 1, ..., k$, any $\boldsymbol{x} \in H$ and any $x'_i \in H_i$,

$$G_i(x_i, \boldsymbol{x}_{-i}) - G_i(x'_i, \boldsymbol{x}_{-i}) = \alpha_i(\Phi(x_i, \boldsymbol{x}_{-i}) - \Phi(x'_i, \boldsymbol{x}_{-i})).$$

Weighted potential normal form games are tractable since the set of their Nash equilibria contains the set of the minimum points to the weighted potential function.

Recent new existence and uniqueness results of Nash equilibria for such games can be found in Caruso & Ceparano & M., JMAA '18 and well-posedness results in Margiocco & Pusillo, Optimization '08.

- Exploiting this feature of weighted potential normal form games, we consider suitable classes of CF games having in mind to overcome the lack of pessimistic solutions. First investigations in this direction: Kulkarni & Shanbhag '14,'15.
- Two particular interesting classes of CF games can be considered but, due to the lack of time, only one will be now presented.

Definition 1 (Weighted Potential CF game)

A CF game is said to be a Weighted Potential CF game if:

- *there exists a real-valued function* π *defined on* $X \times Y$, *called a weighted potential of the CF game,*
- *for* $i = 1, \dots, k$, *there exists* $\beta_i \in \mathbb{R}_{++}$,

such that, for all $(\boldsymbol{x}, y) \in X \times Y$ *and* $(x'_i, y') \in X_i \times Y$:

$$L_i(x_i, \boldsymbol{x}_{-i}, y) - L_i(x'_i, \boldsymbol{x}_{-i}, y') = \beta_i \left[\pi(x_i, \boldsymbol{x}_{-i}, y) - \pi(x'_i, \boldsymbol{x}_{-i}, y')\right].$$

Generalizes the concept defined in Kulkarni & Shanbhag '14, '15 when a single follower is common to all leaders.

Proposition 1

A CF game is a weighted potential CF game if and only:

- *there exists a real-valued function* Π *defined on* $X \times Y$,
- *for* $i = 1, \dots, k$, *there exist* $\beta_i \in \mathbb{R}_{++}$ *and a r.v. function* Ψ_i *defined on* X_{-i}

such that, for all $(\boldsymbol{x}, y) \in X \times Y$: $L_i(\boldsymbol{x}, y) = \beta_i \Pi(\boldsymbol{x}, y) + \Psi_i(\boldsymbol{x}_{-i})$.

Moreover, the function Π *is a weighted potential of the CF game.*

Theorem 1 (Existence of pessimistic solutions of CF games)

- *CF game is a weighted potential CF game with* Π *as a weighted potential;*
- X_i *is compact, for* $i = 1, ..., k$*;*
- $\mathcal{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$, *called weighted value-potential, is lower semicontinuous over* X*;*

$\Rightarrow$ *there exists a pessimistic solution to the CF game*

However, even if the leader's payoffs are linear, last assumption may fail to be satisfied and weighted potential games may fail to have equilibria

Example 2

$$X_1 = X_2 = [0,1], Y = [0,1], F(x_1, x_2, y) = x_1 y, K(x_1, x_2) = [0,1],$$

$$L_1(x_1, x_2, y) = x_1 + y - x_2, \qquad L_2(x_1, x_2, y) = \frac{1}{2}(x_1 + y)$$

Weighted potential CF game with $\Pi(\boldsymbol{x}, y) = x_1 + y$ as a weighted potential and $\beta_1 = 1,\ \beta_2 = \frac{1}{2}$.

However, the problem (PCF) does not have pessimistic solutions.

