Network games with bounded strategies: properties, algorithms, extensions

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Outline

- Network games
- The linear-quadratic model with bounded strategies
- Variational inequality reformulation and properties of Nash equilibria
- A solution algorithm with finite convergence
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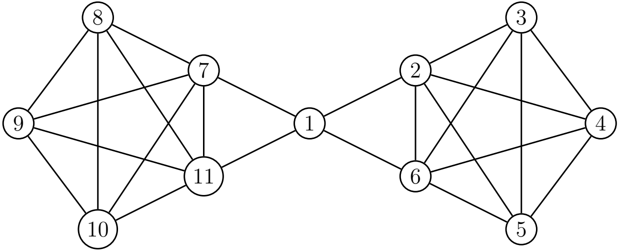
- Extension to Generalized Nash Equilibrium Problems
- Extension to parametric network games

Network games

Given a simple undirected graph (V, E), a network game is a non-cooperative game where

- the set of players is the set of nodes $V = \{1, \dots, n\}$
- the action space of player i is $A_i \subset \mathbb{R}$
- player i has a payoff function $u_i : A = \prod_{i=1}^{n} A_i \to \mathbb{R}$ to be maximized that depends only on the strategies of its neighbors



The linear-quadratic model

In the linear-quadratic model[1] we assume that

- the strategy space is $A_i = \mathbb{R}_+$
- the payoff function is

$$u_i(a) = -\frac{1}{2} a_i^2 + \alpha_i a_i + \phi a_i \sum_{j=1}^{n} g_{ij} a_j, \qquad \alpha_i, \phi > 0,$$

  where G is the adjacency matrix of graph (V, E).

$\phi > 0 \Longrightarrow$ strategic complements.

$a^* \in A$ is a Nash equilibrium if

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall \; a_i \in A_i$$

holds for any $i = 1, \dots, n$.

[1] Ballester, Calvo-Armengol, Zenou, *Who's Who in Networks. Wanted: The Key Player*, Econometrica 74 (2006), 1403–1417.

The linear-quadratic model: existence of Nash equilibria

Nash equilibria are the solutions of the affine variational inequality $VI(F, A)$, where $F(a) = (I - \phi G)a - \alpha$ and $A = \mathbb{R}^n_+$.

Theorem [Ballester et al. 2006]

Let $\rho(G) = \lambda_{max}(G)$ be the spectral radius of G. If $\phi\rho(G) < 1$, then

- a unique Nash equilibrium a^* exists
- a^* is the solution of a linear system: $a^* = (I - \phi G)^{-1}\alpha$
- $a^* = \sum_{p=0}^{\infty} \phi^p G^p \alpha$ (Katz-Bonacich centrality measure)

The (i, j) entry of G^p gives the number of walks of length p between i and j.
If $\alpha_i = 1$, then Katz-Bonacich centrality measure[2] a_i^* counts the total number of walks which start at node i, exponentially damped by ϕ.

[2] Bonacich, *Power and centrality: a family of measures*, Am. J. Sociol. 92 (1987), 1170–1182.

The linear-quadratic model: key player

Given the Nash equilibrium a^*, consider the equilibrium aggregate $\sum_{i=1}^{n} a_i^*$.

Suppose to remove a player k from the network. The game will then have a new equilibrium $a' = (a'_1, \ldots, a'_{k-1}, a'_{k+1}, \ldots, a'_n)$ with corresponding aggregate

$$EA_k = \sum_{i \neq k} a'_i.$$

A key player is a player such that, after its removal, the new equilibrium aggregate is the minimum possible with respect to all possible removals of one player.

The linear-quadratic model with bounded strategies[3]

From now on we assume that the strategies have an upper bound: $A_i = [0, U_i]$.

Theorem

Assume that $\phi\rho(G) < 1$ and exactly k components of the Nash equilibrium a^* take on their maximum value: $a^*_{i_1} = U_{i_1}, \dots, a^*_{i_k} = U_{i_k}$.
Then the subvector $\tilde{a}^* = (\tilde{a}^*_{i_{k+1}}, \dots, \tilde{a}^*_{i_n})$ of the nonboundary components of a^* is

$$\tilde{a}^* = (I_{n-k} - \phi\, G_1)^{-1} w,$$

where G_1 is the submatrix obtained from G choosing the rows $i_{k+1}, \dots, i_n$ and the columns $i_{k+1}, \dots, i_n$, $w = \alpha \mathbf{1}_{n-k} + \phi G_2\, U$, G_2 is the submatrix obtained from G choosing the rows $i_{k+1}, \dots, i_n$ and the columns $i_1, \dots, i_k$, and $U = (U_{i_1}, \dots, U_{i_k})$.

