Optimal portfolio choice with path dependent labor income: the infinite horizon case

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Outline

Asset allocation

- Merton (1971): wealth of the investor is given by risky and non risky assets. Optimal for agents to put a constant fraction of their wealth in the risky asset throughout all their life
- Starting from the '90: models which add labor income to the wealth, see e.g. Bodie et al. ('92), Campbell-Viceira ('02), Fahri-Panageas ('07) Dybvig-Liu ('10). The total wealth of an agent is given by her financial wealth and her *human capital*, i.e. the present value of future labor income. Investors put optimally a constant fraction of their financial wealth in the risky asset.
- Empirical evidence of sticky wages: they respond to shocks in the market, but with a delay (see Meghir-Pistaferri 2004)

Our Goal.

- Study a model of asset allocation where the wages are path-dependent to describe more precisely their real behavior. First step of a project in progress.

Outline

The model of Dybvig, P.H. and Liu, H. (2010)

The market is Black & Scholes type:

$$
\begin{aligned}
dS_0(t) =& rS_0(t)dt \\
dS_1(t) =& S_1(t)\mu dt + S_1(t)\sigma dZ(t),
\end{aligned}
$$

$$
\begin{aligned}
dW(t) = & \left[W(t)r + \theta(t)(\mu - r) - c(t) + \delta\big(W(t) - B(t)\big)\right] dt \\
& + (1 - R(t))y(t)dt + \theta(t)\sigma dZ(t), \quad W(0) = W_0 \\
dy(t) = & y(t)\big(\mu_y dt + \sigma_y dZ(t)\big), \quad y(0) = y_0
\end{aligned}
$$

- $W(t)$ the wealth process, state
- $y(t)$ the labor income, state
- $\theta(t)$ the dollar investment in the risky asset, control
- $c(t)$ the consumption, control
- $B(t)$ the bequest, control
- $R(t) := \mathbb{I}_{\{T \leq t\}}$ and T is the retirement time, control
- δ constant rate of mortality

The death time τ_δ is modeled as a Poisson arrival time, with hazard rate δ.
τ_δ is independent of the Wiener process $(Z(t))$.

We should consider as basic filtration the one generated by $\mathcal{F}^{\tau_\delta} \vee \mathcal{F}^Z$, but we will actually work on $\{\tau_\delta > t\}$.

$B(t)$ is the bequest-target we want to leave to the recipient:

- for $W(t) - B(t) < 0$ we have to finance it buying continuously a life insurance with premium $\delta(B(t) - W(t))$
- for $W(t) - B(t) > 0$ then the term $\delta(B(t) - W(t))$ can be seen as a life annuity since it trades wealth in the event of death for a cash inflow while living.

Goal: maximize over $\big(c(\cdot), B(\cdot), \theta(\cdot), T\big)$

$$\mathbb{E}\left\{\int_0^{\tau_\delta} e^{-\rho t}\left((1-R(t))\frac{c(t)^{1-\gamma}}{1-\gamma}+R(t)\frac{(Kc(t))^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. +e^{-\rho\tau_\delta}\frac{\big(kB(\tau_\delta)\big)^{1-\gamma}}{1-\gamma}dt\right\},$$

$K>1$, different marginal utility for unit of c before and after T.
The constant $k>0$ measures the intensity of preference for leaving a large bequest.
The expectation above can be written as $J(W_0, y_0; c, B, \theta, T) :=$

$$\mathbb{E}\left\{\int_0^{+\infty} e^{-(\rho+\delta)t}\left(\frac{(K^{R(t)}c(t))^{1-\gamma}}{1-\gamma}+\delta\frac{\big(kB(t)\big)^{1-\gamma}}{1-\gamma}\right)dt\right\}$$

The constraints

Problem 1 of Dybvig-Liu 2010

Given retirement time T, no borrowing-without-repayment constraint

$$W(t) \geq -g(t)y(t),$$

where

$$g(t) := \begin{cases} \left(\frac{1-e^{-\beta_1(T-t)}}{\beta_1}\right)^+ & \text{if } \beta_1 \neq 0 \\ (T-t)^+ & \text{if } \beta_1 = 0 \end{cases}$$

with

$$\beta_1 := r + \delta - \mu_y + \frac{(\mu - r)}{\sigma}\sigma_y.$$

Meaning of the constraints

Let $\xi(t)$ be the state price density adjusted to condition on living;

$$\xi(t) := e^{-\left(r+\delta+\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\right)t-\frac{(\mu-r)}{\sigma}Z(t)}.$$

i.e. the solution of

$$\begin{cases} d\xi(t) & = -\xi(t)(r+\delta)dt - \xi(t)\kappa^\top dZ(t), \\ \xi(0) & = 1. \end{cases} \tag{1}$$

