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SYNCHRONIZATION AND ON-OFF INTERMITTENCY PHENOMENA IN A MARKET MODEL WITH COMPLEMENTARY GOODS AND ADAPTIVE EXPECTATIONS

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Synchronization and on-off intermittency phenomena in a market model with complementary goods and adaptive expectations.

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Abstract

In this paper we point out some features of the dynamical models describing the interaction of two similar agents competing in a market. Such models are fitted by a two-dimensional map $M : (x_t, y_t) \mapsto (x_{t+1}, y_{t+1})$ symmetric with respect to the diagonal, i.e., such that $M(x, y) = M(y, x)$. In particular we shall consider a model describing the strategic behavior of two firms that produce complementary goods and have adaptive expectation.

The aim of the paper is to analyze of the trajectories generated by the iteration of M starting from different initial conditions. In particular we are interested in two different problems arising in such situation. First, we analyze the conditions that allow an agent to reach a favourable position (in term of its profit) in the long run. Second, we investigate the mechanisms that lead the agents to behave in the same way in the long run (synchronization) and the phenomena associated with these particular situations, as the on-off intermittency.

JEL Codes: C02, C61, C73.

Keywords: Discrete nonlinear dynamical systems. Synchronization. Strategic competition.

1 Introduction

Several microeconomic models describe the interaction between two (or more) agents that compete in a market to gain the best possible result in terms of profit. The classical example is duopoly, where two producers strategically choose the quantities or the prices that maximize their own profit and such a choice depends generally on that of the competitor, mainly when they act simultaneously.

The agents face a common situation depending on the framework in which they are inserted (the demand function of the market in the duopoly example) and they may either be similar or have different characteristics, as the cost structure in the duopoly. In the first case a *representative agent* can be introduced, and the market will be equally shared between the different agents, while in the case of heterogeneity of agents the share of market of an agent crucially depends on its peculiarity.

Furthermore, an important feature of the agents is the degree of knowledge they have on the market structure and on the characteristic of the competitor. In the simplest case of perfect knowledge (a strong assumption in this kind of models), game theory leads to the equilibrium solution (*Nash equilibrium*), that is, the optimal strategy the agents can adopt, by the intersection point of the so called *reaction curves*. Obviously, in the case of nonlinear reaction curves, many equilibria may exist and in such a case some selection problems may emerge.

Under the perfect knowledge assumption we are dealing with a *static* model, in the sense that the market is described by a one-stage game. The structure of the model can be completely different when the agents are assumed to have *bounded rationality*, allowing them to have some lack of information about the market and/or the competitor. Indeed, in such repeated games the equilibrium in the market can be reached only after some steps, and the model assumes a *dynamic* structure, being described by the iterations of a two-dimensional map (the evolution law of the market). The outcome of the map may also depend on the initial condition of the market. In particular, even if the agents in the market have similar characteristics, it may happen that in the long run an agent has a favorable position with respect to the competitor if the initial conditions are different.

The latter is the case we are interested in, investigating some particular properties of a dynamic competition model with two identical competitors.

In the case of identical competitors, the dynamical system $M : (x_t, y_t) \mapsto (x_{t+1}, y_{t+1})$ must remain the same if the variables x and y are interchanged, that is, if the map M is symmetric with respect to the diagonal Δ of the plane $\mathbb{R}^2$. The symmetry property implies that the diagonal Δ is mapped into itself by the map M and this means that identical agents starting from identical initial conditions obviously behave in the same way.

The trajectories generated by the iteration of the map M and characterized by $x_t = y_t$ for any t, are said *synchronized trajectories* and are governed by the one-dimensional map given by the restriction of M to the invariant submanifold Δ.

Such one-dimensional map describes the behavior of the representative agent, but the analysis of its dynamics is an important tool to understand the dynamical behavior also when the competitors start from different initial conditions.

A trajectory starting out of Δ, that is, with $x_0 \neq y_0$, is said to *synchronize* if $|x_t - y_t| \to 0$ as $t \to +\infty$ (see, e.g. , Bischi et. al., 1998, Agliari et. al., 2002).

A natural question arising in this framework is whether identical competitors starting with different initial conditions synchronize in the long run. And if not, which initial conditions allow an agent to reach a favorable situation? If a competitor starts in a favorable condition, is that condition guaranteed in the long run?

The aim of this paper is to give an answer to these questions, putting also in evidence the local and global bifurcations which give rise to the different situations.

As an illustrative example we consider a model proposed by Matsumoto and Nonaka, 2006, that describes the strategic behavior of two firms producing complementary goods and using an adaptive expectation mechanism, instead of static expectations as in the original paper.

Such a model reduces to the iteration of a nonlinear, and noninvertible, two-dimensional map that exhibits up to four equilibria, coexistence of different attractors and complex dynamics (see, also, Bignami and Agliari, 2007).

In order to obtain a symmetric map we restrict the analysis to the particular case of proportional prices and costs. The corresponding map still exhibits multiple equilibria and through a global analysis of its dynamics we shall investigate the role played by the initial conditions in the selection of one of them. In this context, we shall also see that homoclinic and contact bifurcations (these latter typical of nonlinear maps) can occur, causing important qualitative change in the asymptotic behavior of the model.

The synchronization of the trajectories will be also shown, as well as some interesting phenomena associated with such a situation.

Synchronization phenomena may occur in dynamic systems having an invariant submanifold of lower dimension than the total phase space. In physics such phenomena have been deeply studied associated with the coupled chaotic oscillators dynamics (see, among other, Fujisaka and Yamada, 1983, Pecora and Carrol, 1990), while in economics only a few studies have been elaborated (see, among others, Bischi et al., 1998, Bischi and Gardini, 1998, Bischi et al., 1999, Bischi and Lamantia, 2002, Agliari et al., 2002). In our opinion, it is worth highlighting the peculiarity of some trajectories when synchronization occurs. In particular, we shall show the existence of a *Milnor attractor* and the associated *on-off intermittency* behavior of the trajectories, that is, the chaotic time patterns which are synchronized for several time periods, nevertheless asynchronous fluctuations sometimes take place. We recall that a closed invariant set A is said to be a *Milnor attractor* (or a *weak attractor in the Milnor sense*) if its stable set has positive Lebesgue measure, while A is a *topological attractor* (or *asymptotically stable attractor*) if it is Lyapunov stable, that is for every neighborhood W of A there exist a neighborhood U of W such that $T^t(U) \subset W$ $\forall t \geq 0$ and the distance between $T^t(U)$ and A tends to 0 when t goes to infinity. Note that a topological attractor implies a Milnor attractor but the converse is not true.

Before to enter into the analysis, we emphasize once more that the particular model here consider is only a pedagogical example that allow us to put in evidence the global properties of the more general class of models having the symmetry property, as those allowing the existence of a representative agent.

The paper is organized as follows. In Section 2, we introduce the model describing the time evolution of the production levels of two firms acting in a complementary good market by the iteration of a two-dimensional map T. The main properties of the map T are analyzed in Section 3. In particular, we point out that such a map is a noninvertible one, that it may possess two or four fixed points and we study their local stability. Then, in Section 4, we study some not synchronizing trajectories by the analysis of the basins of attractions of several coexisting attractors that the map T exhibits. In particular some global bifurcations are pointed out, that involve important change in the long run situation of the producers. The aim of Section 5 is to analyze a synchronization phenomena, considering two different ways in which synchronization is attained. A first one in which initially two strange attractors not embedded in Δ exist and we describe the mechanism which gives rise to the appearance of synchronizing trajectories. The second one we propose is related to chaos synchronization and gives us the opportunity to show the on-off intermittency phenomenon. Section 6 is the conclusion.

2 The model

We consider the model proposed by Matsumoto and Nonaka, 2006, describing the strategic behavior of two firms producing complementary goods, x_1 and x_2 respectively. They assume that in each market, the inverse demand function is given by

$$p_i(x_i, x_j) = \alpha_i - \beta_i x_i + \gamma_i x_j^2 \tag{1}$$

where $\alpha_i > 0$, $\beta_i > 0$ and $\gamma_i > 0$ $(i = 1, 2)$ and $i \neq j$.