[3] P., Raciti, *A note on network games with strategic complements and the Katz-Bonacich centrality measure*, in "Optimization and Decision Science", R. Cerulli, M. Dell'Amico, F. Guerriero, D. Pacciarelli, A. Sforza (eds.), AIRO Springer Series, vol. 7 (2021), 51–61.

The linear-quadratic model with bounded strategies[4]

Theorem

If $\phi\rho(G) < 1/2$, then

- a unique Nash equilibrium a^* exists
- the best-response dynamics converges to a^*
- $a_i^* \leq a_i^{so}$ for any $i = 1, \ldots, n$, where $a^{so} = \arg\max_a \sum_{i=1}^n u_i(a)$ is the unique social optimum of the game.

[4] P., Raciti, *A note on network games with strategic complements and the Katz-Bonacich centrality measure*, in "Optimization and Decision Science", R. Cerulli, M. Dell'Amico, F. Guerriero, D. Pacciarelli, A. Sforza (eds.), AIRO Springer Series, vol. 7 (2021), 51–61.

The linear-quadratic model with bounded strategies

The Nash equilibrium can be found by solving a finite number of linear systems.

Algorithm 1

1. Solve the linear system $(I - \phi G)\,\bar{a} = \alpha$
2. **If** $\bar{a}_i \leq U_i$ for any $i = 1, \dots, n$ **then** STOP: $\bar{a}$ is the Nash equilibrium
   else set $V_0 := \{i : \ \bar{a}_i > U_i\}$, $S_0 := \{i : \ \bar{a}_i \leq U_i\}$ and $k = 0$
3. Solve the linear system

$$(I_{S_k S_k} - \phi G_{S_k S_k})\, z^k = \alpha_{S_k} + \phi G_{S_k V_k} U_{V_k}$$

   and define the vector $a_i^k := \begin{cases} U_i & \text{if } i \in V_k \\ z_i^k & \text{if } i \in S_k \end{cases}$

4. Compute $\mu^k = \alpha_{V_k} - (I_{V_k V_k} - \phi G_{V_k V_k}) U_{V_k} + \phi G_{V_k S_k} z^k$
   If $\mu^k \geq 0$ **then** STOP: a^k is the Nash equilibrium
   else set $N_k := \{i \in V_k : \ \mu_i^k < 0\}$
   $V_{k+1} := V_k \setminus N_k, \qquad S_{k+1} := S_k \cup N_k$
   $k = k + 1$ and go to Step 3

The linear-quadratic model with bounded strategies[5]

Theorem

If $\phi\rho(G) < 1$, then Algorithm 1 finds the Nash equilibrium after at most n iterations.

Proof

- The linear systems at Steps 1 and 3 admit a unique solution.
- The sequence $\{a^k\}$ generated by the algorithm is feasible (by induction).
- The cardinality of the set V_k is decreasing at each iteration, thus $\mu^k \geq 0$ holds after at most n iterations.
- When $\mu^k \geq 0$ holds, the vector a^k is the Nash equilibrium since it solves the KKT system associated to the VI.

[5] P., Raciti, *A finite convergence algorithm for solving linear-quadratic network games with strategic complements and bounded strategies*, Optim. Methods Soft. 38 (2023), 1105–1128.

A network-game model of delinquency with random data

Given a network of n players, the action $a_i \in [0, U_i]$ of each player represents her/his effort in delinquent activities.
The utility function of each player is

$$u_i(a) = \left(\pi_i + \phi \sum_{j=1}^{n} g_{ij} a_j \right) a_i - \left(p a_i + \frac{1}{2} a_i^2 \right),$$

where π_i represents the specific ability of player i in criminal activities and can be partially estimated with the help of statistical analysis of data[6].

To take into account the contributions which are not observable to the econometrician, we model π_i as the sum of a deterministic term β_i and a random perturbation $\gamma_i r$, where γ_i is a fixed number and r is a random variable following a given distribution:

$$\pi_i = \beta_i + \gamma_i r.$$

[6]Lee, Liu, Patacchini, Zenou, *Who is the Key Player? A Network Analysis of Juvenile Delinquency*, J. Bus. Econ. Stat. 39 (2021), 849–857.

