R FUNCTIONS AND EXACT DISTRIBUTION OF THE LIKELIHOOD RATIO TEST STATISTICS FOR TESTING A NULL HYPOTHESIS

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Abstract

The aim of this article is that of obtaining an expression for the density for the likelihood ratio test which should be suitable for the computation of percentage points. By extending previous results obtained by Krishnaiah (1976), Rathie (1989) and Pederzoli (1995), the density function is expressed in terms of R-function and G-functions.

Keywords
Multivariate distribution
Null hypothesis
Likelihood ratio
Series expansion

1. Introduction

Complex multivariate distributions play an important rôle in various areas such as Nuclear Physics, see Carmeli (1974), Porter (1965) and Multiple Time Series, see Brillinger (1974), Hannon (1970) for instance.

Let $\underset{\sim}{z}' = (z_1, ..., z_p)$ be distributed as a complex multivariate normal with mean $\underset{\sim}{\mu}' = (\mu_1, ..., \mu_p)$ and covariance matrix Σ. Let H denote the hypotesis

$$H: \Sigma = \underset{\sim}{\Sigma}_0 \ , \ \underset{\sim}{\mu} = \underset{\sim}{\mu}_0 \tag{1.1}$$

where $\underset{\sim}{\mu}_0$ and $\underset{\sim}{\Sigma}_0$ are supposed to be known.

If we denote the likelihood ratio test statistics for H by λ, then it is known [see Krishnaiah (1976), page 17] that

$$\lambda = \frac{(e)^{pN}}{N} \left| A \, \Sigma_0^{-1} \right|^N e^{\operatorname{tr}\left\{-\Sigma_0^{-1}\left[A + N\left(\underset{\sim}{z}_0 - \underset{\sim}{\mu}_0\right)\left(\underset{\sim}{z}_0 - \underset{\sim}{\mu}_0\right)^*\right]\right\}} \tag{1.2}$$

where $\left(z_{1j}, ..., z_{pj}\right)$, $j=1, ..., N$ are N independent observations on z_j;

$Nz_{i.} = \sum_{j=1}^{N} z_{1j}$; $\underset{\sim}{z}.' = \left(z_{1.}, ... z_{p.}\right)$,

$A_{1m} = \sum_{j=1}^{N}\left(z_{1j} - z_{1.}\right)\left(z_{mj} - z_{m.}\right)^*$; $\left(\underset{\sim}{z} - \underset{\sim}{\theta}\right)^* = \left(\overline{\underset{\sim}{z} - \underset{\sim}{\theta}}\right)'$

being the conjugate transpose of the vector $\left(\underset{\sim}{z} - \theta\right)$ and

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ \dots & \dots & \dots & \dots \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{bmatrix}$$

The moments of λ are given by

$$E\left(\lambda^h\right) = \frac{\left(\frac{e}{N}\right)^{phN}}{(i+h)^{pN(i+h)}} \prod_{i=1}^{p} \frac{\Gamma(N-i+Nh)}{\Gamma(N-i)} \tag{1.3}$$

The distribution of certain powers of λ is approximated in Chang, Krishnaiah and Lee (1977) with Pearson's type I distribution by using the first four moments. Using this approximation they have also obtained percentage points for -2 Log λ. Rathie (1994) has expressed the density function of λ in terms of R-function as well as in series of chi-square distributions.

The purpose of this paper is to obtain an alternate series expression for the density function of λ involving Meijer's G-function. A series representation, suitable for the computation of percentage points, is also obtained by using residues theory.

2. Density in Terms of R-and G-Functions

In this section, we will obtain the density function in terms of R-function introduced earlier by Rathie (1989). Using the asymptotic expansion of gamma function, the density will be expressed as a series involving G-function.

On substituting $L=\frac{s}{N}$ in (1.3), denoting the density function of λ^{Y_N} by $f(x)$ and using inverse Mellin transform, we get

$$f(x) = \frac{\left(\frac{N}{e}\right)^{pN} x^{N-1}}{\prod_{i=1}^{p} \Gamma(N-i)} (2\pi\omega)^{-1} \int_L x^{-s} \left(\frac{s}{e}\right)^{-ps} \prod_{i=1}^{p} \Gamma(s-i)\, ds \tag{2.1}$$

where $\omega = \sqrt{-1}$ and L is a suitable contour with c > p.