Regularization of the second stage

Existence of solutions being not necessarily guaranteed for weighted potential CF games, we introduce appropriate regularizations, in line with the case one-leader one-follower:

Let $\varepsilon > 0$ and the ε-minimum map $\mathcal{M}^\varepsilon$ be defined by

$$\mathcal{M}^\varepsilon : \boldsymbol{x} \in X \rightrightarrows \mathcal{M}^\varepsilon(\boldsymbol{x}) = \left\{ y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq \inf_{z \in K(\boldsymbol{x})} F(\boldsymbol{x}, z) + \varepsilon \right\}.$$

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Let $\varepsilon > 0$ and $\mathcal{P}^\varepsilon$ be defined by $\mathcal{P}^\varepsilon(\boldsymbol{x}) = \sup_{y \in \mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$,

called ε-weighted value-potential .

Definition 2 (ε-pessimistic solution)

Let $\varepsilon > 0$, a point in X is an ε-pessimistic solution of the CF game if it is a minimum point of the function $\mathcal{P}^\varepsilon$ on X.

Theorem 2 (Existence of ε-pessimistic solutions)

Consider a weighted potential CF game with Π as a weighted potential s.t.

i) the sets Y and X_i are compact, for $i = 1, .., k$;
ii) K is closed, lower semicontinuous and convex-valued over X;
iii) the function F is continuous over $X \times Y$;
iv) the function $F(\boldsymbol{x}, \cdot)$ is strictly quasi-convex over $K(\boldsymbol{x})$ for every $\boldsymbol{x} \in X$;
v) the weighted potential Π is lower semicontinuous over $X \times Y$;

$\Rightarrow$ there exists an ε-pessimistic solution of the CF game, for all $\varepsilon > 0$.

Definition 3 (pessimistic viscosity solutions)

Consider a weighted potential CF game with Π *as a weighted potential. A point* $\bar{\boldsymbol{x}} \in X$ *is a* **pessimistic viscosity solution** *of the CF game if for every sequence of positive numbers* $(\varepsilon_n)_n$ *decreasing to zero there exists a sequence* $(\bar{\boldsymbol{x}}_n)_n$*, such that:*

V_1) *a subsequence* $(\bar{\boldsymbol{x}}_{n_k})_k$ *converges towards* $\bar{\boldsymbol{x}}$*;*

V_2) *for any* $n \in \mathcal{N}$, $\bar{\boldsymbol{x}}_n$ *is an* ε_n*-pessimistic solution of the CF game*

V_3) $\lim_n \mathcal{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) = \inf_{\boldsymbol{x} \in X} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$

Roughly speaking, a pessimistic viscosity solution of a weighted potential CF game is a cluster point of a sequence of minimum points of suitable regularizations $\mathcal{P}^{\varepsilon_n}$ of the function $\mathcal{P}$ (weighted-value potential), whose values approach the security value of a one-leader-one-follower two-stage game with an hypothetic pessimistic leader having the weighted potential Π as payoff.

Theorem 3 (Existence of pessimistic viscosity solutions)

Weighted potential CF game with Π as a weighted potential.
i) the sets Y and X_i are compact, for $i = 1, .., k$;
ii) K is closed, lower semicontinuous and convex-valued over X;
iii) the function F is continuous over $X \times Y$;
iv) the function $F(\boldsymbol{x}, \cdot)$ is strictly quasi-convex over $K(\boldsymbol{x})$ for every $\boldsymbol{x} \in X$;
v) the weighted potential Π is lower semicontinuous over $X \times Y$;
vi) for every $\boldsymbol{x} \in X$, the function $\Pi(\boldsymbol{x}, \cdot)$ is upper semicontinuous

$\Rightarrow$ there exists a pessimistic viscosity solution of the CF game.

Proposition 2

Let CF be a weighted potential game. If assumptions of Theorem 3 hold and the weighted potential Π is upper semicontinuous, then, any pessimistic viscosity solution of the CF game is a minimum point for cl $\mathcal{P}$, lower semicontinuous regularization of the weighted value-potential $\mathcal{P}$, defined by $\mathcal{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$.