A network-game model of delinquency with random data

The Nash equilibrium is the solution of the following stochastic variational inequality:
for each $r \in \mathbb{R}$, find $a^*(r) \in A$ such that for each $a \in A$ we have:

$$\sum_{i=1}^{n} [a_i^*(r) - \phi \sum_{j=1}^{n} g_{ij} a_j^*(r)] \, [a_i - a_i^*(r)] \geq \sum_{i=1}^{n} [\beta_i + \gamma_i r - p] \, [a_i - a_i^*(r)]. \qquad (1)$$

To compute the expected value of the Nash equilibrium $a^*(r)$ with respect to the probability measure P, we follow the so-called L^2 approach which consists of considering an integral version of (1):
Find $a^* \in L^2(\mathbb{R}, P, \mathbb{R}^n)$ such that for all the functions $a \in L^2(\mathbb{R}, P, \mathbb{R}^n)$, with $0 \leq a_i(r) \leq U_i$ P-almost surely, we have:

$$\begin{aligned} &\int_{-\infty}^{+\infty} \sum_{i=1}^{n} [a_i^*(r) - \phi \sum_{j=1}^{n} g_{ij} a_j^*(r)] \, [a_i(r) - a_i^*(r)] dP \\ &\geq \int_{-\infty}^{+\infty} \sum_{i=1}^{n} [\beta_i + \gamma_i r - p] \, [a_i(r) - a_i^*(r)] dP. \end{aligned} \qquad (2)$$

A network-game model of delinquency with random data

The expectation of the Nash equilibrium

$$E_P[a^*(r)] = \int_{-\infty}^{+\infty} a^*(r) dP$$

can be found by the following approximation procedure[7]:

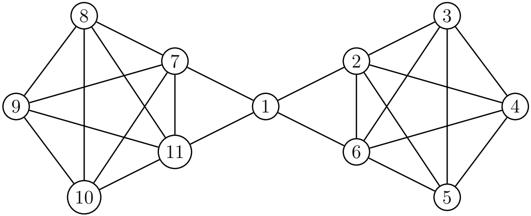
- Discretize the (compact) support of the probability measure P in N subintervals and denote L^2_N the space of step functions on the partition of the support.
- Solve (2) in L^2_N to get the step function $a^*_N(r)$, i.e., solve N finite-dimensional variational inequalities on $\mathbb{R}^n$.

If $N \to \infty$, then the sequence of step functions $\{a^*_N(r)\}$ is norm-convergent to $a^*(r)$ and the approximated mean values $E_P[a^*_N(r)]$ converge to the exact mean value $E_P[a^*(r)]$.

[7] J. Gwinner, B. Jadamba, A.A. Khan, and F. Raciti, *Uncertainty Quantification in Variational Inequalities: Theory, Numerics, and Applications*, Chapman and Hall/CRC, 2021.

Numerical experiments

We consider the following network with 11 nodes (players):



The spectral radius of G is $\rho(G) \simeq 4.4040$.
We set $\phi = 0.2$ so that $I - \phi G$ is positive definite.
We set $\beta = (10, \dots, 10)$, $\gamma = (1, \dots, 1)$, $p = 1$ and $U = (100, \dots, 100)$.

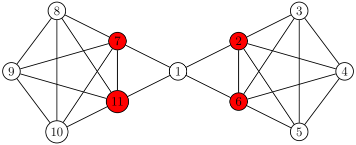
We assume the random variable r varies in the interval $[-5, 5]$ with uniform distribution. The approximation procedure considers a uniform partition of the interval $[-5, 5]$ into N subintervals and solves a deterministic network game for any subinterval by exploiting Algorithm 1.

Convergence of the approximate mean values of the equilibrium aggregate

Convergence of the approximate mean values of the equilibrium aggregate for different values of N, when r varies in the interval $[-5, 5]$ with uniform distribution and $\phi = 0.1$.

Equilibrium aggregate	N = 100	1,000	10,000	100,000
EA_1	149.167	149.917	149.992	149.999
EA_2	145.929	146.663	146.736	146.744
EA_3	150.360	151.116	151.192	151.199
EA_4	150.360	151.116	151.192	151.199
EA_5	150.360	151.116	151.192	151.199
EA_6	145.929	146.663	146.736	146.744
EA_7	145.929	146.663	146.736	146.744
EA_8	150.360	151.116	151.192	151.199
EA_9	150.360	151.116	151.192	151.199
EA_{10}	150.360	151.116	151.192	151.199
EA_{11}	145.929	146.663	146.736	146.744

The key players are the nodes most connected to the others.