Using the definition of R-function, the density (2.1) can be written as

$$f(x) = \frac{\left(\frac{N}{e}\right)^{pN} x^{N-1}}{\prod_{i=1}^{p} \Gamma(N-i)} \; {}^{0,-1}_{0,\,p} R^{\,p,\,0}_{\,0,\,p} \left[(-1)^p x e^{-p} \middle| \; {}_1\overline{(-i,\,1)}_p \right]$$

for $0 < x < 1$. (2.2)

Using the asymptotic expansion [7, p. 32 (5)]

$$\log \Gamma\left(z+\frac{1}{2}\right) = z \log z - z + \frac{1}{2} \log(2\pi)$$

$$+\sum_{k=1}^{m} \frac{(-1)^{k+1} B_{k+1}\left(\frac{1}{2}\right)}{k\,(k+1)\; z^k} + O\left(z^{-m-i}\right) \tag{2.3}$$

for $|\arg z| \leq 1 < \pi - \in, \; \in > 0$

Now

$$\left(\frac{s}{e}\right)^{-ps} \Gamma^p \left(s+\frac{1}{2}\right) = (2\pi)^{\frac{p}{2}} \; e^p \sum_{k=1}^{m} \frac{(-1)^{k+1} B_{k+1}\left(\frac{1}{2}\right)}{k\,(k+1)\, s^k} \tag{2.4}$$

$$= \left(2\pi\right)^{\frac{p}{2}} \sum_{n=0} \frac{1}{n!\, s^n} \; C_n \; \left(y_1, ..., y_n\right)$$

by using formula 21 on page 183 of Rathie (1994) with $x = \frac{1}{s}$ and

$y_i = \dfrac{\left(-1\right)^{i+1} \; p \, B_{i+1} \left(\frac{1}{2}\right)}{\left(i+1\right)}$.

Here C_n (y_1, ..., y_n) is the cyclic indicator of the symmetric group. Note that $y_i = 0$ for i = 2, 4, 6, ... because B_i (a) = 0 when i is an odd integer. Thus (2.1), with the help of (2.4), takes the following form on interchanging the order of summation and integration

$$f(x) = \frac{\left(\frac{N}{e}\right)^{pN} \; x^{N-1}}{\prod_{i=1}^{N} \Gamma\,(N-i)} \; (2\pi\omega)^{-1} \; (2\pi)^{\frac{p}{2}} \sum_{n=0} \frac{C_n \; (y_1, ..., y_n)}{n!}$$

$$x \int_L \frac{x^{-s}}{s^n} \prod_{i=1}^{p} \frac{\Gamma\,(s-i)}{\Gamma\left(s+\frac{1}{2}\right)} \, ds$$

$$= \frac{\left(\frac{N}{e}\right)^{pN} \; \left(2\pi\right)^{\frac{p}{2}}}{\prod_{i=1}^{p} \Gamma\left(N-i\right)} \; x^{N-1} \sum_{n=0} \frac{C_n \; \left(y_i, ..., y_n\right)}{n!} \; G^{p+n,\,0}_{p+n,\,p+n} \left[x \middle| \begin{matrix} 1, ..., 1, \frac{1}{2}, ..., \frac{1}{2} \\ 0, ..., 0, -1, ..., -1 \end{matrix} \right] \tag{2.5}$$

which expresses the density function f(x) as a series involving a G-function.

3. Series Expansion

To get a series expansion suitable for computation of percentage points, we will expand the G-function in (2.5) by using the theory of residues, see Mathai and Rathie (1971), for this we start with

$$G(x) = G^{p+n,\,0}_{p+n,\,p+n} \left[x \middle| \begin{matrix} 1, ..., 1, \frac{1}{2}, ..., \frac{1}{2} \\ 0, ..., 0, -1, ..., -p \end{matrix} \right]$$

$$= (2\pi\omega)^{-1} \int_L \frac{x^{-s}}{s^n} \prod_{i=1}^{p} \frac{\Gamma(s-i)}{\Gamma\left(s+\frac{1}{2}\right)} \, ds \tag{3.1}$$

From (3.1), it is clear that the poles of the integrand on the right hand side are given by the expression

$$(s-p+j)^{\alpha_j} = 0 \; , \; j=0, 1, ... \tag{3.2}$$

and the order of the poles is given by

$$\alpha_j = \begin{cases} j+i, & j = 0, 1, ..., p-1 \\ p+n, & j = p \\ p, & j = p+i, p+2, ... \end{cases} \tag{3.3}$$