[Concerning Example 2]

- let $\varepsilon < 1/4$, the ε-minimum map is defined by:
  $\mathcal{M}^{\varepsilon}(\boldsymbol{x}) = \left\{ y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq \inf_{z \in K(\boldsymbol{x})} F(\boldsymbol{x}, z) + \varepsilon \right\}$
- the marginal function of Π over this regularized map, denoted by $\mathcal{P}^{\varepsilon}$, is continuous on H, where $\mathcal{P}^{\varepsilon}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}^{\varepsilon}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$
- for any $\varepsilon \in]0, 1/4[$, the minimum points of $\mathcal{P}^{\varepsilon}$ are $(\sqrt{\varepsilon}, x_2)$ for any $x_2 \in [0, 1]$ $\Rightarrow$ the minimum value of $\mathcal{P}^{\varepsilon}$ is $2\sqrt{\varepsilon}$.
- Therefore, we can consider the limits of the above approximate pessimistic solutions, i.e. points $(0, x_2)$, as reasonable solutions to (PCF), since we have:

  $$\lim_{\varepsilon \to 0} \mathcal{P}^{\varepsilon}(\sqrt{\varepsilon}, x_2) = \lim_{\varepsilon \to 0} \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}^{\varepsilon}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) = 0 = \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y),$$

  and points $(0, x_2)$ allow both the leaders to realize the security value of a new two-stage game, with only one leader, having X as the set of strategies, namely:

  $$\text{find } \bar{\boldsymbol{x}} \in X \text{ such that } \sup_{y \in \mathcal{M}(\bar{\boldsymbol{x}})} \Pi(\bar{\boldsymbol{x}}, y) = \inf_{\boldsymbol{x} \in X} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$$

What can we say if NO CONVEXITY-like assumptions?

Introduction of an other appropriate regularization

Let $\varepsilon > 0$ and the strict ε-minimum map $\widetilde{\mathcal{M}}^{\varepsilon}$ be defined by

$$\widetilde{\mathcal{M}}^{\varepsilon}(\boldsymbol{x}) = \left\{ y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) < \inf_{z \in K(\boldsymbol{x})} F(\boldsymbol{x}, z) + \varepsilon \right\}$$

Let $\varepsilon > 0$ and the strict ε-weighted value-potential $\widetilde{\mathcal{P}}^{\varepsilon}$ be defined by

$$\widetilde{\mathcal{P}}^{\varepsilon}(\boldsymbol{x}) = \sup_{y \in \widetilde{\mathcal{M}}^{\varepsilon}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$$

Definition 4 (strict ε-pessimistic solution)

Let $\varepsilon > 0$, a point in X is a strict ε-pessimistic solution of the CF game if it is a minimum point of the function $\widetilde{\mathcal{P}}^{\varepsilon}$ on X.

Theorem 4 (Existence of strict ε-pessimistic solutions)

Consider a weighted potential CF game with Π as a weighted potential s.t.
I) the sets Y and X_i are compact, for $i = 1, .., k$;
II) K is closed, lower semicontinuous over X;
III) the function F is continuous over $X \times Y$;
IV) the weighted potential Π is lower semicontinuous over $X \times Y$;

$\Rightarrow$ *there exists a strict ε-pessimistic solution for any $\varepsilon > 0$*

NO CONVEXITY-like assumptions!

Definition 5 (strict pessimistic viscosity solutions)

Consider a weighted potential CF game with Π *as a weighted potential. A point* $\bar{\boldsymbol{x}} \in X$ *is a strict pessimistic viscosity solution of the CF game if for every sequence of positive numbers* $(\varepsilon_n)_n$ *decreasing to zero there exists a sequence* $(\bar{\boldsymbol{x}}_n)_n$*, such that:*

$V_1)$ *a subsequence* $(\bar{\boldsymbol{x}}_{n_k})_k$ *converges towards* $\bar{\boldsymbol{x}}$*;*

$V_2)$ *for any* $n \in \mathcal{N}$, $\bar{\boldsymbol{x}}_n$ *is a strict* ε_n*-pessimistic solution of the CF game*

$V_3)$ $\lim_n \widetilde{\mathcal{P}}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) = \inf_{\boldsymbol{x} \in X} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$

