Investment Horizon Effects on Portfolio Decisions under Uncertainty

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Introduction

- Does the investment horizon matter in portfolio decisions?
- Classical portfolio models prove that if asset returns are i.i.d., an investor with power utility should choose the *same* asset allocation, regardless of the investment horizon.
- But evidence shows that investor's horizon is not irrelevant (Brandt (1999), Barberis (2000), Xia (2000)).
- So, if the investment horizon matters, then the question is: how do portfolio choices depend on it?

Aim and Results

- We investigate the last question in a model where horizon effects arise from parameter uncertainty.
- We study an optimal portfolio choice problem for a CRRA investor when the market price of risk is an unobservable positive random variable (cf., among others, Brennan (1998), Karatzas and Zhao (2001), Rieder and Bäuerle (2005), Longo and Mainini (2016)).
- We prove that the optimal allocation to the risky asset is monotonic with respect to the investment horizon: it is increasing if the investor is more risk tolerant than the logarithmic investor and decreasing otherwise.
- Furthermore, we show that extreme behavior occurs in presence of an exceptionally long investment horizons.

Financial market and budget equation

- An investor with investment horizon $T > 0$ and initial wealth $x > 0$, continuously trades in a financial market with:
  - a risk-free asset with deterministic return $r \geq 0$ and
  - a risky asset with stochastic return

    $$dS_t/S_t = (r + \sigma\Theta)\,dt + \sigma\,dW_t, \quad S_0 > 0,$$

    where σ is a positive constant, $\{W_t\}$ is a standard B.M. on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, and the *market price of risk* (Sharpe ratio) Θ is a real r.v., independent of $\{W_t\}$, with known prior distribution μ.

- Investor's wealth $\{X_t\}$ satisfies the budget equation:

$$dX_t = rX_t dt + \sigma\pi_t(\Theta dt + dW_t), \quad X_0 = x,$$

where π_t is the amount of wealth invested in the risky asset at time t;

Preferences and problem

- Investors observe prices $\{S_t\}$ but do not observe Θ nor $\{W_t\}$;
- They aim at maximizing the expected utility from terminal wealth

$$\mathbb{E}\left[u_\gamma\left(X_T\right)\right],$$

by choosing an investment strategy $\{\pi_t\}$ based on the information provided by the price process $\{S_t\}$.

- u_γ belongs to the class of *Constant Relative Risk Aversion* (CRRA) utility functions:

$$u_\gamma\left(x\right) := \begin{cases} \dfrac{x^\gamma}{\gamma}, & \gamma < 1, \quad \gamma \neq 0, \\ \ln x, & \gamma = 0, \end{cases}$$

with $1 - \gamma$ the investor's coefficient of relative risk aversion.

- Portfolio problem of *partial observation* type.

The optimal investment strategy

- The problem is first reduced to a complete observation framework by means of filtering techniques and then solved by dynamic programming methods.
- The optimal allocation to the risky asset at time 0 is

$$\hat{\pi}(x, \gamma, T) = \frac{x}{\sigma(1-\gamma)} \int_{\mathbb{R}} \theta g(\theta; \gamma, T)\, \mu(d\theta)$$

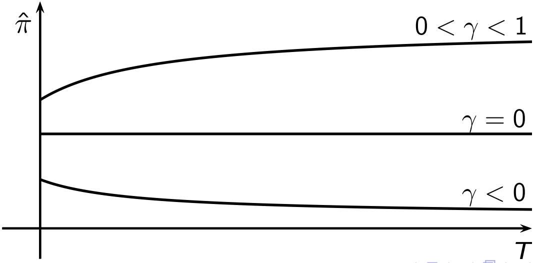
where $g(\theta; \gamma, T)$ is a density with respect to the dominant measure $\mu(d\theta)$,

Monotonicity of the optimal investment in the risky asset

Theorem

If the market price of risk Θ *is positive and bounded,* μ*-a.s., then, for any* $x > 0$, $\hat{\pi}(x, \gamma, T)$, *as a function of the investment horizon* T, *is:*

- *increasing if* $0 < \gamma < 1$;
- *decreasing if* $\gamma < 0$;
- *constant if* $\gamma = 0$ *(logarithmic investor, myopic).*



"Personal conditional expectation" of the Sharpe ratio Θ

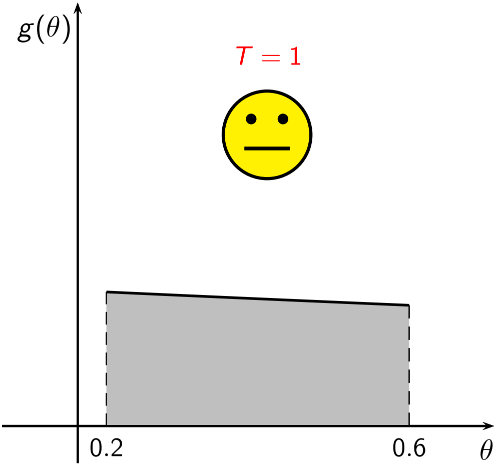
In the representation

$$\hat{\pi}(x,\gamma,T)=\frac{x}{\sigma(1-\gamma)}\underbrace{\int_{\mathbb{R}}\theta g(\theta;,\gamma,T)\,\mu(d\theta)}_{\text{“personal conditional expectation”}},$$

the second factor can be seen as a "personal conditional expectation" of the market price of risk Θ and the density (w.r.t. the measure $\mu(d\theta)$) $g(\theta;\gamma,T)$ as a *"personal conditional density"*.