Then

$$g(t)y(t) = \mathbb{E}\Big(\int_t^T y(s)\xi(s)ds \mid \mathcal{F}_t \Big).$$

$\xi(t)^{-1}E\Big(\int_t^T y(s)\xi(s)ds \mid \mathcal{F}_t \Big)$ is the *human capital* at time t i.e. the value at t of subsequent labor income.

Results of Dybvig-Liu 2010

Find an explicit expression for the value function

$$V(W_0, y_0) := \sup_{\textit{admissible strategies}} J(W_0, y_0; c, B, \theta, T)$$

and for the optimal strategies (consumption, bequest, portfolio).

Also other problems with different constraints are studied pointing out the financial meaning of the results.

Outline

Empirical evidence of sticky wages: they respond to shocks in the market, but with a delay (see Meghir-Pistaferri 2004)

We expect that considering sticky wages will change optimal asset allocation and optimal retirement.
We start with the model with no retirement, for simplicity.

The model

$$dW(t) = \big[W(t)r + \theta(t)(\mu - r) - c(t) + \delta\big(W(t) - B(t)\big)\big]\, dt$$
$$+ y(t)dt + \theta(t)\sigma dZ(t), \quad W(0) = W_0$$
$$dy(t) = \left(y(t)\mu_y + \int_{-d}^{0} \alpha(\xi)y(t+\xi)d\xi\right) dt + y(t)\sigma_y dZ(t),$$
$$y(0) = y_0, \quad y(\xi) = y_1(\xi)\ \forall \xi \in [-d, 0).$$

- $W(t)$, $y(t)$, $\theta(t)$, $c(t)$, $B(t)$, as before;
- $\alpha(\cdot)$ a square integrable (L^2) function.

$$J_1(W_0, y_0, y_1; c, B, \theta) :=$$

$$\mathbb{E}\left\{ \int_0^{+\infty} e^{-(\rho+\delta)t} \left(\frac{c(t)^{1-\gamma}}{1-\gamma} + \delta \frac{\left(kB(t)\right)^{1-\gamma}}{1-\gamma} \right) dt \right\}. \tag{2}$$

Problem

Given T, choose $c(\cdot)$, $\theta(\cdot)$, $B(\cdot)$ to maximize (2), with the following no-borrowing-without-repayment constraint

$$W(t) \geq -Gy(t) - \int_{-d}^{0} H(\xi) y(t+\xi) d\xi,$$

The constant G and the function H are

$$\begin{aligned} G &:= (\beta_1 - \beta_\infty)^{-1} \\ H(\xi) &:= \int_{-d}^{\xi} e^{-(r+\delta)(\xi - s)} \alpha(s) ds \end{aligned}$$

where

$$\beta_\infty := \int_{-d}^{0} e^{-(r+\delta)s} \alpha(s) ds$$

As before

$$Gy(t) + \int_{-d}^{0} H(\xi) y(t+\xi) d\xi = \xi(t)^{-1} E\Big(\int_t^T y(s)\xi(s) ds \mid \mathcal{F}_t \Big)$$

is the *human capital* at time t i.e. the value at t of subsequent labor income. This is a nontrivial result, see Biffis et al '15.
Note. Human capital must take into account the past of y. For $\alpha = 0$, $H = 0$ and G coincides with g.

Stochastic control problem, finite horizon

- state equation

$$\begin{cases} dx(t) & = b\big(x(t), c(t)\big)\,dt + \sigma\big(x(t), c(t)\big)\,dZ(t) \\ x(0) & = x_0 \end{cases}$$

- set of admissible controls (here C is bounded)

$$\mathcal{U} := \{c : [0,T] \times \Omega \longrightarrow C \mid c \text{ is } \mathcal{F}_t\text{-adapted}\}.$$

- objective functional

$$J(c(\cdot)) := \mathbb{E}\Big\{ \int_0^T f\big(t, x(t), c(t)\big)\,dt + h\big(x(T)\big)\Big\}$$

Goal : maximize J subject to the state equation over $\mathcal{U}$.