Theorem 5 (Existence of strict pessimistic viscosity solutions)

Weighted potential CF game with Π *as a weighted potential.*
I) the sets Y *and* X_i *are compact, for* $i = 1, .., k$*;*
II) K *is closed, lower semicontinuous over* X*;*
III) the function F *is continuous over* $X \times Y$*;*
IV) the weighted potential Π *is lower semicontinuous over* $X \times Y$*;*
V) for every $\boldsymbol{x} \in X$*, the function* $\Pi(\boldsymbol{x}, \cdot)$ *is upper semicontinuous on* Y
$\Rightarrow$ *there exists a strict pessimistic viscosity solution of the CF game.*

- "Variational" stability
- Numerical approximation

2. Multi-Leader-Multi-Follower games
(in short MF games)

Wide range of applications. For example in engineering and economics:

- electricity power market (Cardell, Hitt, Hogan, 1997; Hobbs, Metzler, Pang, 2000; Hu, Ralph, 2007; Henrion, Outrata, Surowiec, 2012; Aussel, Cervinka, Marechal 2016; Aussel, Bendotti, Pištěk 2017; Allevi, Aussel, Ricciardi, 2018; . . .),. . .
- spot-forward markets (Allaz, Vila, 1993; Su, 2007;. . .)
- two-period Cournot competitions (Sherali, 1984; Saloner, 1987; Pal, 1991;. . .)
- contract theory (McAfee, Schwartz, 1994; Rey, Vergé, 2004; Pagnozzi, Piccolo, 2011;. . .),
- . . .

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Wide range of applications. For example in engineering and economics:

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- . . .

Few theoretical and numerical investigations. See

- Su, 2005; DeMiguel, Xu, 2009; Hu, Fukushima, 2013, 2015; Kulkarni, Shanbhag 2014; Ceparano, M., 2017; Mallozzi, Messalli 2017; Herty, Steffensen, Thünen, 2020; Aussel, Svensson, 2020; Caruso, Ceparano, M., 2023; Lignola, M., 2024; . . .

Case of **uniqueness of the solution in the second-stage** problem:

- When there is vertical information: **Equilibria under passive beliefs**
  M.C. Ceparano and J.M. *"Equilibrium selection in multi-leader-follower games with vertical information"*, **TOP**, 2017.
- When the Nash equilibrium in the second stage does not depend on the leaders' actions: **Bilevel Nash equilibria**

  F. Caruso, M.C. Ceparano and J.M. *"Bilevel Nash equilibrium problems: numerical approximation via direct-search methods"*, **Dynamic Games and Applications**, 2023.

Case of **non-uniqueness of the solution in the second-stage** problem:

- When the **leaders are pessimistic**:
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Motivating example

Vertical Separation with Private Contracts,
M. Pagnozzi, S. Piccolo. **The Economic Journal**, 2012

Competing markets: Two competing manufacturers (leaders) that produce substitute goods delegate sales of their product through two independent retailers (followers).

- Each retailer sells the good produced by only one manufacturer
- each manufacturer offers a contract to the *exclusive* retailer
- the contract is private: each retailer cannot observe the contract offered to his competitor

Key point: exclusivity between a manufacturer and his retailer

- a manufacturer delegate the sales using an exclusive retailer
- private contract between a manufacturer and his retailer

$$\Downarrow$$

General model

Multi-leader-follower game with vertical information

with possibly discontinuous payoff functions

- **Issue:** The associate game may have an **infinity** of Nash Equilibria (that are also Subgame Perfect Nash Equilibria)

$$\Downarrow$$

- **Selection** using Passive Beliefs: special type of **weak perfect Bayesian equilibria** (imposing restriction on beliefs out of the equilibrium path)
- **Result: Existence of equilibria under passive beliefs**

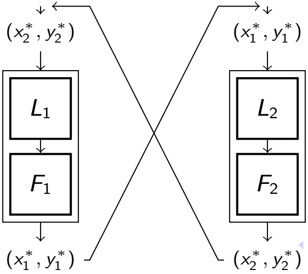
Passive beliefs

If a follower observes an action of his corresponding leader different from one that he expects in equilibrium, then he believes that the rival leaders are still not deviating from their equilibrium strategy

Passive beliefs in economic literature: *multilateral vertical contracting games*, *electoral competition games*, *consumer search*,. . .