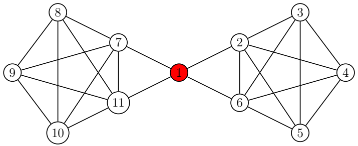


Convergence of the approximate mean values of the equilibrium aggregate

Convergence of the approximate mean values of the equilibrium aggregate for different values of N, when r varies in the interval $[-5, 5]$ with uniform distribution and $\phi = 0.2$.

Equilibrium aggregate	N = 100	1,000	10,000	100,000
EA_1	447.500	449.750	449.975	449.998
EA_2	459.267	461.525	461.750	461.773
EA_3	523.915	526.304	526.543	526.567
EA_4	523.915	526.304	526.543	526.567
EA_5	523.915	526.304	526.543	526.567
EA_6	459.267	461.525	461.750	461.773
EA_7	459.267	461.525	461.750	461.773
EA_8	523.915	526.304	526.543	526.567
EA_9	523.915	526.304	526.543	526.567
EA_{10}	523.915	526.304	526.543	526.567
EA_{11}	459.267	461.525	461.750	461.773

The unique key player is the bridge connecting the two complete subgraphs.



Scalability of Algorithm 1

We consider a set of random instances, where # players varies from 10 to 10,000.

The adjacency matrix of any random network is generated according to the following code:

```
G = rand(n);
G = floor((G+G')/2 + δ);
G = G - diag(diag(G));
```

so that G is an $n \times n$ zero-diagonal binary symmetric matrix and the parameter $\delta \in (0, 1)$ represents the density of the network ($\delta = 0$ corresponds to an empty network, while $\delta = 1$ to a complete network).

We set $\beta = (4, \ldots, 4)$, $\gamma = (1, \ldots, 1)$ and $p = 1$.
We assume r varies in $[-1, 1]$ with uniform distribution and the approximation procedure considers a uniform partition of $[-1, 1]$ into $N = 100$ subintervals.

Scalability of Algorithm 1 - number of linear systems solved

Average number of linear systems solved by Algorithm 1.
[The figures are the average values obtained on a set of five random instances.]

n	$\delta = 0.2$			$\delta = 0.5$		
	$\phi = \frac{0.1}{\rho(G)}$	$\phi = \frac{0.5}{\rho(G)}$	$\phi = \frac{0.9}{\rho(G)}$	$\phi = \frac{0.1}{\rho(G)}$	$\phi = \frac{0.5}{\rho(G)}$	$\phi = \frac{0.9}{\rho(G)}$
10	1.94	2.18	2.38	2.05	2.26	2.54
20	2.05	2.20	2.94	2.12	2.42	2.97
50	2.16	2.73	3.15	2.30	2.78	3.43
100	2.37	2.90	3.77	2.49	2.99	3.83
200	2.59	3.07	4.05	2.64	3.09	4.04
500	2.85	3.21	4.21	2.83	3.30	4.31
1,000	2.94	3.45	4.46	2.94	3.42	4.54
2,000	2.97	3.59	4.64	2.99	3.63	4.68
5,000	3.02	3.77	4.82	3.02	3.78	4.82
10,000	3.03	3.86	4.89	3.03	3.88	4.89

The average number of linear systems solved by Algorithm 1 is very low and quite stable (between 2 to 5).

Scalability of Algorithm 1 - Comparison with other solution approaches

For any subinterval of the discretization, the deterministic network game can also be solved by the following well-known methods:

- exploit the potential function: the game is first reformulated as a convex quadratic optimization problem and then solved by an optimization solver
- exploit the classic best-response method[8].

We compare the performance of Algorithm 1 with the performances of

- the potential-based approach exploiting three different solvers:
  - `Gurobi` (with default options)
  - the MATLAB `quadprog` function with the 'interior-point-convex' algorithm
  - the MATLAB `quadprog` function with the 'trust-region-reflective' algorithm.
- the best-response method with starting point $(U_1, \ldots, U_n)$:
  - Jacobi variant
  - Gauss-Seidel variant, where the order of play is $1, 2, \ldots, n$.

[8] Sagratella, *Computing equilibria of Cournot oligopoly models with mixed-integer quantities*, Math. Meth. Oper. Res. 86 (2017), 549–565.

