QUADERNI DEL DIPARTIMENTO DI SCIENZE ECONOMICHE E SOCIALI

ENDOGENOUS LABOR SUPPLY,

BORROWING CONSTRAINT, AND CREDIT CYCLES

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Endogenous Labor Supply,

Borrowing Constraint, and Credit Cycles

Anna Agliari[*]   George Vachadze[†]

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Abstract

We investigate Matsuyama's (*Econometrica*, 72, pp. 853-84, 2004) model modified only to include endogenous and forward looking labor supply decision. Young agents supply one unit of labor endowment elastically to a competitive labor market. While, old agents of ex-ante identical individuals are divided in equilibrium into depositors and entrepreneurs. Depositors lend funds in the form of interest bearing loans, while entrepreneurs borrow funds in the competitive credit market. We emphasize the interaction between credit and labor markets and show the possibility of occurrence of multiple steady states, local and global indeterminacy, and endogenous fluctuations.

When young agents become optimistic about the future deposit rate then they decide to work harder and invest more. Countercyclical borrowing constraint will help agents to fulfill their initial optimistic expectations, because the next period credit volume and deposit rate can increase simultaneously. By conducting global bifurcation analysis, we show that credit cycles can occur through a self-fulfilling expectation mechanism. History-versus-expectations considerations can exist and escape from underdevelopment as well as fall into poverty can be a self-fulfilling prophecy.

Keywords: Borrowing constraint; Credit cycles; Elastic labor supply; Endogenous fluctuations; Self-fulfilling expectations.

JEL classification: C62; E32; E44; J22; O11; O16; O41

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Contents

1 Introduction

Since the seminar contributions of Azariadis & Drazen (1997), Galor & Zeira (1993), Banerjee & Newman (1993), Freeman (1996), Aghion & Bolton (1997), Matsuyama (2000), Matsuyama (2006), Matsuyama (2007), Glavan (2008) and others, it has been realized that financial factors can play a central role in emergence of development/poverty traps. Collateral value, which affects agents borrowing capacity, is central in such models. High collateral implies higher business activity, higher income, which again reinforces the higher collateral. The opposite happens when the economy starts with low collateral. Countries in such models can not escape underdevelopment trap without any external assistance.

There are other type of models which argue that not only imperfections in the credit market but also the failure to coordinate agents expectations can be a main reason behind underdevelopment trap. Examples of such models include Woodford (1986), Matsuyama (1991), Grandmont (1998), Cazzavillan, Lloyd-Braga, & Pintus (1998), Barinci & Cheron (2001), and Slobodyan (2005). Imperfections in the credit market leads to a possibility of multiple steady states and local and global indeterminacy. Local indeterminacy refers the situation when for a given predetermined variable (lets say capital stock) there are multiple control variables (labor supply for example) causing the economy to converge to the same steady state. In technical terms, the local indeterminacy refers the situation when the dimension of the stable set is higher than the number of predetermined variables and thus there exists a continuum of values of control variables that put the system onto the stable set. Therefore, there exists a continuum of perfect foresight trajectories converging to a given steady state or fluctuating around it. Global indeterminacy refers the situation when there exists two or more steady states and there are multiple trajectories converging to them. In case of global indeterminacy, different choices of control variables might imply different long run behavior and initial conditions do not necessarily determine to which steady state the economy will converge eventually. The economy might fall into poverty only because of failure of economic agents to agree on the control variable value leading to the best equilibrium.

Global indeterminacy naturally arises when the dynamical model under scrutiny is nonlinear and multiple steady states (either locally determinate or indeterminate) exist. Indeed, in such a case, the local indeterminacy of a steady state may induce global

indeterminacy, even if the other stationary equilibria are locally determinate. Furthermore, even though all the steady states are locally determinate, the equilibrium of the model may result globally indeterminate due to the coexistence of the different paths leading to the steady states. To investigate such situations a global analysis of the dynamics generated by the model is necessary, as evidenced in many papers (see, among others, Benhabib, Schmitt-Grohe, & Uribe (2001), Cazzavillan, Lloyd-Braga, & Pintus (1998), Gomis-Porqueras, & Haro (2007), Saidi (2008)). In particular, the global analysis of the perfect foresight equilibria allows us to investigate the stable and unstable sets of each stationary equilibrium and the bifurcations causing their qualitative changes, whorty of an in-depth investigation, as suggested by Chiappori & Guesnerie (1991). Indeed, this kind of study allows us to obtain a global picture of the phase-space and to evidenciate that the local analysis of the determinacy of an equilibrium can be misleading even if we restrict the analysis to a small neighborhood of the steady state. For instance, this is the case when some heteroclinic connection exist between two steady states.

The major motivation of the current paper is to demonstrate the possibility of multiple equilibria due to self-fulfilling expectations in an overlapping generations model with imperfect credit market. Self-fulfilling prophecies of economic recovery can occur when agents labor supply is endogenous and forward looking. "History" alone cannot determine where the economy will end up. Instead, escape from a poverty trap can become possible only through coordinating self-fulfilling expectations about the future credit market conditions. To show this, we consider OLG growth model with credit market imperfection proposed and analyzed by Matsuyama (2004). We modify the model only by including agents' endogenous labor supply decision. Entrepreneurs can hide a portion of their cash flow from financiers due to imperfect investor protection. This causes financiers to set a lending/deposit rates which reflects not only profitability of entrepreneurs but also prevents them from not repaying their debt. When investors protection is perfect, then the model reduces to standard one sector model with endogenous labor supply studied by Reichlin (1986). In contrast, the model reduces to model studied by Matsuyama (2004) when agents labor supply is inelastic. Differently from Reichlin (1986), we show that endogenous fluctuations are still possible even when the elasticity of substitution between capital and labor inputs is sufficiently high, and differently from Matsuyama (2004), we show that monotonic convergence of the economy can be lost as soon as agents labor supply decision becomes more and more elastic

and forward looking.

To see why indeterminacy is possible in the model it is useful to look at the credit market clearing condition. When there is an imperfect investor protection then borrowing/landing rate depends not only on marginal product of capital but also on investors' wage income. Increase of steady state capital stock implies not only lower marginal product of capital, but also higher wage income, relaxed credit constraint and thus higher interest rate. I.e., the relation between the equilibrium interest rate and the capital stock is not necessarily monotonic. This non-standard feature of the interest rate curve is the key for expectations driven fluctuations in the model.[1] In particular, if agents start to expect high interest rate then they increase their savings and thus supply more labor. As a result, agents wage income and thus savings increase and potential investors start to rely less on external finance. This causes the problem of investor protection to become less important, demand on credit and interest rate to go up and the expectation about high interest rate to become a self-fulfilling prophecy. The opposite happen when agents expect economic slowdown. In the long run, the economy can converge to a steady state or cycle indefinitely by switching endogenously between the periods of optimistic/pessimistic self-fulfilling expectations of economic recovery.