A strategy profile is an equilibrium under passive beliefs if each "leader–exclusive follower" block acts as a team that solves a **Parametric Bilevel Optimization problem**, where the parameter is the equilibrium actions of the other "leader–exclusive follower" blocks

The strategic interaction is between the "teams".



"Bilevel Nash Equilibrium Problems: Numerical Approximation Via Direct-Search Methods", F. Caruso, M.C. Ceparano, J. Morgan.
Dynamic Games and Applications, 2024

Bilevel Nash equilibrium problem: hierarchical structure with two levels

(i) a Nash equilibrium problem at the first level

(ii) a Nash equilibrium problem at the second level

Mathematical problem associated to multi-leader-multi-follower games

Our Framework:

- first-stage players (leaders) involved in a potential game
- second-stage players (followers) involved in a ratio-bounded game
- the Nash equilibrium in the second-stage does not depend on the leaders' actions and it is **unique by ratio-bounded game assumption**

Class of **Ratio-Bounded games** introduced and investigated in

"An inverse-adjusted best response algorithm for Nash equilibria", F. Caruso, M.C. Ceparano, J. Morgan, **SIAM Journal on Optimization**, 2020.
"Affine relaxations of the best response algorithm: global convergence in ratio-bounded games", F. Caruso, M.C. Ceparano, J. Morgan, **SIAM Journal on Optimization**, 2023.

to study the convergence to Nash equilibria of the **affine relaxations of the best response algorithm**.

The class **contains games broadly used in literature**: quadratic weighted potential games, quadratic zero-sum games, differential games.

Results on ratio-bounded games:

- Existence and uniqueness of the Nash equilibrium,
- Global convergence of specific types of affine relaxations of the best response algorithm and estimation of the related errors,
- Determination of the algorithm with highest speed of convergence.

Back to our **Bilevel Nash Equilibrium problem**:

- Existence and uniqueness of a Bilevel Nash equilibrium
- Definition of the Diagonal Direct-Search Method (DDBN): numerical method to approximate a bilevel Nash equilibrium which combines
  - affine relaxations of the best response algorithm,
  - direct-search methods (class of derivative-free methods for unconstrained optimization)

Results on DDBN:

- Global convergence of the sequence generated by DDBN to the bilevel Nash equilibrium,
- Error bounds,
- Rate of convergence,
- Numerical examples.

Case of **uniqueness of the solution in the second-stage** problem:

- When there is vertical information: **Equilibria under passive beliefs**
  M.C. Ceparano and J.M. *"Equilibrium selection in multi-leader-follower games with vertical information"*, **TOP**, 2017.
- When the Nash equilibrium in the second stage does not depend on the leaders' actions: **Bilevel Nash equilibria**

  F. Caruso, M.C. Ceparano and J.M. *"Bilevel Nash equilibrium problems: numerical approximation via direct-search methods"*, **Dynamic Games and Applications**, 2023.

Case of **non-uniqueness of the solution in the second-stage** problem:

- When the **leaders are pessimistic**:
  M.B. Lignola and J.M. *"On existence and approximations of solutions in multi-leader-multi-follower games"*, **W. in P.**, 2024.