Scalability of Algorithm 1 - Comparison with other solution approaches

Find the approximated stochastic Nash equilibrium, with $\delta = 0.5$ (CPU times in seconds).

n	Algorithm 1	Potential-based approach			Best-response method	
		`Gurobi`	`quadprog` interior-point	`quadprog` trust-region	Jacobi	Gauss-Seidel
100	0.031	1.018	0.100	0.574	0.045	0.027
200	0.162	3.168	0.899	0.879	0.140	0.082
300	0.316	5.609	2.189	1.485	0.290	0.170
400	0.472	11.615	4.235	2.228	0.474	0.281
500	0.561	14.638	9.232	3.102	0.706	0.421
600	0.682	18.530	15.711	4.389	1.552	0.900
700	0.933	23.384	22.033	5.864	2.065	1.195
800	1.102	28.769	30.683	7.767	3.105	1.814
900	1.139	35.450	40.387	9.538	3.398	2.006
1,000	1.319	42.587	47.510	11.332	4.786	2.816
2,000	3.052			39.689		12.917
3,000	6.163			79.893		46.715
4,000	10.069			135.675		68.777
5,000	14.298			218.514		453.096
6,000	19.006			311.270		575.458
7,000	24.688			405.576		758.007
8,000	31.279			534.597		710.733
9,000	38.238			933.030		941.853
10,000	56.980			1,163.213		1,119.356

Scalability of Algorithm 1 - Comparison with other solution approaches

Find the approximated stochastic key player, with $\delta = 0.5$ (CPU times in seconds).

		Potential-based approach			Best-response method	
n	Algorithm 1	Gurobi	quadprog interior-point	quadprog trust-region	Jacobi	Gauss-Seidel
10	0.021	5.882	0.307	3.484	0.025	0.013
20	0.062	11.924	0.654	7.611	0.118	0.059
30	0.138	18.634	1.088	12.481	0.256	0.134
40	0.251	25.993	1.580	17.471	0.445	0.241
50	0.399	34.005	2.143	21.991	0.857	0.472
60	0.624	43.515	2.951	27.570	1.233	0.674
70	0.952	54.180	4.121	33.312	1.773	0.956
80	1.432	66.265	5.376	39.578	2.643	1.456
90	2.021	80.751	6.987	46.149	3.459	1.945
100	2.883	97.537	10.277	54.596	4.372	2.426
200	29.644	630.424	169.477	173.197	27.461	15.983
300	91.677	1,637.397	621.744	445.772	91.666	51.738
400	192.747			827.213		111.295
500	257.223			1,478.674		209.505
600	381.021			2,718.983		478.374
700	629.128			3,989.585		888.171
800	736.379			6,032.577		1,397.354
900	960.393			8,565.410		1,796.690
1,000	1,243.579			11,498.981		2,064.835

GNEP on networks

Assume the players have a global shared constraint:

$$K = \left\{ a \in \mathbb{R}^n_+ : \sum_{j=1}^{n} a_j \leq C \right\}.$$

a^* is a generalized Nash equilibrium if for any $i = 1, \dots, n$ we have

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall\ a_i \in K_i(a_{-i}^*),$$

where $K_i(a_{-i}^*) = \left\{ a_i \in \mathbb{R}_+ : \quad a_i + \sum_{j \neq i} a_j^* \leq C \right\}.$

GNE are solutions of a quasi-variational inequality.
Variational equilibria are solutions of $VI(F, K)$, where $F(a) = (I - \phi G)a - \alpha$.

GNEP on networks[9]

Theorem

If $\phi\rho(G) < 1$, then the unique variational equilibrium is

$$\bar{a} = \begin{cases} a^* = \sum_{p=0}^{\infty} \phi^p G^p \alpha & \text{if } \sum_{i=1}^{n} a_i^* \leq C, \\ \dfrac{C a^*}{\sum_{i=1}^{n} a_i^*} = \dfrac{C \sum_{p=0}^{\infty} \phi^p G^p \alpha}{\sum_{p=0}^{\infty} \phi^p \alpha^\top G^p \alpha} & \text{if } \sum_{i=1}^{n} a_i^* > C, \end{cases}$$

where $a^* = (I - \phi G)^{-1}\alpha$ is the (non-generalized) Nash equilibrium.

[9] P., Raciti, *A note on generalized Nash games played on networks*, in "Nonlinear Analysis, Differential Equations, and Applications", T.M. Rassias (ed.), Springer Optimization and Its Applications, vol. 173 (2021), 365–380.