Pessimistic agents: $\gamma < 0$

- $\gamma = -2$ and uniform prior μ



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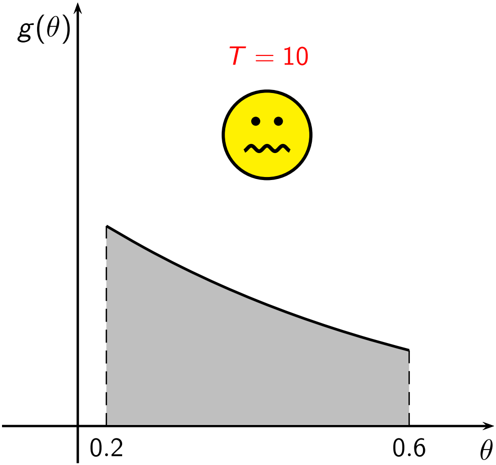

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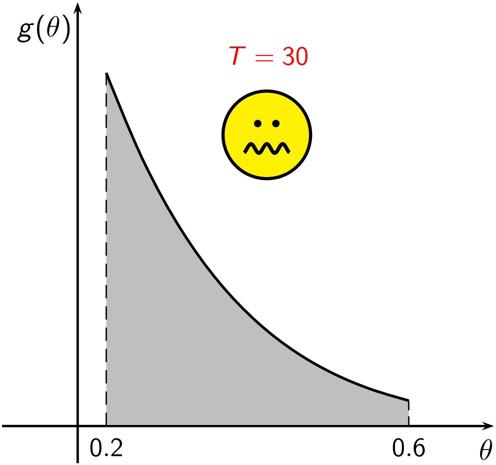
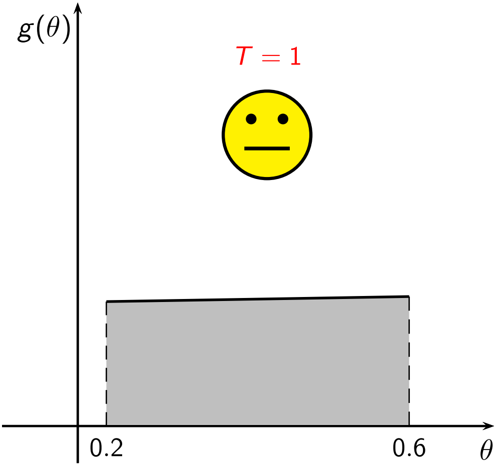


Optimistic agents: $0 < \gamma < 1$

- $\gamma = 0.2$ and uniform prior μ



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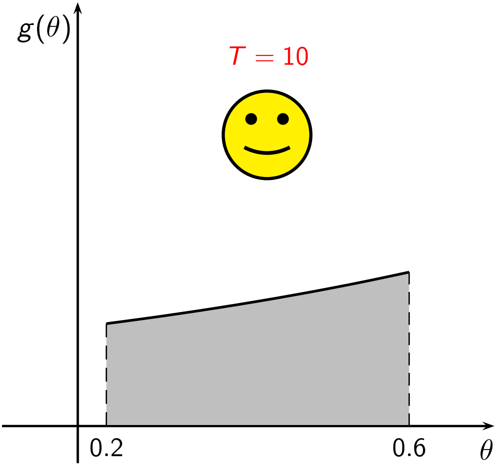

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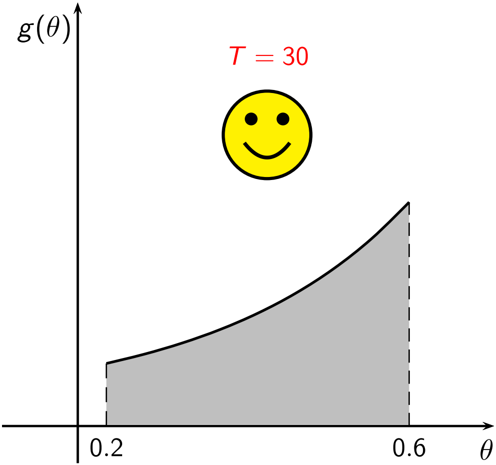


Identikit of an optimistic

- Optimistic agents exhibit low relative risk aversion ($0 < \gamma < 1$).
- Who are the agents with low relative risk aversion?
- According to Morin and Suarez (1983), relative risk aversion increases with age.
- Hence, we expect young investors to be optimistic.
- We expect young investors to invest more on stocks the longer the investment horizon.