The complementary relationship between the two outputs is fitted by the quadratic term in (1), whose effect is to introduce in the model a *positive sale externality*, due to the fact that the sales of one firm are positively influenced by the production of the other firm.

On the supply side (as in Kopel, 1997) it is assumed that a *negative production externality* exists, the production choices of one firm being influenced by the production level of the other one in terms of the cost function. Following Matsumoto and Nonaka, we confine the analysis to the simple case in which the production cost linearly depends on the other firm's output, that is,

$$C_i(x_i, x_j) = c_i x_i x_j \tag{2}$$

where $c_i > 0$, $i = 1, 2$.

Hence, we are considering a situation involving a double externality: A positive externality via the market demand, measured by the quadratic term in x_j in (1) and a negative externality via the cost function, due to the dependence on x_j of the marginal cost in (2).

Each firm acts in a strategic framework since, in choosing its production level, it has to take into account the decision of the other producer. The optimal production choice of each firm is given by the solution of the profit maximization problem. Solving such a problem, each producer obtains its own reaction function $x_i = r_i(x_j)$, $i, j = 1, 2$ and $i \neq j$, given by

$$r_i(x_j) = \frac{\alpha_i - c_i x_j + \gamma_i x_j^2}{2\beta_i}. \tag{3}$$

The reaction functions in (3) are nonlinear, with a unimodal shape and crucially depend on the mutual amplitude of the sale and production externalities. Indeed, if $c_i << \gamma_i$, the curves are quadratically upward sloping, due to the dominance of the sale externality; whereas, if $\gamma_i << c_i$, the reaction functions become linearly downward sloping, due to the dominance of the production externality.

The firms decide simultaneously the production quantity and then they have to anticipate the quantity supplied in the complementary market; thus the quantity of x_i is a function of the expected quantity $x_j^{(e)}$, that is,

$$x_i = r\left(x_j^{(e)}\right) = \frac{\alpha_i - c_i x_j^{(e)} + \gamma_i \left(x_j^{(e)}\right)^2}{2\beta_i}. \tag{4}$$

Obviously, if the firms have a perfect foresight of the production in the complementary market, they would be able to choose their optimal strategy, belonging to the Nash equilibria set. But, given the nonlinearity of the reaction functions and, consequently, the existence of multiple Nash equilibria, some selection problem may arise.

In order to choose among the different equilibria, we assume that the firms adopt a "*learning by doing*" approach. Then in the time evolution of the production decisions, we assume that at each stage, one firm optimally decides by means of its reaction function, supposing that the expected production of the complementary good is the same as in the previous one. This means that the firms have *naive expectations*, that is

$$x_j^{(e)}(t) = x_j(t-1) \qquad \forall t \geq 0 \tag{5}$$

Moreover, in order to offset the lack of information of the firms, we assume that they do not immediately jump to the optimum predicted by the reaction function, but *adaptively adjust* their previous decision in the direction of the new optimum (see, among others, Bischi and Kopel, 2001, Agliari et al., 2005)

$$x_i(t) = (1 - \lambda_i)\, x_i(t-1) + \lambda_i r_i(x_j(t-1)) \tag{6}$$

where $i, j = 1, 2$ ($i \neq j$) and the parameters λ_i ($0 < \lambda_i < 1$) are the adjustment speeds. The assumption in (6) may also be justified by some technology constraint which prevents the producers to immediately jump to the optimal quantity.

Substituting the expression of $r_i(x_j)$ in the adaptive mechanism (6), we obtain the following two-dimensional dynamical system

$$R : \begin{cases} x' = (1-\lambda_1)\, x + \lambda_1 \frac{\alpha_1 - c_1 y + \gamma_1 y^2}{2\beta_1} \\ y' = (1-\lambda_2)\, y + \lambda_2 \frac{\alpha_2 - c_2 x + \gamma_2 x^2}{2\beta_2} \end{cases} \tag{7}$$

where, for the sake of simplicity, we rewrite the state variables (x_i, x_j) as (x, y). In (7), the symbol "$'$" denotes the one-period advancement operator, i.e. if x is the production level at time t, then x' denotes the production level at time $t+1$.

The map R in (7) describes the evolution over time of the two markets and depends on 8 parameters. The particular case $\lambda_1 = \lambda_2 = 1$ has been studied in Agliari and Bignami, 2007, where the asymptotic behavior of the map has been deeply analyzed, considering the same choice of parameters made by Nonaka and Matsumoto (who emphasized on the emergence of chaotic dynamics). Obviously, the methods used in Agliari and Bignami apply even in this framework, but such a study is beyond the scope of this paper.

Then, in the following we restrict the analysis to the case

$$\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} = \frac{\gamma_1}{\gamma_2} = \frac{c_1}{c_2} = k, \tag{8}$$

that is, we assume that the marginal costs are proportional as well as the prices of the two complementary goods. The reason for such an assumption is merely analytic, since this is the first condition in order to study the trajectories which asymptotically behave in the same way (synchronized trajectories).

Indeed, under the assumption in (8), the map R in (7) reduces to

$$T: \begin{cases} x' = \frac{\lambda_1}{2\beta}\left(\alpha - cy + \gamma y^2\right) + (1-\lambda_1)\,x \\ y' = \frac{\lambda_2}{2\beta}\left(\alpha - cx + \gamma x^2\right) + (1-\lambda_2)\,y \end{cases}, \tag{9}$$

where the index 2 has been suppressed in some parameters. In the case of equal adjustment speeds, $\lambda_1 = \lambda_2 = \lambda$, T becomes a symmetric map, being

$$T \circ S = S \circ T$$

where $S : (x, y) \to (y, x)$ is the reflection through the diagonal $\Delta = \left\{(x, y) \in \mathbb{R}^2 : x = y\right\}$.

2.1 The symmetric map

Let us denote with T_λ the map T in (9) with equal adjustment speeds, that is

$$T_\lambda: \begin{cases} x' = \frac{\lambda}{2\beta}\left(\alpha - cy + \gamma y^2\right) + (1-\lambda)\,x \\ y' = \frac{\lambda}{2\beta}\left(\alpha - cx + \gamma x^2\right) + (1-\lambda)\,y \end{cases}. \tag{10}$$

The symmetry property implies that $T_\lambda(\Delta) \subseteq \Delta$, and this means that firms starting from identical initial conditions behave identically in any time. Thus the dynamics generated by the map T_λ on the invariant submanifold Δ can be studied through the restriction of T_λ to Δ, given by the one-dimensional map

$$f(x) = \frac{\lambda\gamma}{2\beta}x^2 + \left(1 - \lambda - \frac{c\lambda}{2\beta}\right)x + \frac{\alpha\lambda}{2\beta}. \tag{11}$$

Looking for the fixed points of the one-dimensional map f in (11), we can see that if $D = (2\beta + c)^2 - 4\alpha\gamma \geq 0$, two fixed points exist and are given by

$$x_1^* = \frac{2\beta + c - \sqrt{D}}{2\gamma}; \qquad x_2^* = \frac{2\beta + c + \sqrt{D}}{2\gamma}. \tag{12}$$

Moreover, if $D \geq 0$ the map f in (11) is topologically conjugate to the standard logistic map

$$z^{'} = az\,(1 - z) = g\,(z) \tag{13}$$

through the homeomorphism h, where

$$h\,(z) = uz + v = \frac{-2\beta - \lambda\sqrt{D}}{\gamma\lambda} z + \frac{2\beta + c + \sqrt{D}}{2\gamma}$$

and the following relation among the parameter holds

$$a = 1 + \frac{\lambda}{2\beta}\sqrt{D}.$$

The topological conjugation between the maps (11) and (13) implies that the dynamics of (11) are completely known, as these can be obtained from those of the logistic map (13). We briefly recall some of such properties. With regard to the fixed points of f, we have that x_2^* is repelling if $D > 0$, while x_1^* is attracting if $0 < D < \frac{16\beta^2}{\lambda^2}$. At $D = \frac{16\beta^2}{\lambda^2}$ a flip bifurcation occurs, which starts the well known *Feigenbaum cascade of period doubling bifurcations* leading to the chaotic behavior of f. In any case, for each $a \in (1, 4]$, i.e. for each $D \in \left(0, 36\beta^2/\lambda^2\right]$, a unique attractor exists. Such an attractor is the ω-limit set of any trajectory starting from a point $x_0 \in \left(x_2^* - \frac{2\beta+\lambda\sqrt{D}}{\gamma\lambda}, x_2^*\right)$, such an interval being equivalent to the interval $(0, 1)$ of the logistic map. Any trajectory starting outside of the interval $\left(x_2^* - \frac{2\beta+\lambda\sqrt{D}}{\gamma\lambda}, x_2^*\right)$ is divergent.