In this paper we present a local and global analysis of the two-dimensional dynamical system implied by the model. We find that local and global indeterminacy can occur under gross substitutability of capital-labor and consumption-leisure, condition often known for ruling out the phenomena of expectations driven fluctuations. By using recent results of higher dimensional dynamical systems, we present computer assisted proofs of the occurrence of heteroclinic connections between the steady states and of homoclinic bifurcations. Pessimistic and optimistic expectations simply select the trajectory out of many, when global indeterminacy occurs. Existence of heteroclinic connections causes indeterminacy of the equilibrium in the neighborhood of a determinate steady state. In contrast, existence of homoclinic points indicates the possibility of indeterminacy due to the contact between stable and unstable manifolds of a given steady state. The main result of the paper requires[2] a careful study of the global dynamics of equilibrium trajectories.

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[1]Similar channel causing the existence of self-fulfilling prophecy of economic recovery to occur is explored in Kikuchi & Vachadze (2009).

[2]In contrast to the local analysis typically offered in the literature.

The rest of the paper is organized as follows. In section 2 we outline the model, derive agent's optimal labor supply decision, and set conditions for a temporary equilibrium in capital and labor markets. In section 3 we obtain the dynamical system which governs the evolution of the economy under perfect foresight dynamics and show the existence and multiplicity of steady states under perfect foresight. We analyze the local bifurcation and stability of each steady state and describe the possible scenarios of global bifurcation. Next we consider a parameterized version of the model and demonstrate numerically the possibility of (a) heteroclinic connections between two saddle points, and (b) homoclinic bifurcation of a saddle point. Finally, section 5 summarizes the results and concludes.

2   The Model

We consider a closed economy version of overlapping generations model with credit market imperfection proposed and analyzed by Matsuyama (2004). The main departure we make in this paper is to allow young agents labor supply decision to be endogenous and forward looking rather than inelastic and exogenously fixed. Time is discrete and extends from zero to infinity. In each period t, there are two generations alive, young and old. Each generation, distinguished by its date of birth, is of equal size and consists of a continuum of risk-neutral agents. There are two goods produced in each period, a consumption good and a capital good. Capital good is produced using the consumption good as an input via an investment technology to be described below. While the consumption good is produced by a large number of identical firms using capital good and labor as inputs.

The technology of the consumption goods producing firm is described by a constant return to scale production function. Output per worker is $y_t = f(k_t)$, where $k_t = K_t/L_t$ denotes capital per worker and K_t and L_t are aggregate supplies of physical capital and labor respectively. We assume that the production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies: $f(0) = 0$; f is twice continuously differentiable, strictly increasing, strictly concave, and Inada conditions hold. Factor markets are competitive and rewards on physical capital and labor are determined by marginal product rule, i.e., $f'(k_t)$ is the rate of return on one unit of capital, and $w_t = W(k_t) := f(k_t) - k_t f'(k_t)$ is the wage rate. Produced commodity can be either consumed or invested in capital, which becomes available in the next period. Capital depreciates fully within a period.

In the first period of life, young agents are working, supplying elastically a portion $l_t \in [0, 1]$ of one unit of labor endowment to the competitive labor market. Agents do not consume at the end of the first period and save their entire wages income. Old agents use their young period saving $s_t = l_t w_t$ in one of two different ways: (a) they may lend it in the competitive credit market, or (b) they may use all of it to finance an investment project. At the end of period, old agents receive their returns from deposits or investment projects, consume and die. We assume that agent at time t will choose its current labor supply l_t, and plan its expected future consumption c_{t+1}^e, in order to maximize a quasi-linear intertemporal utility function $(c_{t+1}, l_t) \mapsto c_{t+1} - u(l_t)$, where the utility from work $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly convex, and satisfies Inada conditions $u'(0) = 0$ and $u'(1) = \infty$.

Old agents are endowed with access to only one investment project. Project undertaken at the end of period t requires a minimum one unit of consumption good[3] for investment in period t and returns $R > 0$ units of capital goods in period $t + 1$. If $s_t < 1$ then agent borrows the amount $1 - s_t$ in the competitive credit market at the rate r_{t+1}. Revenue from an investment project is $Rf'(k_{t+1})$, which is used firstly to repay the debt $(1 - s_t)r_{t+1}$ and the remainder is consumed. Due to the borrowing limit, agents are able to borrow and start the project when the following *Borrowing Constraint* (BC) is satisfied, $(1 - s_t)r_{t+1} \leq \lambda Rf'(k_{t+1})$. Parameter $\lambda \in [0, 1]$ captures the credit market imperfection in a most parsimonious way.[4] When $\lambda = 0$ then there is no investor protection and entrepreneurs can hide entire revenue from financiers. In contrast, when $\lambda = 1$ then investor protection is perfect and entrepreneurs can credibly pledge the entire revenue for repayment to lenders. When $\lambda \in (0, 1)$, then only a fraction of project's revenue can be credibly pledged for repayment to lenders.

2.1 Agents Labor Supply Decision

Suppose old agent's saving, carried from the previous period, is s_t. At the beginning of period $t + 1$, agent observes the capital stock k_{t+1} and deposit rate r_{t+1}, and decides whether to apply for credit. If the credit application is successful agent runs an investment project. Otherwise old agent becomes a depositor. Due to borrowing constraint, credit is rationed and not all credit applicants secure the loan. Let $\pi_{t+1} \in [0, 1]$ denotes

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[3]Investment projects are indivisible such that it requires one unit of consumption good if it is to be undertaken; it is impossible to invest more or less than one unit.

[4]See Matsuyama (2004) for more detailed discussion.

the probability that credit applicant is successful in getting an external funding. Then old agent's random end of period consumption is

$$
c_{t+1} = \begin{cases} Rf'(k_{t+1}) - (1 - s_t)r_{t+1} & \text{with probability} \quad \pi_{t+1} \\ \\ s_t r_{t+1} & \text{with probability} \quad 1 - \pi_{t+1}. \end{cases} \tag{1}
$$

In contrast, old agent's end of period consumption is $c_{t+1} = s_t r_{t+1}$, when agent becomes a depositor. Direct comparison of consumption levels imply that agents are willing to become entrepreneur and apply for credit when the following *Profitability Constraint* (PC) is satisfied, $r_{t+1} \leq Rf'(k_{t+1})$.