- k leaders, $k > 1$, and m followers, $m > 1$
- Y_j nonempty subset of $\mathbb{R}^{h_j}$: set of actions of the follower j, for $j = 1, ..., m$, $Y = \prod_{j=1,..,m} Y_j$
- X_i nonempty closed subset of $\mathbb{R}^{l_i}$: set of actions of leader i, for $i = 1, ..., k$; $X = \prod_{i=1,..,k} X_i$
- No coupled constraints (for the lack of time)

Follower j: payoff function F_j on $X \times Y$, to be minimized

Given $\boldsymbol{x} = (x_1, ..., x_k)$, an action profile of the leaders playing first,

the followers play the m-player non-cooperative normal form game

$$\boldsymbol{\Omega}(\boldsymbol{x}) = \{m, Y_1, .., Y_m, F_1(\boldsymbol{x}, .), .., F_m(\boldsymbol{x}, .)\}$$

that is, for all $j = 1, ..m$, the follower j, knowing the action profile $\boldsymbol{x}$ chosen by the leaders, will choose y_j s.t. $\boldsymbol{y} = (y_1, .., y_m)$ is a Nash equilibria of $\boldsymbol{\Omega}(\boldsymbol{x})$

Let $\mathcal{N}(\boldsymbol{x})$ be the Nash equilibria set of the game $\boldsymbol{\Omega}(\boldsymbol{x})$

Leader i: action set X_i and payoff function L_i from $X \times Y$ to $\mathbb{R}$

All leaders, when prepared for the worst and having in mind to minimize their payoffs, consider the functions $\mathcal{Q}_i$, called pessimistic payoffs and defined by:

$$\mathcal{Q}_i(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} L_i(\boldsymbol{x}, \boldsymbol{y}), \text{ for } i = 1, ..., k$$

Leaders play a k-player non-cooperative normal form game:

$\boldsymbol{\Delta} = \{k, X_1, .., X_k, \mathcal{Q}_1, .., \mathcal{Q}_k\}$

Associated classical Nash Equilibrium Problem:

$$\text{Find } \bar{\boldsymbol{x}} \in X \text{ such that } \mathcal{Q}_i(\bar{\boldsymbol{x}}) = \inf_{x_i \in X_i} \mathcal{Q}_i(x_i, \bar{\boldsymbol{x}}_{-i}) \text{ for } i = 1, .., k$$

called Pessimistic Multi-Leader-Multi-Follower problem (PMF):

(PMF) find $\bar{\boldsymbol{x}} \in X$ such that:

$$\sup_{\boldsymbol{y} \in \mathcal{N}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}}, \boldsymbol{y}) = \inf_{x_i \in X_i} \sup_{\boldsymbol{y} \in \mathcal{N}(x_i, \bar{\boldsymbol{x}}_{-i})} L_i(x_i, \bar{\boldsymbol{x}}_{-i}, \boldsymbol{y}) \ \forall\, i = 1, .., k$$

A solution to (PMF) is called a pessimistic solution to the MF game

Again, the lower semicontinuity and the concavity CANNOT be easily guaranteed for the map $\mathcal{N}$ even for nice data so we consider another class of tractable games.

Definition 6 (Weighted Potential MF game)

A MF game is said to be a Weighted Potential MF game if:
there exists a real-valued function π defined on $X \times Y$, called a weighted potential of the MF game and, for $i = 1, .., k$, there exists $\beta_i \in \mathbb{R}_{++}$ such that, for all $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$ and $(x_i', \boldsymbol{y}') \in X_i \times Y$:

$$L_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) - L_i(x_i', \boldsymbol{x}_{-i}, \boldsymbol{y}') = \beta_i \left[\pi(x_i, \boldsymbol{x}_{-i}, \boldsymbol{y}) - \pi(x_i', \boldsymbol{x}_{-i}, \boldsymbol{y}')\right].$$

Generalizes both the concepts of potential and quasi-potential multi-leader-multi-follower games, defined in Kulkarni & Shanbhag '14, '15

Proposition 3

A MF game is a weighted potential MF game if and only:

- *there exists a real-valued function* Π *defined on* $X \times Y$*;*
- *for* $i = 1, .., k$*, there exist* $\beta_i \in \mathbb{R}_{++}$ *and a real valued function* Ψ_i *defined on* X_{-i}*;*

such that, for all $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$*:* $L_i(\boldsymbol{x}, \boldsymbol{y}) = \beta_i \, \Pi(\boldsymbol{x}, \boldsymbol{y}) + \Psi_i(\boldsymbol{x}_{-i})$.