Let us now return to the two-dimensional map T_λ, in order to put in evidence a noticeable property (useful in the next analysis) of its Jacobian matrix evaluated at the points of the diagonal Δ. Such a matrix assumes the form

$$JT_\lambda\,(x, x) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

where $A = 1 - \lambda$ and $B = \frac{\lambda}{2\beta}\,(2\gamma x - c)$. Then, it is immediate to obtain that the eigenvalues of $JT_\lambda\,(x, x)$ are always real and given by $s_{\|} = A + B$, with associated eigenvector directed along Δ, and $s_{\perp} = A - B$, with associated eigenvector normal to Δ.

In particular, we observe that the rank-1 preimage of the point $\Theta = (\theta, \theta)$, with $\theta = h\left(\frac{a}{4}\right) = \frac{4\beta(c\lambda+\beta(2\lambda-1))-\lambda^2 D}{8\beta\gamma\lambda}$ critical point of the one-dimensional map f, is the point at which the eigenvalue $s_{\|}$ vanishes, that is, the point $\Theta_{-1} = (\theta_{-1}, \theta_{-1})$, with $\theta_{-1} = h\left(\frac{1}{2}\right) = \frac{\beta(\lambda-1)}{\lambda\gamma} + \frac{c}{2\gamma}$.

Thus the fate of any synchronized trajectory (i.e., any trajectory starting from identical initial conditions) is known. Hence, we are now interested in the existence of synchronization phenomena and we analyze producers starting from different initial conditions in order to see if they evolve towards synchronization, so that their dynamical behavior is completely determined by the one-dimensional attractors of the restriction f, or if they converge somewhere else.

Such a problem will be considered in Section 5. In the following section we introduce some properties of the map T to be useful in the next.

3 Properties of the map T

In this section we study some properties of the adjustment process given by the iteration of the map T in (9), which play a crucial role in the study of the global dynamics. In particular we shall analyze the noninvertibility of the map T, useful in the investigation of the topological structures of the attracting sets of the map and related basins of attraction. Moreover, some analytical results about the fixed points will be obtained and their local stability analysis performed.

3.1 Noninvertibility

We recall that a map T is noninvertible if, given a point $p' \in \mathbb{R}^2$, the *rank-1 preimage* of p' (that is, the point $p \in \mathbb{R}^2$ such that $p' = T(p)$), may not exist or may not be unique. In other words, a noninvertible map is a correspondence many-to-one, that is, distinct points of the plane have the same image but there may exist points x having no preimage (see Mira et al., 1996).

Considering the map T in (9), the rank-1 preimages of a given point $(u, v) \in \mathbb{R}^2$ are the solution of the system

$$\begin{cases} u = \frac{\lambda_1}{2\beta}\left(\alpha - cy + \gamma y^2\right) + (1 - \lambda_1)\, x \\ v = \frac{\lambda_2}{2\beta}\left(\alpha - cx + \gamma x^2\right) + (1 - \lambda_2)\, y \end{cases}$$

in the unknown variables x, y. This is a fourth degree algebraic system, which may have four or two real solutions or no real solution at all.

Then, following the terminology introduced in Mira et al., 1996 and in Gumowski and Mira, 1980, we can say that the map T is a $Z_4 - Z_2 - Z_0$ map, since in the plane there is a region of points having four distinct rank-1 preimages, a region of points having two distinct rank-1 preimages and another one whose points have no preimage. Such regions, or *zones*, are separated by the *critical line* LC, i.e. the locus of points having two merging rank-1 preimages. The locus of the merging preimages of the points belonging to the set LC, is the rank-0 critical line LC_{-1}.

The critical lines can be obtained from the Jacobian matrix of the map T, LC_{-1} being included into the locus of points at which the determinant of the

Jacobian matrix vanishes:

$$LC_{-1} \subseteq J_0 = \left\{ (x, y) \in \mathbb{R}^2 : \det JT\,(x, y) = 0 \right\}.$$

From

$$JT\,(x, y) = \begin{bmatrix} 1 - \lambda_1 & \frac{1}{2\beta}\lambda_1\,(-c + 2\gamma y) \\ \frac{1}{2\beta}\lambda_2\,(-c + 2\gamma x) & 1 - \lambda_2 \end{bmatrix},$$

we obtain

$$LC_{-1} = \left\{ (x, y) : y = \frac{1}{\gamma}\left(\frac{2\beta^2\,(1 - \lambda_1)\,(1 - \lambda_2)}{\lambda_1 \lambda_2} \frac{1}{2\gamma x - c} + \frac{c}{2} \right) \right\}.$$

Then LC_{-1} is an equilateral hyperbola in the plane (x, y), made up by two branches, denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$, and $LC = T\,(LC_{-1})$ is made up by the union of two branches, denoted by $LC^{(a)}$ and $LC^{(b)}$, as well. In Fig.1a, related to the map T_λ, we can observe that the branch $LC^{(a)}$ separates the region Z_0 from the region Z_2, while the second branch $LC^{(b)}$ separates Z_2 from Z_4.

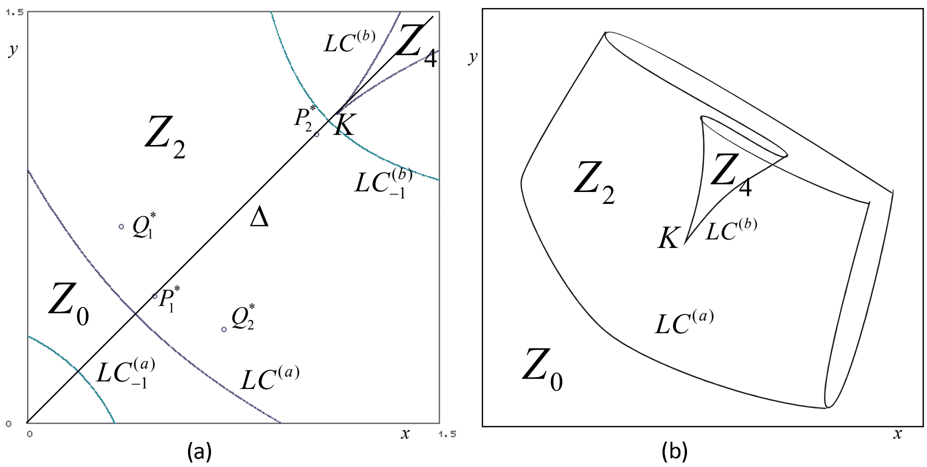

Figure 1: **(a)** *Critical curves and* Z_4, Z_2, Z_0 *regions;* P_1^*, P_2^*, Q_1^*, Q_2^* *are the fixed points of the map* T_λ. **(b)** *Riemann foliation of the plane associated with map* T_λ. *The cusp point* K *belongs to the branch* $LC^{(b)}$.