Parametric network games[10]

Suppose the strategy set $A_i(t) = [0, U_i(t)]$ and the payoff function

$$u_i(t,a) = -\frac{1}{2}a_i^2 + \alpha_i(t)a_i + \phi \sum_{j=1}^{n} g_{ij}a_ia_j,$$

where U_i and α_i are positive Lipschitz continuous functions of a parameter $t \in [0, T]$.

Nash equilibria $a^*(t)$ are solutions of $VI(F(t,\cdot), A(t))$, where

$$F(t,a) = (I - \phi G)a - \alpha(t)$$

and $A(t) = \prod_{i=1}^{n}[0, U_i(t)]$.

[10] P., Raciti, *Lipschitz continuity results for a class of parametric variational inequalities and applications to network games*, Algorithms 16 (2023), Article 458.

Parametric network games - Lipschitz equilibrium

Theorem

Assume F is uniformly τ-strongly monotone on $\mathbb{R}^n$, F is Lipschitz continuous with constant L, i.e.,

$$\|F(t_1, a_1) - F(t_2, a_2)\| \leq L(|t_1 - t_2| + \|a_1 - a_2\|), \qquad \forall a_1, a_2 \in \mathbb{R}^n, \ \forall\ t_1, t_2 \in [0, T],$$

$A(t)$ is a closed and convex set for any $t \in [0, T]$ and there exists $M \geq 0$ such that

$$\|p_{A(t_1)}(a) - p_{A(t_2)}(a)\| \leq M|t_1 - t_2|, \qquad \forall\ a \in \mathbb{R}^n, \ \forall\ t_1, t_2 \in [0, T],$$

where $p_{A(t)}(a)$ denotes the projection of a on the closed convex set $K(t)$.
Then, for any $t \in [0, T]$, $VI(F, A(t))$ has a unique solution $a^*(t)$ which is Lipschitz continuous on $[0, T]$ with estimated constant equal to

$$\Lambda_1 = \begin{cases} \inf\limits_{z \in (0, 2\widetilde{\tau})} \left[\dfrac{M}{1-s} + \dfrac{z(1+z)}{s(1-s)} \right], & \text{if } \widetilde{\tau} < 1, \\[2ex] M + 2\sqrt{2M} + 1, & \text{if } \widetilde{\tau} = 1, \end{cases}$$

where $\widetilde{\tau} = \tau / L$ and $s = \sqrt{z^2 - 2\widetilde{\tau} z + 1}$.

Parametric network games - Lipschitz equilibrium

Theorem

Assume $F(t,a) = G(a) + H(t)$ holds for any $t \in [0,T]$, $a \in \mathbb{R}^n$, where $G : \mathbb{R}^n \to \mathbb{R}^n$ is τ-strongly monotone on $\mathbb{R}^n$ and Lipschitz continuous on $\mathbb{R}^n$ with constant L_a, and H is Lipschitz continuous on $\mathbb{R}^n$ with constant L_t. Moreover, we assume that $A(t)$ is a closed and convex set for any $t \in [0,T]$ and there exists $M \geq 0$ such that

$$\|p_{A(t_1)}(a) - p_{A(t_2)}(a)\| \leq M|t_1 - t_2|, \qquad \forall\, a \in \mathbb{R}^n,\ \forall\, t_1, t_2 \in [0,T].$$

Then, for any $t \in [0,T]$, $VI(F, A(t))$ has a unique solution $a^*(t)$ which is Lipschitz continuous on $[0,T]$ with estimated constant equal to

$$\Lambda_2 = \begin{cases} \inf\limits_{z \in (0, 2\hat{\tau})} \left[\dfrac{M}{1-\hat{s}} + \dfrac{\hat{L}z(1+z)}{\hat{s}(1-\hat{s})} \right], & \text{if } \hat{\tau} < 1, \\[2ex] M + 2\sqrt{2M\hat{L}} + \hat{L}, & \text{if } \hat{\tau} = 1, \end{cases}$$

where $\hat{\tau} = \tau / L_a$, $\hat{L} = L_t / L_a$ and $\hat{s} = \sqrt{z^2 - 2\hat{\tau}z + 1}$.