Dynamic programming

Consider a family of problems, $s \in [0, T]$

- state equation

$$(*) \begin{cases} dx(t) & = b\big(x(t), c(t)\big)dt + \sigma\big(x(t), c(t)\big)dZ(t) \quad t \in [s, T \\ x(s) & = y \end{cases}$$

- set of admissible controls

$$\mathcal{U}^s := \{c : [s, T] \times \Omega \longrightarrow C \mid c \text{ is } (\mathcal{F}_t^s)_{t \geq s}\text{-adapted}\}.$$

- objective functional

$$J\big(s, y; c()\big) := \mathbb{E}\Big\{ \int_s^T f\big(t, x^{(s,y)}(t), c(t)\big)dt + h\big(x^{(s,y)}(T)\big)\Big\},$$

where $x^{(s,y)}(t)$ is the solution at time t of $(*)$.

Define the value function

$$V(s,y) := \sup_{c(\cdot)\in\mathcal{U}^s} J\big(s,y;c(\cdot)\big), \text{ for any } (s,y)\in[0,T]\times\mathbb{R}$$

Dynamic Programming Principle, $\hat{s}\in[s,T]$

$$V(s,y) = \sup_{c(\cdot)\in\mathcal{U}^s} \mathbb{E}\Big\{ \int_s^{\hat{s}} f\big(t, x^{(s,y)}(t), c(t)\big)\,dt + V\big(\hat{s}, x^{(s,y)}(\hat{s})\big)\Big\}$$

If $\bar{c}(\cdot)\mid_{[s,T]}$ is optimal on $[s,T]$ with initial value (s,y), then $\bar{c}(\cdot)\mid_{[\hat{s},T]}$ is optimal on $[\hat{s},T]$ with initial value $(\hat{s}, x^{(s,y)}(\hat{s}))$.

Dynamic programming cont'd

Passing to the limit we get the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} -v_t & = \mathcal{H}\big(t, x, v_x, v_{xx}\big) \text{ for any } (s, y) \in [0, T] \times \mathbb{R} \\ v(T, y) & = h(y) \quad \text{for any } y \in \mathbb{R} \end{cases}$$

where

$$\mathcal{H}\big(t, x, p, P\big) = \sup_{c \in C}\{f(t, x, c) + b(x, c)p + \frac{1}{2}\sigma^2(x, c)P\}$$

If V is $\mathcal{C}^{1,2}\big([0, T] \times \mathbb{R}\big)$ + reasonable conditions, then V solves the HJB equation.

Dynamic programming cont'd

Verification Theorem

Let v a $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$-solution of the HJB and let exist a function $\bar{c}:[0,T]\times\mathbb{R}\longrightarrow C$ such that for any (t,x)

$$\bar{c}(t,x)=argmax\{f(t,x,\bar{c})+b(x,\bar{c})v_x+\frac{1}{2}\sigma^2(x,\bar{c})v_{xx}\}$$

and $\bar{c}(\cdot)$ is admissible, then v is the value function and $\bar{c}()$ is an optimal control.

Infinite horizon

- state equation and admissible controls as before
- objective functional

$$J(s,y;c(\cdot)) := \mathbb{E}\Big\{ \int_s^{+\infty} e^{-\rho t} f\big(x^{(s,y)}(t), c(t)\big)\,dt \Big\},$$

- value function

$$V(s,y) := \sup_{c(\cdot)\in\mathcal{U}^s} J(s,y;c(\cdot)), \text{ for any } (s,y)\in[0,+\infty)\times\mathbb{R}$$

  we have

$$V(s,y) = e^{-\rho s} V(0,y) = e^{-\rho s} V_0(y).$$

- Hamilton-Jacobi-Bellman equation for V_0

$$\rho v \quad = \mathcal{H}(x, v_x, v_{xx}) \text{ for any } y\in\mathbb{R}$$

  where

$$\mathcal{H}(x,p,P) = \sup_{c\in C}\Big\{ f(x,c) + b(x,c)p + \frac{1}{2}\sigma^2(x,c)P \Big\}$$