Geometrically, the action of a noninvertible map can be expressed by saying that it "folds and pleats" the plane, so that distinct points are mapped into the same point and several inverses are defined in some point $x \in \mathbb{R}^2$. These inverses "unfold" the plane. The geometrical interpretation of the action of the map T, and of the multivalued inverse relation T^{-1}, can be understood by considering a region Z_k as the superposition of k "sheets" each one associated with a different inverse. Such a representation is known as *Riemann foliation* of the plane associated with the map T (see e.g., Mira et al., 1996), that is the number of superimposed "sheets" which cover the plane explaining the number of preimages existing in the different regions. The sheets are connected by folds joining two sheets and the projections of such folds on the phase plane are the arcs of LC. The Riemann foliation of the plane $\mathbb{R}^2$ defined by the map T_λ is shown in Fig.1b. From this figure we may also observe the existence of a *cusp* point K, belonging to the branch $LC^{(b)}$. Then, still following the terminology of Mira et al., 1996, we may be more precise, by saying that the T_λ (and T too) is a noninvertible $Z_4 > Z_2 - Z_0$ map, where the symbol "$>$" denotes the existence of a *cusp point* in the branch $LC^{(b)}$ (see also Arnold et al., 1968).

The coordinate of the cusp point can be easily computed in the symmetric case $\lambda_1 = \lambda_2 = \lambda$, since, in such a case, K belongs to the diagonal Δ (see Bischi and Kopel, 2001).

As we can see in Fig.1a, the critical line LC intersects the diagonal in two points, K and Θ, rank-1 image of the points K_{-1} and Θ_{-1}, respectively, belonging to LC_{-1}.[1] Observing that in any point of LC_{-1} at least one eigenvalue of the Jacobian matrix of the map T_λ, given in (10), vanishes, we obtain that

$$K_{-1} = LC_{-1}^{(b)} \cap \Delta = (k_{-1}, k_{-1})$$

with $k_{-1} = \frac{\beta(1-\lambda)}{\lambda\gamma} + \frac{c}{2\gamma}$, as in such a point the eigenvalue $s_\perp = 1 - \lambda - \frac{\lambda}{2\beta}(2\gamma x - c)$ vanishes. Consequently, the cusp point is given by

$$K = LC^{(b)} \cap \Delta = (k, k) \tag{14}$$

with $k = f(k_{-1}) = \frac{4\beta\left(3\beta - 6\beta\lambda + 4\beta\lambda^2 + c\lambda\right) - \lambda^2 D}{8\beta\gamma\lambda}$.

The coordinates of the cusp point K allow us to detect a noticeable global bifurcations arising in the basins of attraction of the stable equilibria and causing important qualitative changes in it, as we shall see in the next section.

3.2 Fixed points and local stability

The equilibrium points of the map T in (9) are the solutions of the system

$$\begin{cases} x = \frac{\lambda_1}{2\beta}\left(\alpha - cy + \gamma y^2\right) + (1 - \lambda_1)\, x \\ y = \frac{\lambda_2}{2\beta}\left(\alpha - cx + \gamma x^2\right) + (1 - \lambda_2)\, y \end{cases} \tag{15}$$

in the unknown variables x and y. The possible solutions of the fourth degree system (15) are independent of the adjustment speeds λ_1 and λ_2, hence the steady states of the map T and T_λ are the same.

Solving (15), we obtain that the map T (and T_λ) may have up to four fixed points; in particular, setting $D = (2\beta + c)^2 - 4\alpha\gamma$ and $D_1 = (c - 2\beta)(c + 6\beta) - 4\alpha\gamma < D$, we obtain that no fixed point exists if $D < 0$. If $D_1 < 0 \leq D$, two fixed points exist and belong to the diagonal Δ, given by $P_1^* = (x_1^*, x_1^*)$ and

[1] The point Θ_{-1} is the same point already considered in subsection 2.1

$P_2^* = (x_2^*, x_2^*)$ with x_1^* and x_2^* defined in (12). Finally, if $D_1 \geq 0$, two further fixed points appear, Q_1^* and Q_2^***,** whose coordinates are

$$\begin{aligned} Q_1^* &= \left(\frac{1}{2\gamma}\left(c - 2\beta - \sqrt{D_1}\right), \frac{1}{2\gamma}\left(c - 2\beta + \sqrt{D_1}\right) \right), \\ Q_2^* &= \left(\frac{1}{2\gamma}\left(c - 2\beta + \sqrt{D_1}\right), \frac{1}{2\gamma}\left(c - 2\beta - \sqrt{D_1}\right) \right). \end{aligned}$$

Note that the equilibria points Q_1^* and Q_2^* are symmetric with respect to the diagonal Δ.

The local stability of the four fixed points of the map T is summarized in the following proposition, where we consider as parameter space the set $\Omega = \{(\alpha\gamma, \lambda_1, \lambda_2) : \alpha\gamma > 0, 0 < \lambda_i < 1, i = 1, 2\}$.

Proposition 1 **(i)** *The fixed point $P_1^* \in \Delta$ exists if $D \geq 0$ (i.e., if $\alpha\gamma \leq (2\beta + c)^2/4$) and is a stable node in the parameter region*

$$\Omega_s^{(P)} = \left\{ (\alpha\gamma, \lambda_1, \lambda_2) \in \Omega : \frac{(6\beta + c)(c - 2\beta)}{4} < \alpha\gamma \leq \frac{(2\beta + c)^2}{4} \right\}$$

(ii) *The fixed point $P_2^* \in \Delta$ exists if $D \geq 0$ and is always unstable. In particular, if $8\beta^2(2 - \lambda_1 - \lambda_2) - \lambda_1\lambda_2\sqrt{D}\left(\sqrt{D} + 4\beta\right) > 0$, P_2^* is a saddle point with stable set along the direction of the eigenvector*

$$v_{\|} = \left[\sqrt{\beta^2(\lambda_1 + \lambda_2)^2 + \sqrt{D}\left(4\beta + \sqrt{D}\right)\lambda_1\lambda_2} - \beta(\lambda_1 - \lambda_2), \quad \lambda_2\left(2\beta + \sqrt{D}\right) \right].$$

(iii) *The two fixed points Q_1^* and Q_2^* exist if $4\alpha\gamma - (c - 2\beta)(c + 6\beta) < 0$ and are stable in the parameter region*

$$\Omega_s^{(Q)} = \left\{ \begin{array}{c} (\alpha\gamma, \lambda_1, \lambda_2) \in \Omega : \\ \frac{1}{\lambda_1} + \frac{1}{\lambda_2} > \frac{(c-2\beta)(c+6\beta) - 4\alpha\gamma}{4\beta^2} \cap \alpha\gamma < \frac{(6\beta+c)(c-2\beta)}{4} \end{array} \right\}.$$

The proof of Proposition 1 is quite standard and based on the localization of the eigenvalues of the Jacobian matrix J evaluated at the fixed points. In particular we have used the local stability conditions (see Gumowski and Mira, 1980; Medio and Lines, 2001):

$$\left\{ \begin{array}{c} 1 - trJ + \det J > 0 \\ 1 + trJ + \det J > 0 \\ 1 - \det J > 0 \end{array} \right. \tag{16}$$

From Proposition 1 we deduce that when $\alpha\gamma = (2\beta + c)^2/4$ a saddle-node bifurcation occurs and two fixed points appear, a saddle P_2^* and a stable node P_1^*. Both these fixed points belong to the diagonal Δ. As the parameter $\alpha\gamma$ crosses the plane of equation $\alpha\gamma = \frac{(6\beta+c)(c-2\beta)}{4}$ a pitchfork bifurcation occurs and causes the appearance of two stable nodes Q_1^* and Q_2^*. Immediately after the crossing, the fixed point P_1^* becomes a saddle point whose stable set is the diagonal Δ and separates the basins of attraction of the two stable nodes. Finally, as the parameters approach the condition

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{(c-2\beta)(c+6\beta) - 4\alpha\gamma}{4\beta^2}, \tag{17}$$

the fixed points Q_1^* and Q_2^*, turn into stable foci, and undergo a Neimark-Sacker bifurcation at the crossing of the condition in (17). Through numerical simulation, it is possible to verify that such a bifurcation is of *supercritical* type, that is, after its occurrence two attracting closed curves appear, each one surrounding an unstable focus Q_1^* or Q_2^*.