Since young agents born in period t can't observe the quantities $(r_{t+1}, k_{t+1}, \pi_{t+1})$, they forms point expectation $(r_{t+1}^e, k_{t+1}^e, \pi_{t+1}^e)$ about them while making labor supply decision. After observing the wage rate w_t, young agents solve the following optimization problem

$$
\max_{l_t \in [0,1]} \mathbb{E}c_{t+1} - u(l_t), \tag{2}
$$

where

$$
\mathbb{E}c_{t+1} = \begin{cases} \pi_{t+1}^e \left[Rf'\left(k_{t+1}^e\right) - r_{t+1}^e \right] + l_t w_t r_{t+1}^e & \text{if agent plans to run a project} \\ \\ l_t w_t r_{t+1}^e & \text{if agent wants to become a depositor.} \end{cases} \tag{3}
$$

F.O.C. of the above optimization problem is: $u'(l_t) = w_t r_{t+1}^e$, which with the properties of u' implies a well defined optimal labor supply decision

$$
l_t = (u')^{-1} \left[w_t r_{t+1}^e \right]. \tag{4}
$$

Moreover, monotonicity of u' implies gross substitutability between first period leisure and the second period consumption. Young agents observe the wage rate w_t, form expectation about next period interest rate r_{t+1}, and supply labor according to (4). At the beginning of period $t + 1$, old agents observe the interest rate r_{t+1}, and per capita capital stock k_{t+1}, and make decision whether to apply for credit or to become a depositor.

Since agents care about current level of work effort and about the expected next period consumption, it follows that when all agents hold the same expectations about the future deposit rate then agents would choose the same level of work effort and no agent would benefit by working and saving a little bit more in order to increase her chance of receiving the credit. As a result all agents save the same and the only credit allocation rule can be one described above.

2.2 Equilibrium in the Capital Market

Since the size of young agents is constant and normalized to unity and young agents are homogeneous and save s_t, it follows that s_t is also the aggregate saving in the economy. Capital market clearing condition (aggregate savings is equal to aggregate investment) implies that the next period capital stock is $K_{t+1} = Rs_t$. Interest rate adjusts until either borrowing or profitability constraint binds, implying the next period interest rate to be

$$r_{t+1} = \min\left\{\frac{\lambda}{1-s_t}Rf'\left(\frac{Rs_t}{L_{t+1}}\right), Rf'\left(\frac{Rs_t}{L_{t+1}}\right)\right\}, \tag{5}$$

where L_{t+1} is the next period aggregate employment. It follows from (5) that, when $s_t < 1 - \lambda$ then $r_{t+1} < Rf'(k_{t+1})$ and thus all young agents in the next period would strictly prefer to become entrepreneurs and apply for credit. Total demand for credit in such a case is one, while the total supply of credit is s_t. Since domestic credit demand exceeds the domestic credit supply, it follows that credit rationing must occur. Since each project requires one unit of good for investment, it follows that the number of projects which can be finances through borrowing is s_t. This with the assumptions (a) all old agents are ex-ante homogeneous; and (b) size of old agents and thus the size of credit applicants is unity; implies that the probability that a randomly chosen agent will be successful in obtaining loan is $\pi_{t+1} = s_t$.

When the aggregate saving satisfies $s_t \geq 1 - \lambda$, then $r_{t+1} = Rf'(k_{t+1})$ and thus agents are indifferent between becoming a depositor or running a project. Since $r_{t+1} = Rf'(k_{t+1})$, it follows that the borrowing constraint is not binding and credit is no longer rationed. This means that all credit applicants will be able to become entrepreneurs; so that whenever $s_t \geq 1 - \lambda$ then $\pi_{t+1} = 1$.

2.3 Equilibrium in the Labor Market

For a given non-negative pair (w_t, K_t), labor demand schedule is

$$\mathcal{L}^d(w_t, K_t) = \frac{K_t}{W^{-1}(w_t)}, \tag{6}$$

where W^{-1} denotes the inverse of the wage function. Properties of the production function implies that the labor demand curve is well behaved, monotonically decreasing function with respect to wage rate, w_t.

Since young agents are homogeneous with unit mass it follows that the aggregate employment is $L_t = l_t$. This implies that the aggregate saving in the economy, for a given non-negative pair (w_t, K_t) is $s_t = L_t w_t = K_t S(w_t)$, where

$$S(w) := \frac{w}{W^{-1}(w)} \tag{7}$$

describes the relation between the wage rate and the aggregate saving.

Assumption 1 *Suppose f is such that*

$$\sigma(k) > \frac{k f'(k)}{f(k)} \quad where \quad \sigma(k) := \frac{f'(k) W(k)}{f(k) W'(k)} \tag{8}$$

denotes the elasticity of substitution between capital and labor inputs.

Assumption 1 restricts the production function to satisfy the minimum elasticity of substitution requirement. This condition is trivially satisfied when $\sigma(k) \geq 1$ for all $k \geq 0$. However, Assumption 1 may hold also for production functions with $\sigma(k) < 1$ for some k. Assumption 1 implies that the function $k \mapsto W(k)/k$ is strictly decreasing.

Lemma 1 *If Assumption 1 is satisfied then $S' < 0$.*

It follows from Lemma 1 and from equation (7) that when Assumption 1 is satisfied then for a given K_t, increase of the wage rate implies the decline of the aggregate saving and vice versa. When agents make their labor supply decision they observe the current wage rate w_t, and make point forecast about the next period interest rate. It follows from (5) and (7) that the expected next period interest rate is

$$r_{t+1}^e = \begin{cases} \dfrac{\lambda}{1 - K_t S(w_t)} R f'\left(\dfrac{R K_t S(w_t)}{L_{t+1}^e}\right) & \text{if} \quad K_t S(w_t) < 1 - \lambda \\[4mm] R f'\left(\dfrac{R K_t S(w_t)}{L_{t+1}^e}\right) & \text{if} \quad K_t S(w_t) \geq 1 - \lambda, \end{cases} \tag{9}$$

where L_{t+1}^e is the expected aggregate employment in the next period. It follows from (4) and (9) that individual labor supply curve is

$$l_t = \mathcal{L}^s(w_t, K_t, L_{t+1}^e) := \begin{cases} (u')^{-1}\left[\dfrac{\lambda w_t}{1 - K_t S(w_t)} R f'\left(\dfrac{R K_t S(w_t)}{L_{t+1}^e}\right)\right] & \text{if} \quad S(w_t) < \dfrac{1 - \lambda}{K_t} \\[4mm] (u')^{-1}\left[w_t R f'\left(\dfrac{R K_t S(w_t)}{L_{t+1}^e}\right)\right] & \text{if} \quad S(w_t) \geq \dfrac{1 - \lambda}{K_t}. \end{cases} \tag{10}$$