Moreover, the function Π *is a weighted potential of the MF game.*

Theorem 6 (Existence of pessimistic solutions of MF games)

- *MF is a weighted potential game with Π as a weighted potential;*
- *X_i is compact, for $i = 1, \dots, k$;*
- $\mathcal{Q}(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y})$, *called weighted value-potential, is lower semicontinuous over X;*

$\Rightarrow$ *there exists a pessimistic solution to the MF game*

However, even if the leader's payoffs are linear, last assumption may fail to be satisfied and weighted potential MF games may fail to have equilibria

Regularization of the second stage

Now, we introduce appropriate regularizations of the set of Nash equilibria $\mathcal{N}(\boldsymbol{x})$, as used in the one-leader one or two-follower cases

- J. Morgan and R. Raucci, New convergence results for Nash equilibria: Journal of Convex Analysis 6(2), 377-385,(1999)
- J. Morgan and R. Raucci, Lower semicontinuity for approximate social Nash equilibria, Int. J. of Game Theory 31(4), 499-509, (2003)
- M.B. Lignola and J. Morgan: Viscosity solutions for bilevel problems with Nash equilibrium constraints, Far East J. Appl. Math. 88 (1), 15-34, (2014).

Let $\varepsilon > 0 \Rightarrow$ Approximate Nash equilibria

strict ε-approximate equilibrium map $\widetilde{\mathcal{N}}^{\varepsilon}$ defined by

$$\widetilde{\mathcal{N}}^{\varepsilon}(\boldsymbol{x}) =: \{\boldsymbol{y} \in Y : F_1(\boldsymbol{x},\boldsymbol{y}) + ... + F_m(\boldsymbol{x},\boldsymbol{y}) < \inf_{z_1 \in Y_1} F_1(\boldsymbol{x}, z_1, \boldsymbol{y}_{-1}) + ... + \inf_{z_m \in Y_m} F_m(\boldsymbol{x}, z_m, \boldsymbol{y}_{-m}) + \varepsilon\}$$

ε-approximate equilibrium map $\mathcal{N}^{\varepsilon}$ defined by:

$$\mathcal{N}^{\varepsilon}(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : F_1(\boldsymbol{x},\boldsymbol{y}) + ... + F_l(\boldsymbol{x},\boldsymbol{y}) \leq \inf_{z_1 \in Y_1} F_1(\boldsymbol{x}, z_1, \boldsymbol{y}_{-1}) + ... + \inf_{z_m \in Y_m} F_m(\boldsymbol{x}, z_m, \boldsymbol{y}_{-m}) + \varepsilon\}$$

ε-equilibrium map $\hat{\mathcal{N}}^{\varepsilon}$ defined by:

$$\hat{\mathcal{N}}^{\varepsilon}(\boldsymbol{x}) = \Big\{\boldsymbol{y} \in Y : F_1(\boldsymbol{x},\boldsymbol{y}) \leq \inf_{z_1 \in Y_1} F_1(\boldsymbol{x}, z_1, \boldsymbol{y}_{-1}) + \varepsilon ; ... ; ... \text{ and } F_m(\boldsymbol{x},\boldsymbol{y}) \leq \inf_{z_m \in Y_m} F_m(\boldsymbol{x}, z_m, \boldsymbol{y}_{-m}) + \varepsilon\}$$

For the lack of time consider only the first type of approximate equilibria

strict ε-approximate equilibrium map $\widetilde{\mathcal{N}}^{\varepsilon}$ defined by

$$\widetilde{\mathcal{N}}^{\varepsilon}(\boldsymbol{x}) =: \{\boldsymbol{y} \in Y : F_1(\boldsymbol{x}, \boldsymbol{y}) + \dots + F_m(\boldsymbol{x}, \boldsymbol{y}) < \inf_{z_1 \in Y_1} F_1(\boldsymbol{x}, z_1, \boldsymbol{y}_{-1}) + \dots + \inf_{z_m \in Y_m} F_m(\boldsymbol{x}, z_m, \boldsymbol{y}_{-m}) + \varepsilon\}$$