Since in the following we confine our study to the case of identical adjustment speeds, that is, we analyze the dynamical behavior of the map T_λ defined in (10), we specialize Proposition 1 to such a particular case, obtaining

Proposition 2 **(i)** *The fixed point $P_1^* \in \Delta$ exists if $D \geq 0$ (i.e., if $\alpha\gamma \leq (2\beta+c)^2/4$) and is a stable node in the parameter region*

$$\Omega_s^{(P)} = \left\{(\alpha\gamma, \lambda) \in \Omega : \frac{(6\beta+c)(c-2\beta)}{4} < \alpha\gamma \leq \frac{(2\beta+c)^2}{4}\right\}$$

(ii) *The fixed point $P_2^* \in \Delta$ exists if $D \geq 0$ and is always unstable. In particular, if $\lambda < 4\beta/\left(\sqrt{D}+4\beta\right)$, P_2^* is a saddle with stable set transverse to Δ.*

(iii) *The two fixed points Q_1^* and Q_2^* exist if $4\alpha\gamma - (c-2\beta)(c+6\beta) < 0$ and are stable in the parameter region*

$$\Omega_s^{(Q)} = \left\{(\alpha\gamma, \lambda) \in \Omega : \lambda < \lambda_N(\alpha\gamma) \cap \alpha\gamma < \frac{(6\beta+c)(2\beta-c)}{4}\right\}$$

where $\lambda_N(\alpha\gamma)$ is an hyperbola of equation $\lambda_N(\alpha\gamma) = \frac{8\beta^2}{(c-2\beta)(c+6\beta)-4\alpha\gamma}$.

Obviously, in Proposition 2 the parameter set Ω is a subset of the plane $(\alpha\gamma, \lambda)$.

A qualitative sketch of the stability regions $\Omega_s^{(P)}$ and $\Omega_s^{(Q)}$ of the equilibria P_1^*, Q_1^* and Q_2^* of the map T_λ is given in Fig.2.

It is worth to recall also that, being the map T_λ symmetric, either an invariant set (attractors, basins of attractions etc.) of the map is symmetric with respect to Δ, either its symmetric set is invariant as well.

4 Basins of attraction and global bifurcations

In this section we consider the situation in which the firms have different initial productions and do not synchronize. As we have seen in the previous section, our

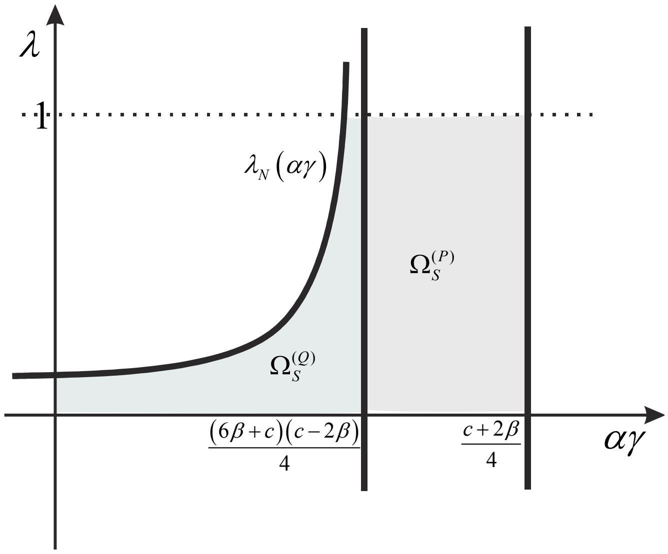

Figure 2: *Stability regions* $\Omega_s^{(P)}$, $\Omega_s^{(Q)}$ *of the map* T_λ.

model admits up to two equilibria outside the diagonal Δ, and each producer has its own preference between them, determined by the profit functions, $\pi_1(x, y)$ and $\pi_2(x, y)$.

More precisely, producer 1 prefers the equilibrium Q_2^* (below the diagonal) and, conversely, producer 2 prefers the equilibrium Q_1^* (above the diagonal), as a straightforward computation permits to obtain (see Table 1).

	$\pi_1(x, y)$	$\pi_2(x, y)$
Q_1^*	$\frac{k\beta}{4\gamma^2}\left(c-2\beta-\sqrt{D_1}\right)^2$	$\frac{\beta}{4\gamma^2}\left(c-2\beta+\sqrt{D_1}\right)^2$
Q_2^*	$\frac{k\beta}{4\gamma^2}\left(c-2\beta-\sqrt{D_1}\right)^2$	$\frac{\beta}{4\gamma^2}\left(c-2\beta-\sqrt{D_1}\right)^2$

Table 1. Profit functions with $D_1 = 4c\beta - 4\alpha\gamma + c^2 - 12\beta^2$

The natural task in such a context is to determine the initial conditions that enables the producers to reach the preferred equilibrium in the long run. We may expect that if the initial conditions are above the diagonal then producer 2 will be satisfied in the long run, and the opposite situation occurs if the initial condition is chosen below the diagonal. This is exactly the case shown in Fig.3a, where the two fixed points Q_1^* and Q_2^* are attracting and have connected basins of attraction separated by the diagonal Δ, stable set of the saddle point P_1^*. In such a figure, the connected basins of attraction of the fixed points Q_1^* and Q_2^* are represented in red and yellow, respectively, the grey area representing the basin of attraction, $B(\infty)$, of an attracting set located at infinity on the *Poincaré equator*.

Then, in the parameter constellation of Fig.3a, an initial difference in the

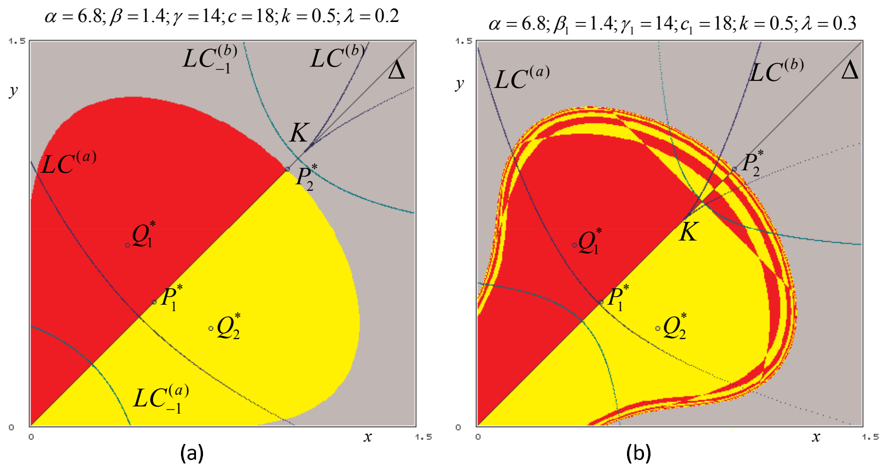

Figure 3: **(a)** *Connected basins of attraction of the equilibria Q_1^* and Q_2^*, in red and yellow, respectively. The gray area represents the set of points giving rise to divergent trajectories.* **(b)** *After the occurrence of a contact bifurcation, the two basins of attraction are disconnected.*

production determines the equilibria selected in the long run and the producer having a larger production preserves its advantage in the long-run. Indeed, any bounded trajectory starting with $x_0 > y_0$ converges to the fixed point Q_2^* and any bounded trajectory starting with $x_0 < y_0$ converges to the equilibrium Q_1^*.

Such a situation can be radically modified if a *contact bifurcation* between the critical curves and the boundary of the basins takes place, causing the breakdown of the connected structure of the basins of attraction.

As an illustration of the occurrence of this global bifurcation, we start from the parameter constellation of Fig.3a and increase the adjustment speed. So doing, the cusp point K of the critical curve $LC^{(b)}$, belonging to the set $B(\infty)$ in Fig.3a, approaches more and more the boundary of the set of bounded trajectories until it merges with the fixed point P_2^* (contact bifurcation). The adjustment speed value at which such a bifurcation occurs is $\lambda_{cb} = 2\beta / \left(\sqrt{D} + 4\beta\right)$. As a result of such a contact we obtain that two further rank-1 preimages of the fixed point P_2^* appear, P_2^* belonging to the Z_2 region if $\lambda < \lambda_{cb}$ and to the Z_4 region otherwise (compare Fig.3a and Fig.3b). Then the portion KP_2^* of Δ belongs to the region Z_4 and this implies that besides its two rank-1 preimages on Δ, each point of KP_2^* has two further rank-1 preimages located on the line Δ_{-1} of equation

$$y + x = \frac{1}{\gamma}\left(2\beta\left(\frac{1}{\lambda} - 1\right) + c\right)$$

locus of preimage of rank-1 of Δ. Obviously, all these preimages, and their preimages of increasing rank, belong to the stable set of P_1^*, since the segment KP_2^* belongs to it, and makes the boundary separating the basins of attraction of Q_1^* and Q_2^* more complex.