9

Properties of functions u, f, and S, imply that for a given non-negative pair (K_t, L_{t+1}^e), optimal labor supply decision, in general, to depend non-monotonically on wage rate. Non-monotonic labor supply implies multiple labor market clearing wage and aggregate employment and causes the local indeterminacy of equilibrium discussed later. It follows from (10) that imperfections in the credit market is a necessary condition for the local indeterminacy of equilibria. When credit market is perfect, i.e., $\lambda = 1$, then direct and indirect effects act on the same direction (see second equation of (10)) and implies monotonic labor supply function. This with monotonic demand function implies a unique labor market clearing wage and employment. Thus credit market imperfection is the only reason of backward banding labor supply curve.

To see how indeterminacy comes about in this model with (a) constant returns to scale production technology; (b) sufficiently high substitutability between capital-labor inputs; and (c) positive substitutability between first period leisure and second period consumption; we observe that the constant returns to scale production function implies downward sloping labor demand curve, substitutability between first period leisure and second period consumption implies the positive relation between labor supply and wage rate and labor supply and expected interest rate. The reason behind indeterminacy is non-monotonic interest rate function, which is due to imperfection in the credit market. Agent's labor supply decision depends not only on current wage rate but also on expected next period interest rate. When wage rate increases then there are two effects. The direct effect (which is always positive as long as first period leisure and second period consumption are gross substitutes) is that high wage means high opportunity cost on leisure and thus more labor supply. The indirect effect is that high wage rate means lower aggregate savings, tighter credit market conditions, less entrepreneur activity and thus lower expectation about the next period interest rate. As a result, when there is imperfections in the credit market then direct and indirect effects of wage increase can act in opposite directions. In case of domination of indirect effect this leads to a downward sloping labor supply curve.

3   Perfect Foresight Dynamics

In order to obtain perfect foresight dynamics we assume that the expected next period labor supply is perfectly known and we determine k_{t+1}. Capital and Labor market

clearing conditions, $K_{t+1} = Rs_t$ and $L_t = l_t = s_t/w_t$, with (4) and (5) imply that

$$u'\left(\frac{s_t}{w_t}\right) = \begin{cases} \dfrac{\lambda}{1-s_t} Rw_t f'(k_{t+1}) & \text{if} \quad s_t < 1-\lambda \\[2ex] Rw_t f'(k_{t+1}) & \text{if} \quad s_t \geq 1-\lambda. \end{cases} \tag{11}$$

It follows from (11) that the next period capital per capita under perfect foresight is

$$k_{t+1} = \xi(w_t, s_t) := \begin{cases} (f')^{-1}\left[\dfrac{1-s_t}{\lambda}\dfrac{1}{Rw_t}u'\left(\dfrac{s_t}{w_t}\right)\right] & \text{if} \quad s_t < 1-\lambda \\[3ex] (f')^{-1}\left[\dfrac{1}{Rw_t}u'\left(\dfrac{s_t}{w_t}\right)\right] & \text{if} \quad s_t \geq 1-\lambda. \end{cases} \tag{12}$$

(12) implies that the evolution of the pair (w_t, s_t), under perfect foresight dynamics, to be described by the following two dimensional dynamical system

$$M : \begin{cases} w_{t+1} = m_1(w_t, s_t) \\[2ex] s_{t+1} = m_2(w_t, s_t), \end{cases} \tag{13}$$

where

$$m_1(w, s) := W\left[\xi(w, s)\right] \quad \text{and} \quad m_2(w, s) := W\left[\xi(w, s)\right]\frac{Rs}{\xi(w, s)}. \tag{14}$$

3.1   Steady State Analysis

In order to find the steady states of the dynamical system M, we solve the following system of equations

$$w = W\left[\xi(w, s)\right] \quad \text{and} \quad s = w\frac{Rs}{\xi(w, s)}. \tag{15}$$

Second equation of (15) implies that at any steady state the following equation holds, $\xi(w, s) = Rw$. After substituting this into first equation, we obtain that the steady state wage rate satisfies the equation

$$w = W(Rw). \tag{16}$$

Assumption 2 *The function f is such that $W'(0) = \infty$ and $W'' < 0$.*

Assumption 2 implies the existence of one corner (not acceptable) and one interior solution, $w^* = W^*(R)$.

Lemma 2 *If Assumption 1 is satisfied then W^* is a monotonically increasing function*

Assumption 3 *Suppose f is such that*

$$\sigma(k) > 1 - \frac{kf'(k)}{f(k)} \tag{17}$$

where σ defined in (17) denotes the elasticity of substitution between capital and labor inputs.

Assumption 3, similarly to Assumption 1, restricts the elasticity of substitution between capital and labor incomes, so that capital income in production $\rho(k) := kf'(k)$, is strictly increasing. Strict monotonicity properties of ρ, W^*, and u', imply the existence and uniqueness of such R^+ solving equation

$$W^*(R) \cdot (u')^{-1} \{\rho[RW^*(R)]\} = 1. \tag{18}$$

Assumption 4 *Suppose $R \in (0, R^+)$.*

As shown below, Assumption 4 guarantees firstly the existence of at least one interior steady state, and secondly the aggregate saving in any steady state to satisfy $s^* \in (0, 1)$; i.e., agents need to borrow funds in order to run the investment project. (15) implies that the steady state saving satisfies the equation $\xi(w^*, s) = Rw^*$. This with (11) implies that the steady state saving solves the following equation

$$\rho(Rw^*) = \begin{cases} \dfrac{1-s}{\lambda} u'\left(\dfrac{s}{w^*}\right) & \text{if} \quad s < 1 - \lambda \\[4mm] u'\left(\dfrac{s}{w^*}\right) & \text{if} \quad s \geq 1 - \lambda, \end{cases} \tag{19}$$

where $\rho(k) := kf'(k)$ is the capital income in production.