Hence by residue theorem, we have from (3.1) that

$$G(x) = \sum_{j=0}^{p-1} \frac{1}{j!} \frac{\partial^j}{\partial s^j} \left[x^{-s} (s-p+j)^{j+1} A(s) \right] \Big|_{s=p-j}$$

$$+ \frac{1}{(p+n-1)!} \frac{\partial^{p+n-1}}{\partial s^{p+n-1}} \left[x^{-s} \; s^{p+n} A(s) \right] \Big\|_{s=0}$$

$$+ \sum_{j=p+1}^{\infty} \frac{1}{(p-1)!} \frac{\partial^{p+n-1}}{\partial s^{p-1}} \left[x^{-s} (s-p+j)^{p} A(s) \right] \Big|_{s=p-j}$$

$$= \sum_{j=0}^{p-1} R_{1j} + R_{2p} + \sum_{j=p+1}^{\infty} R_{3j} \quad \text{(say)} \tag{3.4}$$

$$\text{where } A(s) = \frac{1}{s^n} \frac{\prod_{i=0}^{p-1} \Gamma(s-p+1)}{\Gamma^p\left(s+\frac{1}{2}\right)} \tag{3.5}$$

Now

$$R_{1j} = \lim_{s \to p-j} \frac{1}{J!} \frac{\partial^j}{\partial s^j} \left[x^{-s} \frac{\Gamma^{j+1}(s-p+j+1) \prod_{i=j+1}^{p-1} \Gamma(s-p+i)}{s^n \, \Gamma^p \left(s+\frac{1}{2}\right) \prod_{i=0}^{j-1} (s-p+i)^{i+1}} \right]$$

$$= \frac{1}{J!} \lim_{s \to p-j} \frac{\partial^j}{\partial s^j} \left[x^{-s} \, g_1(s) \right] \tag{3.6}$$

where

$$g_1(s) = \frac{\Gamma^{j+1}\ (s-p+j+1) \prod_{i-j+1}^{p-1} \Gamma\ (s-p+i)}{s^n\ \Gamma^p\ \left(s+\frac{1}{2}\right) \prod_{i=0}^{j-1} (s-p+i)^{i+1}} \tag{3.7}$$

Now, we have

$$\frac{\partial^j}{\partial s^j}\left[x^{-s} g_1(s)\right] = x^{-s}\left[\frac{\partial}{\partial s} + (-\log x)\right]^j g_1(s) \tag{3.8}$$

and

$$\left[\frac{\partial}{\partial s} + (-\log x)\right]^j g_1(s) = \sum_{r=0}^{j} \binom{j}{r} (-\log x)^{j-r}\ \frac{\partial}{\partial s}\ g_1(s) \tag{3.9}$$

Also

$$\frac{\partial}{\partial s}\ g_1(s) = \frac{\partial^{r-1}}{\partial s^{r-1}}\left[\frac{\partial}{\partial s}\ g_1(s)\right] = \frac{\partial^{r-1}}{\partial s^{r-1}}\left[g_1(s)\ h_1(s)\right] \tag{3.10}$$

where

$$h_1(s) = \frac{\partial}{\partial s}\ \log\left[g_1(s)\right] = (j+1)\ \Psi\ (s-p+j+1) + \sum_{i=j+1}^{p-1}\ \Psi\ (s-p+1) - \frac{n}{s} =$$

$$- p\ \Psi\ \left(s-\frac{1}{2}\right) - \sum_{i=0}^{j-1} \frac{i+1}{s-p+i} \quad (3.11)$$

Thus all the derivatives of $g_1(s)$ are given by

$$\frac{\partial}{\partial s}\ g_1(s) = \sum_{t=0}^{r-1} \binom{r-1}{t} \frac{\partial^{r-1-t}}{\partial s^{r-1-t}}\ h_1(s)\ \frac{\partial^t}{\partial s^t}\ g_1(s) \tag{3.12}$$

where

$$\frac{\partial^t}{\partial s^t}\ h_1(s) = (-1)^{t+1}\ t!\ \Big[(j+1)\ \zeta\ (t+1, s-p+j+1) - p\zeta\ \left(t+1, s+\frac{1}{2}\right)$$

$$+ \sum_{i=j+1}^{p-1} \zeta\ (t+1, s-p+i) + ns^{-t-1} + \sum_{i=0}^{j-1} \frac{i+1}{(s-p+i)^{t+1}}\Big] \tag{3.13}$$