Investment for retirement (a comparative statics exercise)

- τ agent's age;
- $\bar{\tau}$ retirement age;
- investment horizon $T = \bar{\tau} - \tau =: T(\tau)$ shrinks as the agent ages;
- $\tau \longmapsto \gamma(\tau)$ decreases;
- $\gamma \longmapsto \hat{\pi}(x, \gamma, T)$ increases.

Then, for $0 < \gamma < 1$,

$$\tau \longmapsto \hat{\pi}\left(x, \gamma\left(\tau\right), T\left(\tau\right)\right) \quad \text{decreases,}$$

this also applies to $\gamma < 0$ and reasonable values for the parameters (i.e. $-10 < \gamma < 0$ and $0 < T < 40$).

Extremal behavior

Let θ_{min} and θ_{max}, with $0 < \theta_{\text{min}} < \theta_{\text{max}} < \infty$, be the minimum and the maximum of the support of μ, then $\hat{\pi}(x, \gamma, T)$ exists for all $T > 0$ and, for all $x > 0$, $\gamma < 1$ and $T \geq 0$,

$$\pi_{\text{min}} \leq \hat{\pi}(x, \gamma, T) \leq \pi_{\text{max}},$$

where

$$\pi_{\text{min}} := \frac{x\theta_{\text{min}}}{\sigma(1-\gamma)} \quad \text{and} \quad \pi_{\text{max}} := \frac{x\theta_{\text{max}}}{\sigma(1-\gamma)}.$$

are the *extremal allocations*.

The monotonicity of $\hat{\pi}(x, \gamma, T)$ with respect to T and the previous bounds imply that $\hat{\pi}(x, \gamma, T)$ converges, as $T \to +\infty$, for any fixed $x > 0$ and $\gamma < 1$.

Extremal behavior

Theorem

Let $\theta_{\min}$ and $\theta_{\max}$, with $0 < \theta_{\min} < \theta_{\max} < \infty$, be the minimum and the maximum of the support of μ, then, for any $x > 0$,

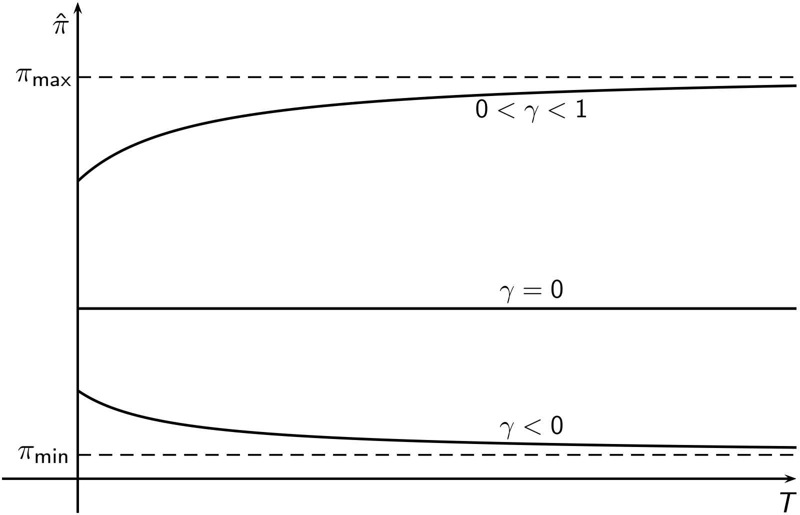
$$\lim_{T \to \infty} \hat{\pi}(x, \gamma, T) = \pi_{\max},$$

if $0 < \gamma < 1$, and

$$\lim_{T \to \infty} \hat{\pi}(x, \gamma, T) = \pi_{\min},$$

if $\gamma < 0$.

Extremal behavior



Extremal behavior

In presence of an exceptionally long investment horizons:

- pessimistic agents (*i.e.*, agents with $\gamma < 0$) become so pessimistic as to believe that the worst market price of risk will eventually arise (*Murphy's law* followers: "Things will eventually go wrong, if you give them a chance");
- optimistic agents (*i.e.*, agents with $0 < \gamma < 1$) become as optimistic as to believe that the best market price of risk will eventually prevail (*Yhprum's law* followers).

Conclusions

- In a simple continuous-time portfolio management model with CRRA agents, if the market price of risk is an unobservable positive random variable, the optimal allocation to the risky asset depends on the investment horizon and, in particular, it is:
  - increases if $0 < \gamma < 1$ (optimistic agents),
  - decreases if $\gamma < 0$ (optimistic agents),
  - is constant if $\gamma = 0$ (logarithmic agent, myopic agent).
- The model sheds light on the common but controversial advise that investors with long horizons should allocate more heavily on stocks: Not everybody invests more on stocks the longer the investment horizon, but if this is the case, then probably behind that strategy there is a young investor.

Conclusions

- For exceptionally long investment horizons, optimistic agents will behave as if the best market price of risk will eventually arise; whereas, pessimistic agents will behave as if the worst market price of risk will prevail.
- Power/logarithmic utility functions (*i.e.*, *CRRA* preferences) are as simple and suitable for computations as they are ductile to represent very different behaviors.