As a result of the contact bifurcation occurring at λ_{cb} we observe the transition from a connected basin into a disconnected one and it may occur that the producer having initially a larger production loses its advantage in the long run.

A second situation, due to a different global bifurcation, in which the preferred position of a producer is not predetermined by the initial conditions, is illustrated in Fig.4.

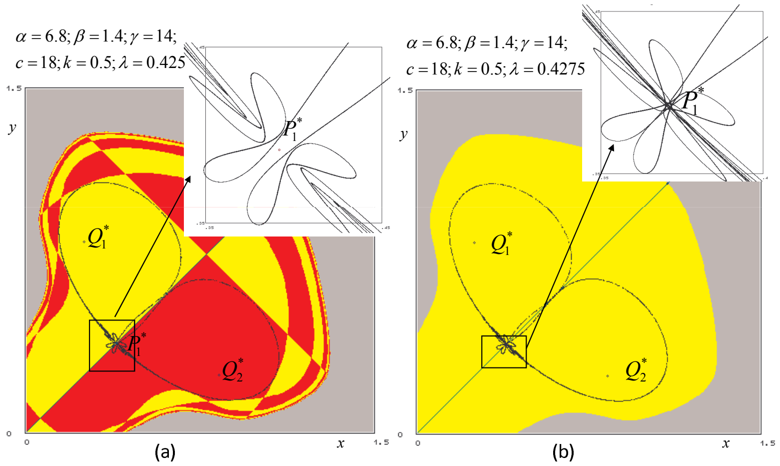

Figure 4: **(a)** *Coexistence of two attractors surrounding the fixed points* Q_1^* *and* Q_2^**. The enlargement shows many convolutions in the proximity of the saddle point* P_1^**.* **(b)** *At* $\lambda = .4275$*, an unique chaotic ring exists, appeared after a homoclinic bifurcation.*

As we have seen in Sec.3.2, the fixed points Q_1^* and Q_2^* lose their stability through a *supercritical Neimark-Sacker* bifurcation occurring at $\lambda = \lambda_N(\alpha\gamma)$, and immediately after the bifurcation two coexisting stable closed curves appear around them. Then, after the Neimark-Sacker bifurcation, the generic trajectory exhibits a quasi-periodic or a periodic behavior. In Fig. 4a, obtained quite far from the Neimark-Sacker bifurcation, the two coexisting attractors are represented together with their disconnected basins of attraction. Due to the nonlinearity of the map T_λ, they exhibit a large number of oscillations close to the diagonal Δ, as the enlargement shows. This implies that the unstable set of the saddle point P_1^*, converging to the closed curves, involves more and more and approaches the stable set of the same saddle, and in particular the diagonal Δ. As the adjustment speed is slightly increased (from $\lambda = .425$ to $\lambda = .4275$) the stable and the unstable sets of the saddle P_1^* have a tangential contact and a *homoclinic bifurcation* occurs. As a consequence, the two attractors merge, giving rise to a unique *chaotic ring*, shown in Fig. 4b. Then, after the occurrence of the homoclinic bifurcation, the coexistence of the two attractors is lost and the generic trajectory fluctuates in a chaotic way.

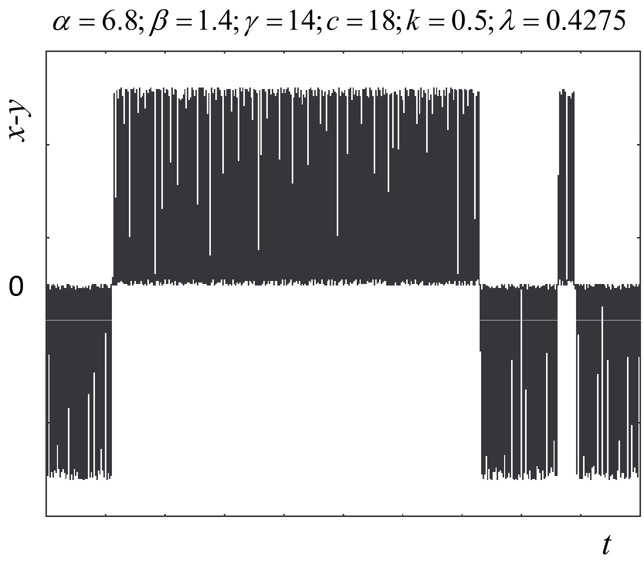

Figure 5: *The time evolution of the difference $x(t) - y(t)$ when the chaotic ring exists.*

To understand the long-run evolution of the quantities in such a case, where a unique symmetric attractor exists, we consider the difference of the quantities $x(t) - y(t)$, represented versus time in Fig.5. We observe that a larger initial production does not favor a producer, but, quite unexpectedly, stages in which a producer has a good position alternate in an irregular way with stages in which the opposite situation is established.

5 Synchronization

In this section, still considering producers starting with different initial condition, we study the mechanisms which can lead to the synchronization of the trajectories. Obviously, we achieve synchronization when on the diagonal Δ there exists a transversely stable orbit (in the sense that it attracts points not belonging to the diagonal itself). Such an attractor can be also coexisting with non synchronizing trajectories and in such a case it becomes important to know the initial conditions leading to synchronization.

To better understand this particular situation we propose an example, illustrated in Fig.6, in which the transversely stable orbit is a cycle of period 2.

We start from the parameter constellation of Fig.6a, which is linked to two symmetric strange attractors. They have been obtained through a standard route to chaos, starting from the two attracting closed curves appeared after the Neimark-Sacker bifurcation of the two equilibria Q_1^* and Q_2^*. The generic bounded not synchronized trajectory converges to one of them, as the basins of

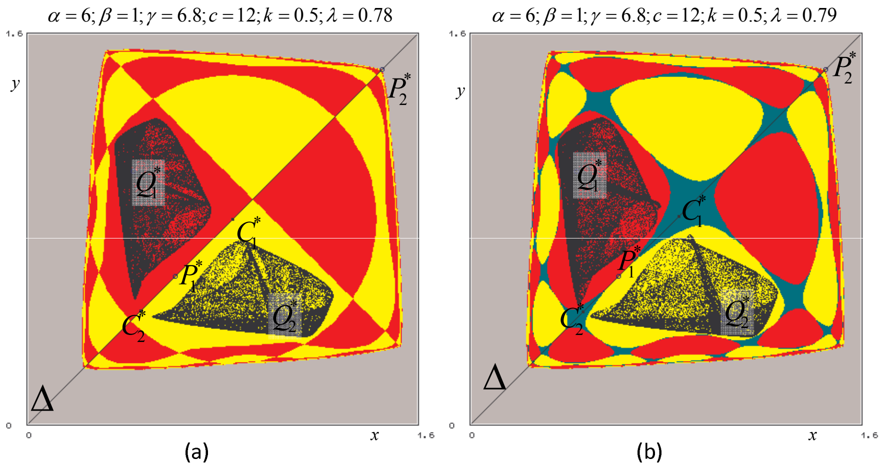

Figure 6: **(a)** *The coexistence of two symmetric strange attractors. Along the diagonal a saddle cycle $C = \{C_1, C_2\}$ has appeared through a flip bifurcation of the fixed point P_1^*.* **(b)** *An increase of the adjustment speed causes the appearance of synchronizing trajectories (green points)*

attraction in Fig.6a show. The separatrix of the two basins of attraction is the stable set of a saddle cycle $C = \{C_1, C_2\}$ belonging to diagonal Δ, appeared through a *flip bifurcation* of the fixed point P_1^*. It is worth noting here that the eigenvalue associated with the cycle C and transverse to the diagonal, $s_\perp$, is larger than 1 in modulus, while the parallel one, $s_\|$, is smaller than 1 in modulus. At this parameter constellation, if the producers have different initial conditions, they cannot synchronize in the long run. A quite different situation is represented in Fig.6b, obtained at a slightly larger value of the adjustment speed λ. Indeed the green points in such a figure correspond to initial condition that give rise to synchronized trajectories. Here, the opportunity of synchronization is due to a simply local bifurcation of the cycle C of period 2, now attracting. Indeed, at a value of λ belonging to the range $(0.78, 0.79)$ the eigenvalue $s_\perp$ assumes the value 1, causing a *subcritical pitchfork bifurcation.* Immediately after the bifurcation the cycle C becomes stable and two cycles of period two (not belonging to the diagonal) appear. They are saddles and their stable sets separate the basin of attraction of the cycle C from those of the two chaotic attractors. As a consequence, we obtain a situation in which three attractors coexist and the synchronization of the producers is a possible outcome.