Hence

$$R_{1j} = \frac{1}{j!} \sum_{r=0}^{j} \binom{j}{r} (-\log x)^{j-r}\ \frac{\partial}{\partial s}\left[g_{10}(s)\right] x^{-p+j} \tag{3.14}$$

where

$$\frac{\partial^r}{\partial s^r} g_{10} = \operatorname*{Lim}_{s\to p-j} \frac{\partial^r}{\partial s^r} g_1(s)$$

$$= \sum_{t=0}^{r-1} \binom{r-1}{t} \frac{\partial^{r-1-t}}{\partial s^{r-1-t}} h_{10}(s) \frac{\partial^t}{\partial s^t} g_{10}(s) \tag{3.15}$$

with

$$g_{10} = \operatorname*{Lim}_{s\to p-j} g_{1s} = \frac{\prod_{i=j+1}^{p-1} \Gamma(i-j)}{(p-j)^n \Gamma^p\left(p-j+\frac{1}{2}\right) \prod_{i=0}^{j-1} (i-j)^{i+1}} \tag{3.16}$$

$$h_{10} = \operatorname*{Lim}_{s\to p-j} h_1(s)$$

$$= (j+1)\,\Psi(1) + \sum_{i=j+1}^{p-1} \Psi(i-j) - \frac{n}{p-j} - p\,\Psi\left(p-j+\frac{1}{2}\right) - \sum_{i=0}^{j-1} \frac{i+1}{i-j} \tag{3.17}$$

and

$$\frac{\partial^t}{\partial s^t} h_{10} = \operatorname*{Lim}_{s\to p-j} \frac{\partial^t}{\partial s^t} h_1(s)$$

$$= (-1)^{t+1}\, t! \Bigg[(j+1)\,\zeta(t+1, 1) + \sum_{i=j+1}^{p-1} \zeta(t+1, i-j)$$

$$+ \frac{n}{(p-j)^{t+1}} - p\,\zeta\left(t+1, p-j+\frac{1}{2}\right) + \sum_{i=0}^{j-1} \frac{i+1}{(i-j)^{t+1}} \Bigg] \tag{3.18}$$

Now we have

$$R_{2p} = \operatorname*{Lim}_{s\to 0} \frac{1}{(p+n-1)!} \frac{\partial^{p+n-1}}{\partial s_{p+n-1}} \left[x^{-s} \frac{\Gamma^p(s+1)}{\Gamma^p\left(s+\frac{1}{2}\right) \prod_{i=0}^{p-1} (s-p+1)^{i+1}} \right]$$

$$= \frac{1}{(p+n-1)!} \operatorname*{Lim}_{s\to 0} \frac{\partial^{p+n-1}}{\partial s_{p+n-1}} \left[x^{-s} g_2(s) \right] \tag{3.19}$$

where $$g_2(s) = \frac{\Gamma^p(s+1)}{\Gamma^p\left(s+\frac{1}{2}\right) \prod_{i=0}^{p-1} (s-p+i)^{i+1}} \tag{3.20}$$

Furthermore

$$\frac{\partial^{p+n-1}}{\partial s_{p+n-1}} \left[x^{-s} g_2(s) \right] = x^{-s} \left[\frac{\partial}{\partial s} + (-\log x) \right]^{p+n-1} g_2(s) \tag{3.21}$$

and

$$\left[\frac{\partial}{\partial s}+(-\log x)\right]^{p+n-1} g_2(s)$$

$$= \sum_{r=0}^{p+n-1} \binom{p+n-1}{r} (-\log x)^{p+n-1-r} \frac{\partial^r}{\partial s^r} g_2(s) \qquad (3.22)$$

Also

$$\frac{\partial^r}{\partial s^r} g_2(s) = \frac{\partial^{r-1}}{\partial s_{r-1}} \left[\frac{\partial}{\partial s} g_2(s)\right] = \frac{\partial^{r-1}}{\partial s_{r-1}} \left[g_2(s)\, h_2(s)\right] \qquad (3.23)$$

where

$$h_2(s) = \frac{\partial}{\partial s} \log g_2(s)\, p\Psi\ (s+1) - p\Psi\ \left(s+\tfrac{1}{2}\right) - \sum_{i=0}^{p-1} \frac{i+1}{s-p+i} \qquad (3.24)$$