Proposition 1 *Suppose Assumptions 1, 2, 3, and 4 are satisfied and let u be such that the function*

$$H(s) := \frac{1-s}{\lambda} u'\left(\frac{s}{w^*}\right) \tag{20}$$

has at most two critical points $\underline{s}^c$ and $\overline{s}^c$ on $(0, w^*)$. Then (19) admits either one or three interior steady states on (0,1).

(a) *if* $w^* < 1 - \lambda$ *or* $w^* \geq 1 - \lambda$ *and* $H(1 - \lambda) \geq \rho(Rw^*)$ *then either a unique steady state or multiple steady states* s_i^* *may exist. Each* s_i^* *solves* $H(s) = \rho(Rw^*)$ *and satisfies* $s_i^* < 1 - \lambda$;

(b) *if* $w^* \geq 1 - \lambda$ *and* $H(1 - \lambda) < \rho(Rw^*)$ *then a solution* $s^{**} = w^* \cdot (u')^{-1} [\rho(Rw^*)]$ *larger than* $1 - \lambda$ *always exists and either none or two solutions of* $H(s) = \rho(Rw^*)$, *smaller than* $1 - \lambda$ *may exist;*

3.2   Local Bifurcation and Stability

This section we start by analyzing the local dynamics around each stead state. Jacobian matrix at any steady state is

$$
J = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} W'\xi_1 & W'\xi_2 \\ W'\xi_1 \dfrac{Rs}{\xi} - W\xi_1 \dfrac{Rs}{\xi^2} & W'\xi_2 \dfrac{Rs}{\xi} + RW \dfrac{\xi - s\xi_2}{\xi^2} \end{pmatrix}, \tag{21}
$$

where W', ξ_1, and ξ_2, are derivatives of functions W and ξ all evaluated at a given steady state. Since steady state pair (w, s) solves $w = W(Rw)$ and $\xi(w, s) = Rw$, it follows that the Trace and Determinant of Jacobian matrix are

$$
T = W'\xi_1 + RW'\frac{s\xi_2}{\xi} + 1 - \frac{s\xi_2}{\xi} \quad \text{and} \quad D = W'\xi_1. \tag{22}
$$

(22) implies that

$$
1 - T + D = \frac{s\xi_2}{\xi}(1 - RW'). \tag{23}
$$

Based on (22), we can evaluate

$$
\frac{s\xi_2}{\xi} = \begin{cases} \dfrac{1}{\epsilon_{f'}(\xi)}\left(\dfrac{s}{1-s} - \epsilon_{u'}\left(\dfrac{s}{w}\right)\right) & \text{if} \quad s < 1 - \lambda \\[4mm] -\dfrac{1}{\epsilon_{f'}(\xi)}\epsilon_{u'}\left(\dfrac{s}{w}\right) & \text{if} \quad s \geq 1 - \lambda, \end{cases} \tag{24}
$$

where

$$
\epsilon_{f'}(k) := -\frac{kf''(k)}{f'(k)} \quad \text{and} \quad \epsilon_{u'}(l) = \frac{lu''(l)}{u'(l)} \tag{25}
$$

are elasticities of marginal product of capital with respect to capital and marginal utility of labor with respect labor respectively. Since $\epsilon_{f'} > 0$ and

$$
\frac{sH'(s)}{H(s)} = -\frac{s}{1-s} + \epsilon_{u'}\left(\frac{s}{w}\right), \tag{26}
$$

13

it follows from (24) that $\dfrac{s\xi_2}{\xi} > 0$ only at intermediate steady state s_2^* (since intermediate steady state solves $H(s) = \rho(Rw^*)$, it follows that $H' < 0$ at the intermediate steady state) and $\dfrac{s\xi_2}{\xi} < 0$ at either unique steady state or at highest and lowest steady states (when there are multiple steady states).

Lemma 3 *Determinant of the Jacobian matrix is always positive, $D > 0$.*

(23) and inequality $RW' < 1$ (see proof of Lemma 2) implies that $1 - T + D < 0$. This with Lemma 3 implies that $T > 1$. As a result, any unique steady state or steady states with highest and lowest saving (in case of multiple solutions) are locally saddle and thus locally determinate. In case of local stability the dimension of the locally stable manifold exactly coincides with the dimension of control variables and thus for any predetermined capital stock K_0 there exists a unique control variable L_0 such that the pair (K_0, L_0) is on the stable manifold and thus is consistent with perfect foresight equilibrium.

In contrast, the middle steady state (whenever it exists) can be locally stable implying local indeterminacy and this existence of continuum of possible choices for L_0 which are consistent with rational expectations. As shown in numerical example, the middle steady state can loose its local stability property either by undergoing flip or Neimark-Sacker bifurcations.

3.3 Global Dynamics

Local stability analysis is not sufficient for fully characterization of model's behavior. Evermore, drawing conclusions based solely on local analysis can be wrong in general. As shown later, the model under consideration exhibits global indeterminacy even when steady states are locally determinate steady states. This is why global dynamics can be dramatically different from local dynamics.[5]

The map M is defined in the set

$$D = \{(s; w) : s \le w \text{ and } s \le 1\}. \tag{27}$$

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[5]Our analysis reinforces the concerns expressed by Grandmont, Pintus, & de Vilder (1998), Christiano & Harrison (1999), Pintus, Sands, & de Vilder (2000), Benhabib & Eusepi (2005), and other.

Such a set is larger than the region where the dynamics (i.e., the forward iterations of the map) take place, since some trajectory may exit D. Then, in order to study the asymptotic behavior of the map, we define the *feasible set* F as the set of points (s_0, w_0) such that $M^n(s_0, w_0) \in D$ for any n. The set F is a subset of D and includes the basins of attraction of the attracting sets of the map, while the set $D \backslash F$ contains all the points that in finite number of iterations reach the set of non definition of the map (*unfeasible trajectories*). The main goal of the global analysis of the map M is the investigation of the topological structure of the set F and the bifurcations that may cause important changes in it. At this aim, we start by studying the invertibility of the map.

Proposition 2 *The map M defined in (13) is invertible.*

The invertibility of the map M is an important result to take into account when we perform the global analysis of M. For instance, it implies that the basins of attraction of any attracting set of the map is a connected set. Furthermore, making use of the inverse map we may obtain the boundary of the set F of feasible trajectories and study how it changes with the change of parameter values. In the rest of this section we describe two possible global bifurcation scenarios, heteroclinic and homoclinic bifurcations, leading to important qualitative changes of perfect foresight dynamics. Existence of such bifurcation scenarios will later confirmed by a numerical example given in the following section.