From Fig.6b, we also observe that the two chaotic attractors are very close to their basin boundary and this fact suggests that a *final bifurcation* caused by the contact between the boundary of the basins and the attractors, is going to occur, causing the disappearance of the two strange attractors. After such a bifurcation, a generic trajectory having different initial conditions will evolve towards synchronization.

The example shown above is related to cyclical synchronizing trajectories, nevertheless more interesting phenomena in this framework can be detected when a chaotic attractor A_s exists on the diagonal Δ.

Indeed, in such a case, A_s includes infinitely many periodic orbits which are unstable in the direction along Δ. For any of these cycles the associated eigenvalues can be obtained. As we have seen in subsection 2.1, due to the symmetry property, the Jacobian matrix of map T, $JT\left(x,x\right)$, evaluated at any point of the diagonal Δ, is such that $J_{11}=J_{22}$ and $J_{12}=J_{21}$. Since the product of matrices with the structure of $JT\left(x,x\right)$ has the same structure as well, a k-cycle $\{x_1,...x_k\}$ embedded into the diagonal has the eigenvalue $s_{\parallel}^{k}=\Pi_{i=1}^{k}s_{\parallel}\left(x_i\right)$, associated with eigenvectors parallel to Δ, and the eigenvalue $s_{\perp}^{k}=\Pi_{i=1}^{k}s_{\perp}\left(x_i\right)$, associated with eigenvectors normal to Δ, where $s_{\parallel}\left(x_i\right)$ and $s_{\perp}\left(x_i\right)$ are the eigenvalues of $JT\left(x_i,x_i\right)$.

Making use of the eigenvalue $s_{\perp}^{k}$ and of the *transverse Lyapunov exponent* we can study the *transverse stability* of the chaotic attractor A_s. That is, we can investigate if not synchronized initial conditions give trajectories converging to it. We recall that, given a chaotic set $A_s\subset\Delta$, the *transverse Lyapunov exponent* is defined as

$$S_{\perp}=\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau}\ln\left|s_{\perp}\left(f^{t}\left(x_0\right)\right)\right| \tag{18}$$

where $f^{t}\left(x_0\right)$, $t\geq0$, is a generic trajectory embedded in A_s. If x_0 belongs to a k-cycle, then $S_{\perp}=\ln\left|s_{\perp}^{k}\right|$, and the cycle is transversely stable if $S_{\perp}<0$. If x_0 belongs to a generic aperiodic trajectory embedded inside the chaotic set A_s then $S_{\perp}$ in (18) is the *natural transverse Lyapunov exponent* $S_{\perp}^{nat}$, where "*natural*" indicates that the exponent is computed for a typical trajectory taken in the chaotic attractor A_s (see Bischi and Gardini, 2000). Since infinitely many cycles, all unstable along the diagonal Δ, are embedded inside a chaotic attractor A_s a spectrum of transverse Lyapunov exponents can be defined (see Buescu, 1997) by the inequality

$$S_{\perp}^{\min}\leq...\leq S_{\perp}^{nat}\leq...\leq S_{\perp}^{\max}$$

and the natural transverse Lyapunov exponent expresses a sort of "weighted balance" between the transversely repelling and transversely attracting cycles (see Nagai and Lai, 1997). If all the cycles embedded in A_s are transversely stable, that is if $S_{\perp}^{\max}<0$, then A_s is asymptotically stable in the Lyapunov sense, for the two-dimensional map T_λ ; nevertheless it may occur that some cycles embedded in the chaotic set A_s become transversely unstable, that is $S_{\perp}^{\max}>0$, while $S_{\perp}^{nat}<0$. In such a case A_s is not stable in the Lyapunov sense but it is a *stable* attractor in the *Milnor sense* (see Milnor, 1985 and Alexander et al., 1992).

If a *Milnor attractor* of the map T_λ exists we obtain that some transversely repelling trajectories can be embedded into a chaotic set which is attracting only "on average" (see Bischi et al., 1998, Agliari et al., 2002). Furthermore, such transversely repelling trajectories can be re-injected towards Δ, so that their behavior is characterized by some "*burst*" far from the diagonal Δ, before the synchronization (or before converging to a different attractor). This situation is called *on-off intermittency* (see Ott and Sommerer, 1994).

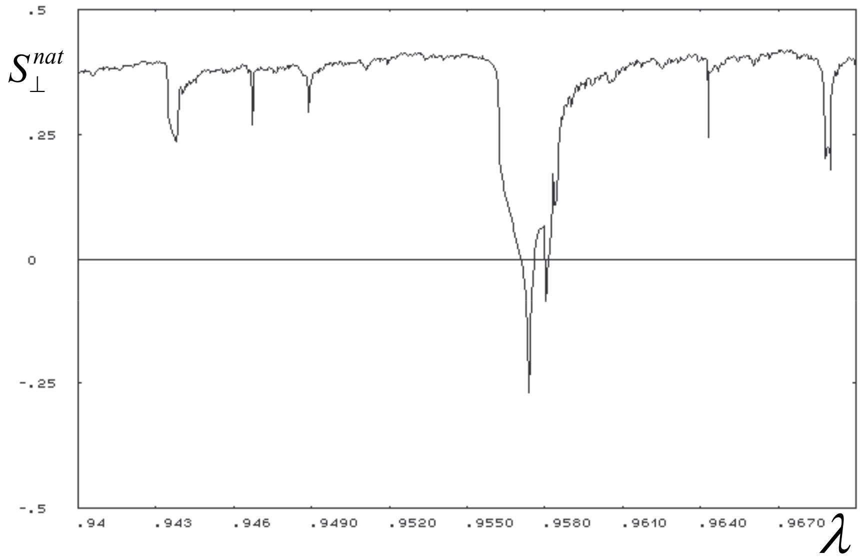

Figure 7: *The natural transverse Lyapunov exponent as a function of the parameter* λ.

Let us come back to our particular model under study. In order to investigate the existence of a Milnor attractor A_s, we estimate the natural transverse Lyapunov exponent $S_\perp^{nat}$, represented as a function of the parameter λ in Fig.7, and observe that it can assume negative values. As an example, we consider $\lambda = .9581$, at which $S_\perp^{nat} = -0.00878$ and the one-dimensional map f in (11) exhibits a the 5-band chaotic intervals A_s. The invariant set A_s is the unique attractor at finite distance of the map T, as shown in Fig.8a, but any trajectory with initial conditions not belonging to the diagonal Δ has a long transient before converging to the 5-band chaotic set. A whole trajectory, starting from $x_0 = .78158$, $y_0 = .68486$, is shown in Fig.8b. Considering the difference $x_t - y_t$ for any t, we can observe that the transient part of the trajectory is characterized by several "bursts" away from Δ (see Fig.9). Then it exhibits the typical *on-off intermittency phenomenon*, so proving that A_s is a Milnor attractor.