Thus all the derivatives of $g_2(s)$ are given by

$$\frac{\partial^r}{\partial s^r} g_2(s) = \sum_{t=0}^{r-1} \binom{r-1}{t} \frac{\partial^{r-1-t}}{\partial s^{r-1-t}} h_2(s) \frac{\partial^t}{\partial s^t} g_2(s) \qquad (3.25)$$

where

$$\frac{\partial t}{\partial s^t} h_2(s) = (-1)^{t+1}\ t! \left[p\zeta(t+1, s+1) - p\zeta\ (t+1, s+\tfrac{1}{2}) + \sum_{i=0}^{p-1} \frac{i+1}{(s-p+i)^{t+1}}\right] \qquad (3.26)$$

Hence

$$R_2 p = \frac{1}{(p+n-1)!} \sum_{r=0}^{p+n-1} \binom{p+n-1}{r} \left(-\log x\right)^{p+n-1-r} \frac{\partial^r}{\partial s^r} g_{20}(s) \qquad (3.27)$$

where

$$\frac{\partial^r}{\partial s^r} g_{20} = \lim_{s\to 0} \frac{\partial t}{\partial s^t} g_2(s)$$

$$= \sum_{t=0}^{r-1} \binom{r-1}{t} \frac{\partial^{r-1-t}}{\partial s^{r-1-t}} h_{20}(s) \frac{\partial t}{\partial s^t} g_{20}\ (s) \qquad (3.28)$$

with

$$g_{20}\ =\ \lim_{s\to 0} g_2(s) = \left[\Gamma^p \left(\tfrac{1}{2}\right) \prod_{i=0}^{p-1} (i-p)^{i+1}\right]^{-1} \qquad (3.29)$$

$$h_{20} = \lim_{s\to 0} h_2(s) = p\Psi(1) - p\Psi\left(\tfrac{1}{2}\right) - \sum_{i=0}^{p-1} \frac{i+1}{i-p} \qquad (3.30)$$

and

$$\frac{\partial t}{\partial s^t} h_{20} = \lim_{s\to 0} h_2(s)$$

$$= (-1)^{t+1}\ t! \left[p\zeta\,(t+1,\,1) - p\zeta\left(t+1,\,\tfrac{1}{2}\right) + \sum_{i=0}^{p-1} \frac{i+1}{(i-p)^{t+1}} \right] \tag{3.31}$$

Now we have

$$R_{3j} = \lim_{s \to p-j} \frac{1}{(p-1)!} \frac{\partial^{p-1}}{\partial s^{p-1}} \left[x^{-s} \frac{\Gamma^p\ (s-p+j+1)}{s^n\ \Gamma^p\left(s+\frac{1}{2}\right) \prod_{i=0}^{p-1} (s-p+i) \prod_{i=p}^{j-1} (s-p+i)^p} \right]$$

$$= \lim_{s \to p-j} \frac{1}{(p-1)!} \frac{\partial^{p-1}}{\partial s^{p-1}} \left[x^{-s}\ g_3(s) \right] \tag{3.32}$$

where

$$g_3(s) = \frac{\Gamma^p\ (s-p+j+1)}{s^n\ \Gamma^p\left(s+\frac{1}{2}\right) \prod_{i=0}^{p-1} (s-p+i)^{i+1} \prod_{i=p}^{j-1} (s-p+i)^p} \tag{3.33}$$

we have

$$\frac{\partial^{p-1}}{\partial s^{p-1}} \left[x^{-s} g_3(s) \right] = x^{-s} \left[\frac{\partial}{\partial s} + (-\log x) \right]^{p-1} g_3(s) \tag{3.34}$$

and

$$\left[\frac{\partial}{\partial s} + (-\log x) \right]^{p-1} g_3(s) = \sum_{r=0}^{p-1} \binom{p-1}{r} (-\log x)^{p-1-r} \frac{\partial^r}{\partial s^r} g_3(s) \tag{3.35}$$