3.3.1 Heteroclinic Connections of Two Saddle Points

Before introducing the concept of heteroclinic intersection we have to define the stable

$$W^s(p) = \{x : M^n(x) \to p \text{ as } n \to \infty\}, \tag{28}$$

and unstable

$$W^u(p) = \{x : M^n(x) \to p \text{ as } n \to -\infty\}, \tag{29}$$

manifolds of a fixed point, p. If the fixed point $p \in R^2$ is a saddle then the stable (respectively unstable) manifold is a smooth curve through p, tangent at p to the eigenvector of the Jacobian matrix evaluated at p corresponding to the eigenvalue λ with $|\lambda| < 1$ (respectively $|\lambda| > 1$) (see for example Guckenheimer & Holmes (1983),
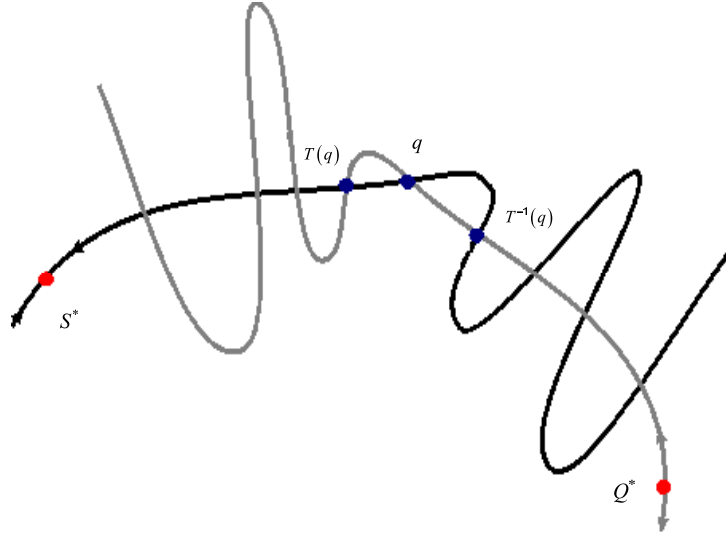
Figure 1: Heteroclinic Connections of Two Saddle Points

p. 18, Theorem 1.4.2). When there exists three steady states in the economy, $S^* = (w^*, s_1^*)$, $E^* = (w^*, s_2^*)$, and $Q^* = (w^*, s_3^*)$, with $s_1 < s_2 < s_3$, then S^* and Q^* are saddles, while an attracting set exists (E^* or some different set) whose basin of attraction may be bounded by the stable manifold of at least a saddle point. In such a case, there can exist a point q in a neighborhood of Q^*, such that $q \in W^u(Q^*) \cap W^s(S^*)$. When such q exists then it is called the *heteroclinic point* from Q^* to S^*. The heteroclinic orbit associated with q is given by

$$\mathcal{O}(q) = \{..., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}, \tag{30}$$

where $q_n = M^n(q) \to S^*$ and $q_{-n} = M^{-n}(q) \to Q^*$.

The occurrence of the heteroclinic bifurcation, as shown in Figure 1, involving the saddle S^* and Q^* is reflected in qualitative change in the basin of attraction of E^*, since after the disappearance of the heteroclinic points one of saddle point may belong (or no longer belong) to its boundary and a heteroclinic connection between such a saddle and E^* appears (or disappears). Global indeterminacy takes place and the economy can move smoothly from low/hight to high/low steady state only due to change in a self-fulfilling expectation, whenever a heteroclinic connection occurs.

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3.3.2 Homoclinic Bifurcation of a Saddle Point

Homoclinic bifurcation, which plays an important role in understanding complexity of the global dynamics, is one of the most fundamental concepts in nonlinear discrete dynamical systems. Let Q^* be a saddle point. A point $q \neq Q^*$ is called a homoclinic point if it is a point of intersection of the stable and unstable manifolds, i.e., $q \in W^u(Q^*) \cap W^s(Q^*)$. If these manifolds intersect transversely at q, then q is called a transversal homoclinic point; if they intersect tangentially at q, then q is called a point of homoclinic tangency.

The homoclinic points accumulate in a neighborhood of Q^* and their existence, intuitively, can be understood observing that the forward orbit of q and the backward sequence is also made up of homoclinic points, and converge to Q^*. The union of the forward and backward orbit of a homoclinic point q is called a *homoclinic orbit of Q^**, or *orbit homoclinic* to Q^*:

$$\tau(q) = \{..., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}, \tag{31}$$