Delay equations as ODEs in infinite dimensional spaces

The state equation of $y(\cdot)$ is a stochastic delay differential equation. Classical theory works for Markovian state-equation. Consider the Hilbert space

$$\mathcal{H} := \mathbb{R} \times L^2\big([-d, 0]; \mathbb{R}\big),$$

with inner product for $x = (x_0, x_1), z = (z_0, z_1) \in \mathcal{H}$

$$\langle x, z \rangle_{\mathcal{H}} := x_0 z_0 + \int_{-d}^{0} x_1(\xi) z_1(\xi) d\xi$$

$$= x_0 z_0 + \langle x_1, z_1 \rangle_{L^2}$$

Set

$$X(t) = \big(X_0(t), X_1(t)\big) := \big(y(t), y(t+\xi)_{|\xi\in[-d,0]}\big),$$

$X(t)$ is an element of $\mathcal{H}$ for all $t \in [0, +\infty)$. Let X satisfy

$$dX(t) = AX(t)dt + CX(t)dZ(t), \quad X(0) = (y_0, y_1) \in \mathcal{H}$$

with

$$\begin{aligned} A(x_0, x_1) &:= \big(\mu_y x_0 + \langle \alpha(\cdot), x_1(\cdot)\rangle_{L^2}, x_1'(\cdot)\big), \\ C(x_0, x_1) &:= (x_0 \sigma_y, 0) \end{aligned}$$

Then the original problem is equivalent to the control problem with state X in the infinite dimensional space $\mathcal{H}$
(see Gozzi-Marinelli '04).

Our result

Theorem

The value function V_0 is

$$V_0(W, x_0, x_1) := f_\infty^\gamma \frac{\Gamma^{1-\gamma}}{1-\gamma},$$

where

$$f_\infty := (1 + \delta k^{\frac{1}{\gamma}-1})\nu,$$

$$\nu := \frac{\gamma}{\rho + \delta - (1-\gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})} > 0.$$

$$\Gamma := W_0 + Gx_0 + \langle H, x_1 \rangle_{L^2} \geq 0,$$

The optimal strategies are

$$c^*(t) := f_\infty^{-1}\Gamma^*(t)$$
$$B^*(t) := k^{-b} f_\infty^{-1}\Gamma^*(t)$$
$$\theta^*(t) := \frac{(\mu - r)\Gamma^*(t)}{\gamma\sigma^2} - \frac{G\sigma_y y(t)}{\sigma},$$

where $\Gamma^*(t) := W^*(t) + GX_0(t) + \langle H, X_1(t,\cdot)\rangle_{L^2}$.
We have

$$\frac{d\Gamma^*(t)}{\Gamma^*(t)} = \Big[r + \delta + \frac{1}{\gamma}(\frac{\mu - r}{\sigma})^2 - f_\infty^{-1}\big(1 + \delta k^{-b}\big)\Big]dt + \frac{\mu - r}{\gamma\sigma} dZ(t).$$

First findings

- with no-labor risk ($\sigma_y = 0$), the optimal ratio $\frac{\theta^*}{\Gamma^*}$ and $\frac{c^*}{\Gamma^*}$ are constant, as in the Merton model
- taking $\alpha = 0$, we recover Dybvig-Liu result
- taking $\alpha \neq 0$, we recover Dybvig-Liu result, but substituting to the optimal total wealth Λ^* (financial wealth + human capital) of Dybvig-Liu the quantity $\Gamma^*(t) = W^*(t) + G(t)X_0 + \langle H(t,\cdot), X_1(t,\cdot)\rangle$
- Λ^* and Γ^* evolve with the same dynamic

Sketch of the proof

- Guess the value function to be

$$V(W_0, x_0, x_1) := f_\infty^\gamma \frac{(W_0 + Gx_0 + \rangle H, x_1 \langle_{L^2})^{1-\gamma}}{1-\gamma},$$

- putting V in the HJB equation, gives equations for f, G, H
- solving these equations, we get that f, G, H are the constant as in the main Theorem
- V is $\mathcal{C}^{1,2}$
- Verification Theorem holds and the optimal feedback strategies are admissible

Thank you for your attention

References

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