Let $\varepsilon > 0$ and $\widetilde{\mathbb{Q}}^{\varepsilon}$ be defined by $\widetilde{\mathbb{Q}}^{\varepsilon}(\boldsymbol{x}) = \sup_{\boldsymbol{y} \in \widetilde{\mathcal{N}}^{\varepsilon}(\boldsymbol{x})} \Pi(\boldsymbol{x}, \boldsymbol{y})$,

called ε-weighted value-potential

Definition 7 (ε-pessimistic solution)

Let $\varepsilon > 0$, a point in X is a strict ε-pessimistic solution of the MF game if it is a minimum point of the function $\widetilde{\mathbb{Q}}^{\varepsilon}$ on X.

That is $\bar{\boldsymbol{x}} \in X$ is a strict ε-pessimistic solution if:

$$\sup_{y \in \widetilde{\mathcal{N}}^{\varepsilon}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}}, \boldsymbol{y}) = \inf_{x_i \in X_i} \sup_{\boldsymbol{y} \in \widetilde{\mathcal{N}}^{\varepsilon}(x_i, \bar{\boldsymbol{x}}_{-i})} L_i(x_i, \bar{\boldsymbol{x}}_{-i}, \boldsymbol{y}) \ \forall\, i = 1, .., k$$

Theorem 7 (Existence of strict ε-pessimistic solutions)

Consider a weighted potential MF game with Π as a weighted potential s.t.
i) the sets Y_j and X_i are compact, for $i = 1, .., k$ and $j = 1, ..m$
ii) the function F_j is continuous over $X \times Y$ $j = 1, .., m$;
iii) the weighted potential Π is lower semicontinuous over $X \times Y$;

$\Rightarrow$ *there exists a strict ε-pessimistic solution of the CF game, for all $\varepsilon > 0$.*

Definition 8 (strict pessimistic viscosity solutions)

Consider a weighted potential MF game with Π as a weighted potential.
A point $\bar{\boldsymbol{x}} \in X$ is a strict-pessimistic viscosity solution *of the MF game if for every sequence of positive numbers $(\varepsilon_n)_n$ decreasing to zero there exists a sequence $(\bar{\boldsymbol{x}}_n)_n$, such that:*

V_1) *a subsequence $(\bar{\boldsymbol{x}}_{n_k})_k$ converges towards $\bar{\boldsymbol{x}}$;*

V_2) *for any $n \in \mathcal{N}$, $\bar{\boldsymbol{x}}_n$ is a strict ε_n-pessimistic solution of the MF game*

V_3) $\lim_n \mathcal{Q}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) = \inf_{\boldsymbol{x} \in X} \sup_{y \in \mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$

Theorem 8 (Existence of strict pessimistic viscosity solutions)

Weighted potential MF game with Π as a weighted potential.
i) the sets Y_j and X_i are compact, for $j = 1, .., m$ and $i = 1, .., k$;
ii) the function F_j is continuous over $X \times Y$ for $j = 1, .., m$;
iii) the weighted potential Π is lower semicontinuous over $X \times Y$;
iv) for every $\boldsymbol{x} \in X$, the function $\Pi(\boldsymbol{x}, \cdot)$ is upper semicontinuous on Y;

$\Rightarrow$ *there exists a strict pessimistic viscosity solution of the MF game.*

Proposition 4

Let MF be a weighted potential game.
If assumptions i) and ii) hold and the weighted potential Π is continuous, then, any strict pessimistic viscosity solution of the MF game is a minimum point for cl $\mathcal{Q}$, lower semicontinuous regularization of the weighted value-potential $\mathcal{Q}$, defined by $\mathcal{Q}(\boldsymbol{x}) = \sup_{y \in \mathcal{N}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$.

Thank you!