Exploiting the geometrical properties of the critical lines, we may also estimate the maximum amplitude of the burst, by obtaining the boundary of a compact trapping region of the phase space in which the on-off intermittency phenomena are confined. Indeed such a trapping region is given by a *minimal absorbing area* (see Bischi and Gardini, 1998) including the Milnor attractor.

We recall that an *absorbing area* D is a bounded region of the phase plane whose boundary is given by a finite number of critical lines segments (i.e, by segments of LC and their increasing rank forward images), such that $T(D) \subseteq D$

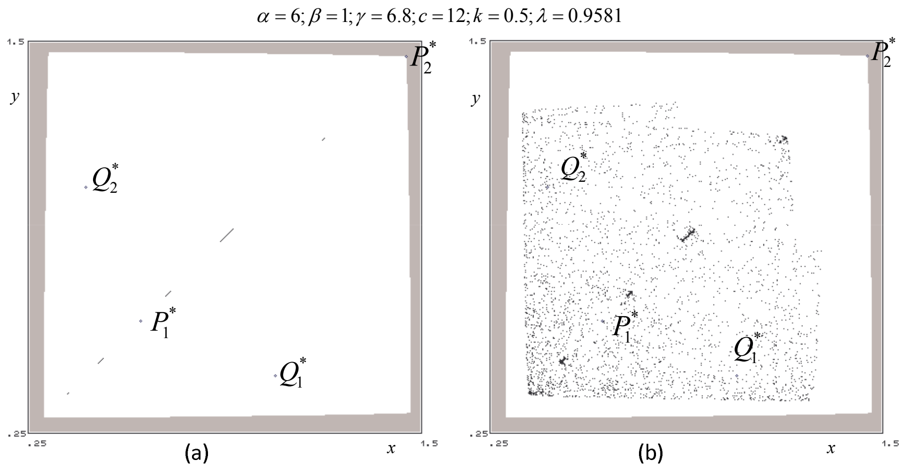

Figure 8: **(a)** *The 5-band chaotic attractor A_s belonging to the diagonal. Its basins of attraction is the set of white points.* **(b)** *A whole trajectory, starting from the initial condition $x_0 = .78158$, $y_0 = .68486$ and converging to A_s.*

and a neighborhood U of D exists whose points enter D after a finite number of iterations and then never escape (see, e.g. Mira et al., 1996, Abraham et al., 1997, Bischi and Gardini, 2000).

As shown in Fig.10a, for the map T_λ at the parameter constellation at which the 5-band chaotic attractor A_s exists, an absorbing area can be obtained by considering a convenient segment of LC_{-1} and its forward images $LC_k = T^{k+1}(LC_{-1})$, with $k = 0, 1, 2$ and $LC_0 = LC$, and it contains the whole trajectory of Fig.8b.

In order to check that the region D so obtained is the region in which the on-off intermittency phenomena are confined, we need to verify that it is the smallest one including the Milnor attractor A_s. In order to do that we introduce a *parameter mismatch* in the speeds of adjustment, breaking the symmetry of the map. Indeed, while generally the symmetry of a map is a *structurally unstable* property (that is, it is lost after an arbitrarily small variation of some parameter), the existence of a minimal invariant absorbing area is *structurally stable*, persisting under a small perturbation of the parameters, even if such a perturbation breaks the symmetry. Then we introduce a slight difference in the adjustment speed and consider the map T in (9) with $\lambda_1 = 0.9581$ and $\lambda_2 = 0.95$. As a consequence of the symmetry breaking, we obtain that the invariance of the diagonal is lost, and the Milnor attractor A_s as well, but the map T exhibits a strange attractor whose shape is exactly that of the absorbing area D, as we can appreciate comparing Fig.10a and Fig.10b. This proves that D is a minimal absorbing area, since it is completely covered by the attractor of T, and that the amplitude of the bursts arising in the trajectories of the map T_λ can be estimated by considering the critical segments LC, LC_1 and LC_2.

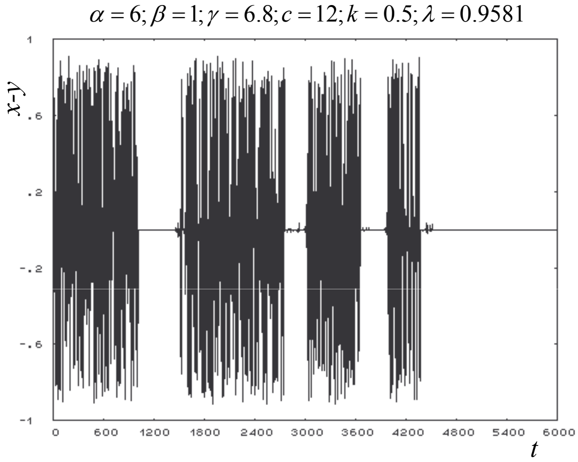

Figure 9: *Bursts away from the diagonal before synchronization occurs. Such a phenomenon is known as on-off intermittency.*

6 Conclusions

In this paper we have studied some global properties of the dynamic behavior of two firms producing complementary goods. They are assumed to make the production choices in a strategic context, adaptively adjusting their previous decisions in the direction of the optimum predicted by the best reply functions. By making some restrictions on the parameters of the model we obtain that the evolution of the quantities over time is described by a two-dimensional noninvertible symmetric map. The symmetry property, that naturally arises when identical competitors are considered, has allowed us to investigate whether identical competitors, starting with different initial conditions, synchronize in the long run. Even if competitors don't synchronize, we are able to determine the initial conditions that allow the agent to reach a favorable situation.

In particular, we have pointed out that, due to the noninvertibility of the model, an initial favorable position in the market of a producer can be lost in the long run. This result has been achieved by the detection of some global bifurcations, as contact and homoclinic bifurcations. The first type causes the transition from connected to disconnected basins. The second type leads changes in the attractors.

The synchronization of the producers, that is, producers behaving in the same way in the long run, has been considered as well, by the analysis of two different situations. In the first one, the attractor A_s is a cycle belonging to the diagonal and in the second A_s is a chaotic set embedded on the invariant submanifold.

The latter case is the more interesting one, since we have proved that A_s is an attractor in Milnor sense, since the trajectories converging to it exhibit a transient part where several "bursts" exist. In different words, in this case,

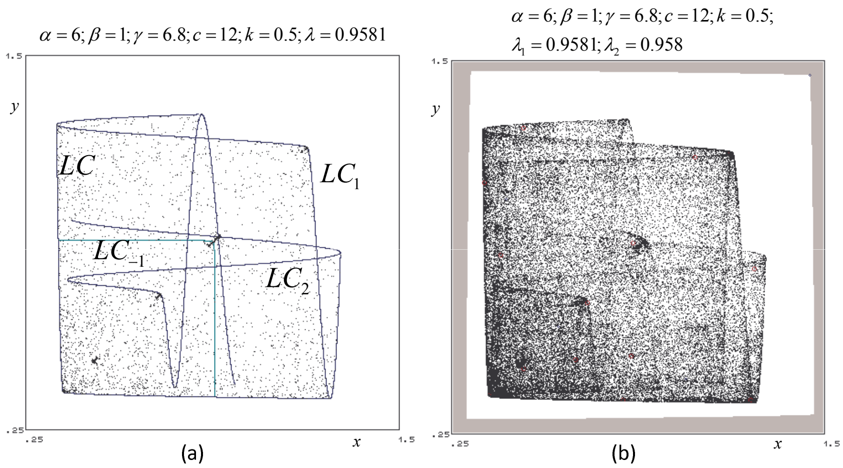

Figure 10: **(a)** *The minimal absorbing area D, bounded by segment of critical curves LC, LC_1 and LC_2, in which the on-off intermittency phenomenon occurs.* **(b)** *The introduction of parameter mismatch, that breaks the symmetry, permits to check the absorbing area properties. Indeed we obtain a strange attractor having exactly the same shape as D.*

the competitor behavior can be summarized as follows. Before definitely converging to A_s, they exhibit intermittent behaviors between synchronous and asynchronous regimes (the so-called on-off intermittency). The amplitude of such oscillations can be estimated, since they are confined inside a minimal absorbing area that we have obtained making use of some critical segments.

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