Parametric network games - Lipschitz equilibrium

Remark

When $\phi\rho(G) < 1$, the map $F(t,a) = (I - \phi G)a - \alpha(t)$, satisfies the assumptions of above Theorems with constants $\tau = 1 - \phi\rho(G)$, $L_a = \|I - \phi G\|_2$, $L_t = L_\alpha \sqrt{n}$, where L_α is the Lipschitz constant of $\alpha(t)$, and $L = \max\{L_a, L_t\}$.
Moreover, the feasible region $A(t)$ satisfies the assumption

$$\|p_{A(t_1)}(a) - p_{A(t_2)}(a)\| \leq M|t_1 - t_2|, \qquad \forall\ a \in \mathbb{R}^n,\ \forall\ t_1, t_2 \in [0, T]$$

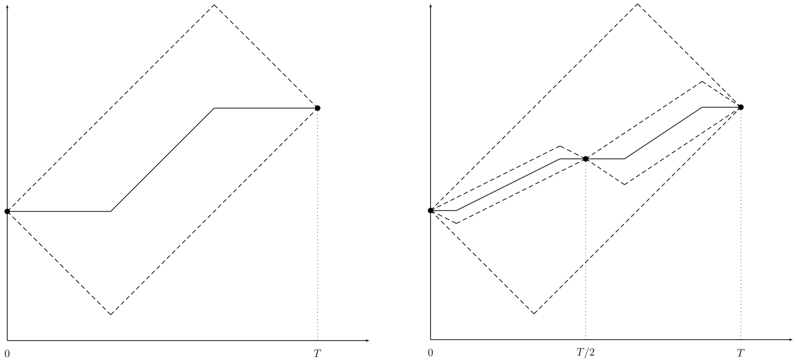
with $M = \|(L_1, \ldots, L_n)\|_2$, where L_i is the Lipschitz constant of $U_i(t)$ for any $i = 1, \ldots, n$.

Parametric network games - Approximation algorithm

Estimate of the Lipschitz constant of Nash equilibrium $a^*(t)$

$\downarrow$

approximate $a^*(t)$ and its mean value on $[0, T]$



Parametric network games

Approximation algorithm

0. Set $\varepsilon > 0$, evaluate $a_j^*(t)$ at 0 and T, compute the area of parallelogram $P_{0,T}$.
1. Find the parallelogram P_{t_1,t_2} with the largest area,
   add the new evaluation point $p = (t_1 + t_2)/2$,
   compute the Lipschitz constants of $a_j^*(t)$ in $[t_1, p]$ and $[p, t_2]$,
   update the parallelograms' areas list by removing the area of P_{t_1,t_2} and inserting the areas of $P_{t_1,p}$ and P_{p,t_2}.
2. Compute the worst case error $E_{tot} =$ (sum of areas of all parallelograms)/2.
   If $E_{tot} \leq \varepsilon$ then stop; otherwise go to Step 1.

Theorem

The algorithm stops after at most $\lceil 2E_0/\varepsilon \rceil$ iterations, where

$$E_0 = \frac{\Lambda^2 T^2 - [a_j^*(0) - a_j^*(T)]^2}{4\Lambda}$$

is the worst case error before the algorithm starts.

Parametric network GNEPs

The previous results can be applied to approximate variational equilibria of network GNEPs where players share a global constraint:

$$A(t) = \left\{ a \in \mathbb{R}^n_+ : \sum_{i=1}^{n} a_i \leq C(t) \right\}.$$

The assumption

$$\|p_{A(t_1)}(a) - p_{A(t_2)}(a)\| \leq M|t_1 - t_2|, \qquad \forall\ a \in \mathbb{R}^n,\ \forall\ t_1, t_2 \in [0, T]$$

is satisfied with M equal to the Lipschitz constant of the function $C(t)$.

The parametric quadratic model[11]

Suppose the strategy set $A_i = [0, U_i]$ and the payoff of player i is

$$u_i(a) = -\frac{\beta}{2} a_i^2 + \alpha_i a_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij}(\alpha) a_i a_j, \qquad \alpha, \beta > 0.$$

If $f_{ij}(\alpha) \geq 0$, then the game falls in the class of games with strategic complements; if $f_{ij}(\alpha) \leq 0$, then it falls in the class of games with strategic substitutes.

The pseudo-gradient of this game is

$$T(a) = [\beta I - \mathcal{F}(\alpha)]a - \alpha,$$

where $\mathcal{F}(\alpha)$ is a zero-diagonal matrix whose off-diagonal entries are equal to $f_{ij}(\alpha)$. We assume $\mathcal{F}(\alpha)$ is symmetric for any α.

[11] P., Raciti, *Some properties of a class of Network Games with strategic complements or substitutes*, in "Mathematical Analysis, Differential Equations and Applications", T.M. Rassias and P.M. Pardalos (eds.), in press, doi: 10.1142/9789811267048_0023.