where $q_n = M^n(q)$ and $M^n(q) \to Q^*$ while $q_{-n} = M^{-n}(q)$ and $M^{-n}(q) \to Q^*$. More generally, an orbit homoclinic to a cycle approaches the cycle asymptotically both through forward and backward iterations, so that it always belongs of the stable and unstable sets of the cycle. The appearance of homoclinic orbits of a saddle point Q^* corresponds to a *homoclinic bifurcation* and implies a very complex configuration of stable and unstable manifold of the saddle, W^s and W^u, called *homoclinic tangle*, due to their winding in proximity of Q^*. The existence of a homoclinic tangle is often related to a sequence of bifurcations occurring in a suitable parameter range, and qualitatively shown in Figure 2. First, a homoclinic tangency between one branch, say ω_1, of the stable set of the saddle and one branch of the unstable one, say α_1, followed by a transversal crossing between ω_1 and α_1, that gives rise to a homoclinic tangle, and by a second homoclinic tangency of the same stable and unstable branches, occurring at opposite side with respect to the previous one, which closes the sequence. Furthermore, in the parameter range in which the manifolds intersect transversely, an invariant set exists such that the restriction of the map to this invariant set is chaotic, that is, the restriction is topologically conjugated with the shift map, as stated in the Smale-Birkhoff Theorem (see for example in Guckenheimer & Holmes (1983), Mira (1987), Wiggins (1988), Bai-Lin (1989), Kuznetsov (1983)). Thus we say that the map possesses a *chaotic repellor*, made up of infinitely many (countable) repelling cycles
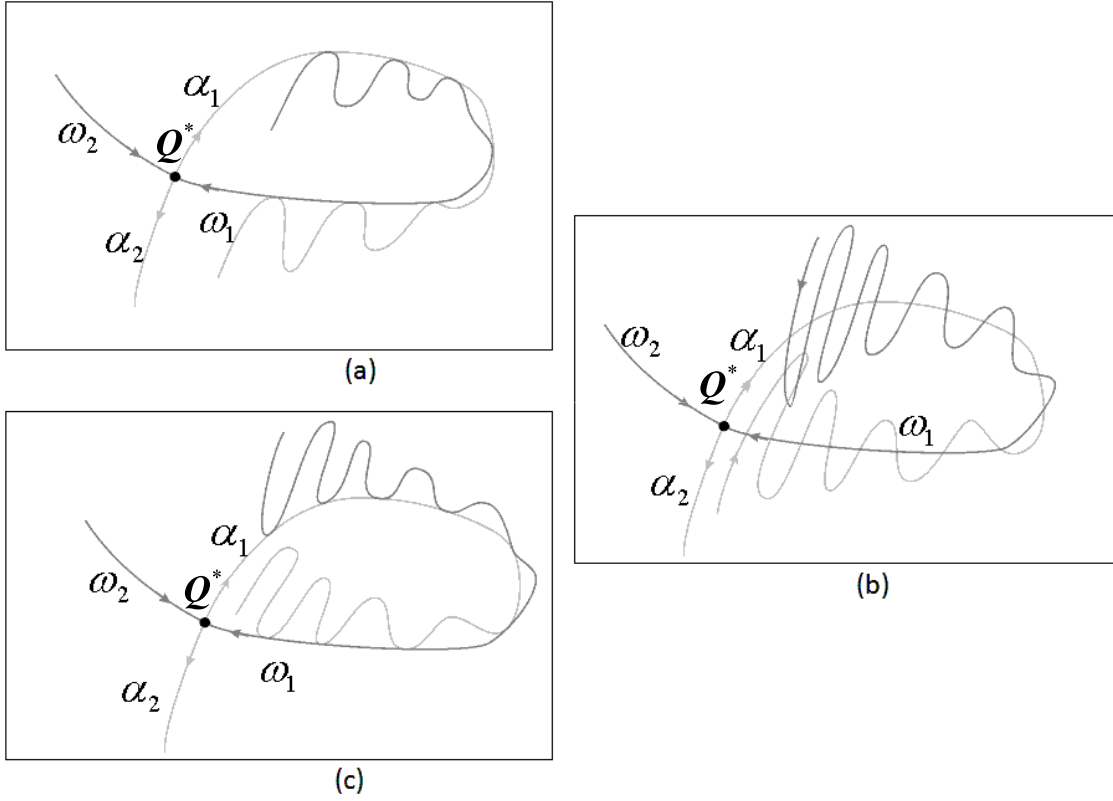
Figure 2: Homoclinic Bifurcation of a Saddle Point

and uncountable aperiodic trajectories. In the case shown in Figure 2 such a chaotic repellor certainly exists after the first homoclinic tangency and disappears after the second one. Before and after the homoclinic tangle (i.e. before the first and after the last homoclinic tangencies), the dynamic behavior of the two branches involved in the bifurcation must differ: The invariant set towards which α_1 tends to (or equivalently the ω-limit set of the points of α_1) and the invariant set from which ω_1 comes from (or equivalently the α-limit set of the points of ω_1) before and after the two tangencies are different, as the comparison of Figures 2.(a) – 2.(c) shows. Thus we can detect the occurrence of such a sequence of bifurcations looking at the asymptotic behavior of W^s and W^u. Whenever homoclinic point appears then as above global indeterminacy takes place and the economy can fluctuate around the determinate steady state involved in the homoclinic bifurcation even only through fluctuations of a self-fulfilling expectation.

4 Numerical Example

To fix ideas, we consider a parameterized version of the above economy. Suppose that production and marginal utility functions are:

$$f(k) = Ak^\alpha \quad \text{and} \quad u'(l) = \beta^{\frac{1}{\theta}} \left(\frac{l}{1-l} \right)^{\frac{1-\theta}{\theta}}, \tag{32}$$

where $A > 0$ is the Hicks' neutral productivity level, $\alpha \in (0,1)$ is the capital share in production, and $\theta \in (0,1)$ is the parameter measuring the elasticity of labor supply. The above production function satisfies all the assumptions given in 1, 2, and 3. It is also clear that $u' > 0$, $u'' > 0$, $u'(0) = 0$, and $u'(1) = \infty$ when $\theta \in (0,1)$.

We fix parameters values to levels given in Table 1. We take these values as benchmark values and keep them constant unless it is otherwise indicated.

A	α	R	λ	β	θ
2.88	0.33	0.12	0.05	0.076	0.78

Table 1: Standard parameter set

Existence of Heteroclinic Connections of Two Saddle Points

When $\theta = 0.78$ then there exists a unique steady state $Q^* = (0.938466, 0.938173)$. Q^* is saddle and its stable manifold separates the region of unfeasible (dark grey region) from the feasible one (light grey region). No bounded trajectories exist in such a case, since the feasible set F contains only divergent trajectories, as shown in Figure 3.(a). Bounded trajectories emerge when the parameter θ increases to $\theta = 0.80$, due to a saddle-node bifurcation causing the appearance of two fixed points, a saddle S^* and a stable node E^*, coexisting with Q^*. Immediately after the saddle-node bifurcation, the basin of attraction, $\mathcal{B}(E^*)$, of the stable steady E^* is bounded by the two branches of the stable manifold of the saddle S^*, while the stable manifold of Q^* persists in separating the feasible and unfeasible sets, as shown in Figure 3.(b). The branch of the unstable manifold of Q^* that enters the feasible set F (that is, α_Q^1) goes to infinity as well as the branch α_S^1 of the unstable manifold of S^*. The stable node E^* is reached by α_S^2, the second branch of the unstable manifold[6] of S^*. When the steady state