Also, $$\frac{\partial^r}{\partial s^r} g_3(s) = \frac{\partial^{r-1}}{\partial s^{r-1}} \left[\frac{\partial}{\partial s} g_3(s) \right] = \frac{\partial^{r-1}}{\partial s^{r-1}} \left[g_3(s)\ h_3(s) \right] \tag{3.36}$$

where

$$h_3(s) = \frac{\partial}{\partial s}\left[\log g_3(s) \right]$$

$$= p\ \Psi\ (s-p+j+1) - \frac{n}{s} - p\ \Psi\left(s+\tfrac{1}{2}\right) - \sum_{i=0}^{p-1} \frac{i+1}{s-p+i} - \sum_{i=p}^{j-1} \frac{p}{s-p+i} \tag{3.37}$$

Thus all the derivatives of $g_3(s)$ are given by

$$\frac{\partial^r}{\partial s^r} g_3(s) = \sum_{t=0}^{r-1} \binom{r-1}{t} \frac{\partial^{r-1-t}}{\partial s^{r-1-t}} h_3(s) \frac{\partial^t}{\partial s^t} g_3(s) \tag{3.38}$$

where

$$\frac{\partial^t}{\partial s^t} h_3(s) = (-1)^{t+1}\ t! \left[p\ \zeta\ (t+1,\, s-p+j+1) + \frac{n}{s^{t+1}} - p\ \zeta\left(t+1,\, s+\tfrac{1}{2}\right) \right.$$

$$+ \sum_{i=0}^{p-1} \frac{i+1}{(s-p+1)^{t+1}} + \sum_{i=p}^{j-1} \frac{p}{(s-p+i)^{t+1}} \Bigg] \tag{3.39}$$

Hence

$$R_{3j} = \frac{1}{(p-1)!} \sum_{r=0}^{p-1} \binom{p-1}{r} (-\log x)^{p-1-r} \frac{\partial^r}{\partial s^r} g_{30}(s)\, x^{j-p} \tag{3.40}$$

where

$$\frac{\partial^r}{\partial s^r} g_{30} = \lim_{s \to p-j} \frac{\partial^r}{\partial s^r} g_3(s) = \sum_{t=0}^{r-1} \binom{r-1}{t} \frac{\partial^{r-1-t}}{\partial s^{r-1-t}} h_{30}(s) \frac{\partial^t}{\partial s^t} g_{30}(s) \tag{3.41}$$

with

$$g_{30} = \lim_{s \to p-j} g_3(s) \left[(-j+p)^n\, \Gamma^p \left(p-j+\tfrac{1}{2}\right) \prod_{i=0}^{p-1} (i-j)^{j+1} \prod_{i=p}^{j-1} (i-j)^p \right]^{-1} \tag{3.42}$$

$$h_{30} = \lim_{s \to p-j} h_3(s) = p\, \Psi(1) - \frac{n}{p-j} - p\, \Psi\left(p-j+\tfrac{1}{2}\right) - \sum_{i=0}^{p-1} \frac{i+1}{i-j} - \sum_{i=p}^{j-1} \frac{p}{i-j} \tag{3.43}$$

and

$$\begin{aligned} \frac{\partial^t}{\partial s^t} h_{30} &= \lim_{s \to p-j} \frac{\partial^t}{\partial s^t} h_3(s) \\ &= (-1)^{t+1}\, t! \left[p\, \zeta(t+1, 1) + \frac{n}{(p-j)^{t+1}} - p\, \zeta\left(t+1, p-j+\tfrac{1}{2}\right) \right. \\ &\quad \left. + \sum_{i=0}^{p-1} \frac{i+1}{(i-j)^{t+1}} + \sum_{i=p}^{j-1} \frac{p}{(i-j)^{t+1}} \right] \end{aligned} \tag{3.44}$$

It follows that (2.5) and (3.4) yield this expression for the density function, namely

$$\begin{aligned} f(x) &= \frac{\left(\frac{N}{e}\right)^{pN} (2\pi)^{\frac{p}{2}}\, x^{N-1}}{\prod_{i=1}^{p} \Gamma(N-i)} \sum_{n=0} \frac{C_n(y_1, \ldots, y_n)}{n!} \\ &\times \left[\sum_{j=0}^{p-1} R_{1j} + R_{2p} + \sum_{j=p+1}^{\infty} R_{3j} \right], \quad \text{for } 0 < x < 1. \end{aligned} \tag{3.45}$$

Integration of $f(x)$ in the above expression between 0 and y for $0 < y < 1$ yields the distribution function F(y).

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