The parametric quadratic model

Theorem

- Let $f_{ij}(\alpha) \geq 0$ for any i, j. The matrix $\beta I - \mathcal{F}(\alpha)$ is positive definite iff

$$\beta > \lambda_{max}(\mathcal{F}(\alpha)) = \rho(\mathcal{F}(\alpha))$$

  where $\lambda_{max}(\mathcal{F}(\alpha))$ is the maximum eigenvalue of $\mathcal{F}(\alpha)$ and $\rho(\mathcal{F}(\alpha))$ its spectral radius.

- Let $f_{ij}(\alpha) \leq 0$ for any i, j. The matrix $\beta I - \mathcal{F}(\alpha)$ is positive definite iff

$$\beta > \lambda_{max}(\mathcal{F}(\alpha))$$

  Moreover, the condition $\beta > \rho(\mathcal{F}(\alpha))$ is, in general, stronger than the latter condition.

The parametric quadratic model

Theorem

Let $\beta > \rho(\mathcal{F}(\alpha))$ and a^* be the unique Nash equilibrium.

- Assume that $f_{ij}(\alpha) \geq 0$ for any $i, j \in \{1, \dots, n\}$. Then $a_i^* > 0$ for any $i \in \{1, \dots, n\}$. Moreover, if exactly k components of a^* take on their maximum value: $a_{i_1}^* = L_{i_1}, \dots, x_{i_k}^* = L_{i_k}$, then the subvector $\tilde{a}^* = (\tilde{a}_{i_{k+1}}^*, \dots, \tilde{a}_{i_n}^*)$ of the non-boundary components is

$$\tilde{a}^* = [\beta I_{n-k} - \mathcal{F}_1(\alpha)]^{-1} w \tag{3}$$

  where $\mathcal{F}_1(\alpha)$ is the submatrix obtained from $\mathcal{F}(\alpha)$ choosing the rows $i_{k+1}, \dots, i_n$ and the columns $i_{k+1}, \dots, i_n$; $w = \alpha_{n-k} + \mathcal{F}_2(\alpha)\, U$; $\mathcal{F}_2(\alpha)$ is the submatrix obtained from $\mathcal{F}(\alpha)$ choosing the rows $i_{k+1}, \dots, i_n$ and the columns $i_1, \dots, i_k$; $U = (U_{i_1}, \dots, U_{i_k})$, $\alpha_{n-k} = (\alpha_{i_{k+1}}, \dots, \alpha_{i_n})$.
- Assume now that $f_{ij}(\alpha) \leq 0$ for any $i, j \in \{1, \dots, n\}$, and there are no zero components of a^*. If exactly k components of a^* take on their maximum value, then formula (3) also applies to this case.

The parametric quadratic model

Theorem

Assume that $f_{ij}(\alpha) \geq 0$ for any i, j, and $\beta > 2\rho(\mathcal{F}(\alpha))$.
Then,

$$a_i^* \leq a_i^{so} \qquad \forall\ i = 1, \ldots, n, \tag{4}$$

where a^{so} is the social optimum of the game.

Inequality (4) does not hold in general in the case of strategic substitutes.
Consider a game with $n = 5$ players, $U = (1, 1, 1, 1, 1)$, $\alpha = (1, 2, 1, 2, 1)$, $\beta = 2.5$ and the interaction matrix given by

$$f_{ij}(\alpha) = -\frac{1}{2}\,|\alpha_i - \alpha_j| \qquad \forall\ i, j = 1, \ldots, 5.$$

Player	Constrained NE	Social Optimum
1	0.1053	0.0000
2	0.7368	0.8000
3	0.1053	0.0000
4	0.7368	0.8000
5	0.1053	0.0000

References

- Belhaj M., Bramoullé Y., Deroïan F., *Network games under strategic complementarities*, Games Econ. Behav. 88 (2014), pp. 310–319.
- Ballester C., Calvo-Armengol A., Zenou Y., *Who's Who in Networks. Wanted: The Key Player*, Econometrica 74 (2006), 1403–1417.
- Bonacich P., *Power and centrality: a family of measures*, Am. J. Sociol. 92 (1987), pp. 1170–1182.
- Jackson M.O., Zenou Y., *Games on Networks*, in: Handbook of Game Theory with Economic Applications, H.P. Young and S. Zamir (eds.), 2015, 95–163.
- Parise F., Ozdaglar A., *A variational inequality framework for network games: Existence, uniqueness, convergence and sensitivity analysis*, Games Econ. Behav. 114 (2019), 47–82.