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[6] Observe that at this parameter constellation all the fixed points belong to the region where the borrowing constraint is binding.
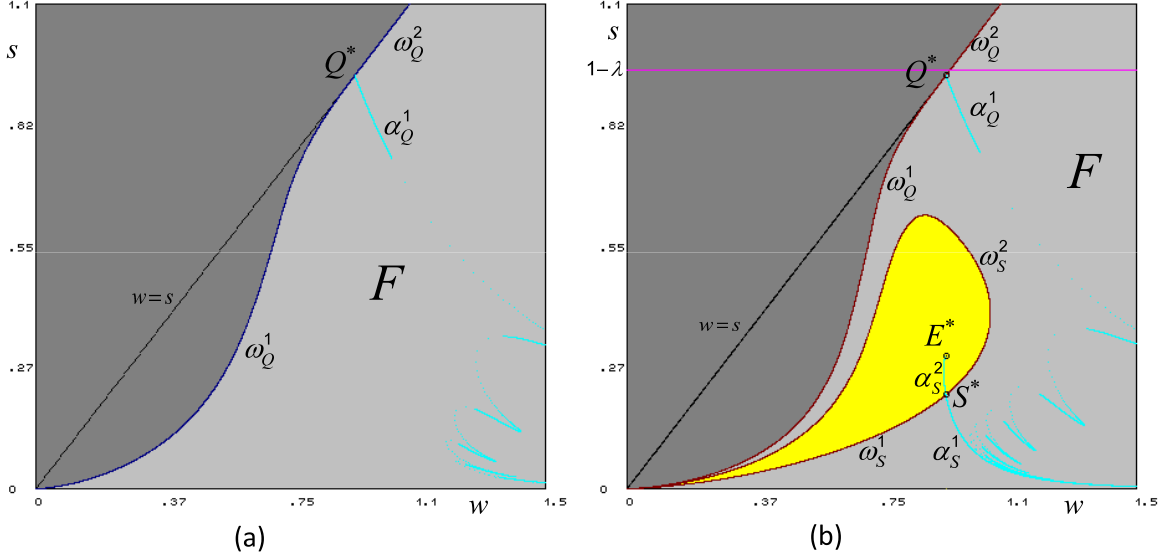
Figure 3: Saddle-node Bifurcation Causing the Appearance of Three Fixed Points

is unique it is a saddle and then locally and globally determinate. The occurrence of the saddle-node bifurcation causes the appearance of two new steady states, one locally determinate (the saddle S^*) and the second locally indeterminate (the stable node E^*). Moreover a heteroclinic connection between S^* and E^* exists, made up by the branch of the unstable manifold of the saddle that converges to E^* (this fact always occurs if the saddle belongs to the boundary of the basin of attraction of the attracting steady state). The existence of multiple steady states immediately involves the indeterminacy of the perfect foresight equilibrium. Moreover we can also observe that even if we restrict our study to a small neighborhood of the locally determinate S^*, the equilibrium is indeterminate due to the coexistence of paths converging to E^*.

As the parameter θ increases further, the branch ω_S^2 starts to oscillate in proximity of the saddle Q^* and the branch α_Q^1 is closed to the boundary of the basin of E^*. This is a preliminary phase preparing the appearance of *heteroclinic points*. Figure 4 demonstrates the existence of heteroclinic points from Q^* to S^*, so that heteroclinic orbits exists. In this figure, we can appreciate the appearance of heteroclinic points
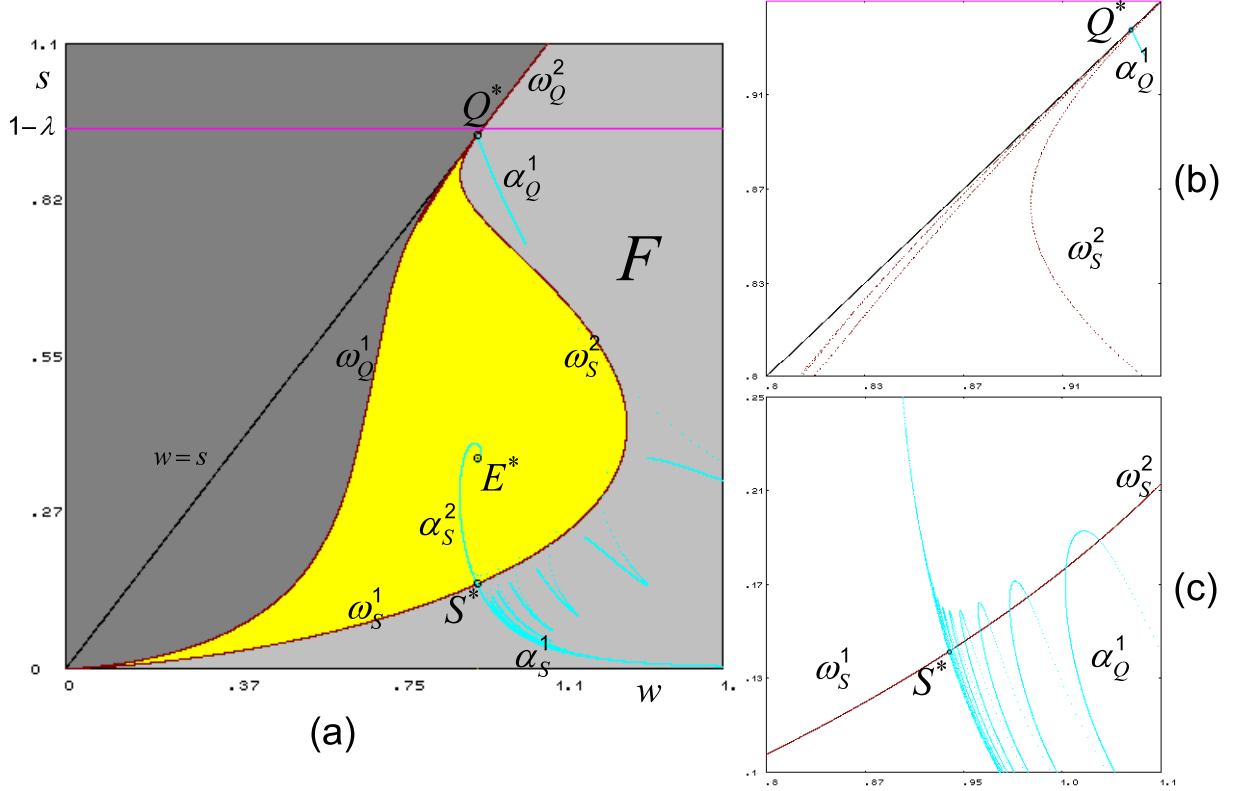
20

Figure 4: Emergence of Heteroclinic Points from Q^* to S^*

from saddle Q^* to the saddle S^*, caused by the contact of the branch α_Q^1 of the unstable manifold of Q^* with the branch w_S^2 of the stable manifold S^*. In particular, in the enlargement of 4.(b) and 4.(c) only the branches involved in the bifurcation are represented and we can observe that there exist points of the unstable manifold of Q^* converging to E^*, now turned in stable focus.

The heteroclinic points appear when the two branches α_Q^1 and w_S^2 have a tangential contact, exist in a certain parameter range and disappear when a second tangential contact take place at opposite side with respect to the previous one. When the heteroclinic points are disappeared, we observe a first important qualitative change in the basin of attraction of E^*, as shown in Figure 5.(a). Indeed, now both the saddles Q^*
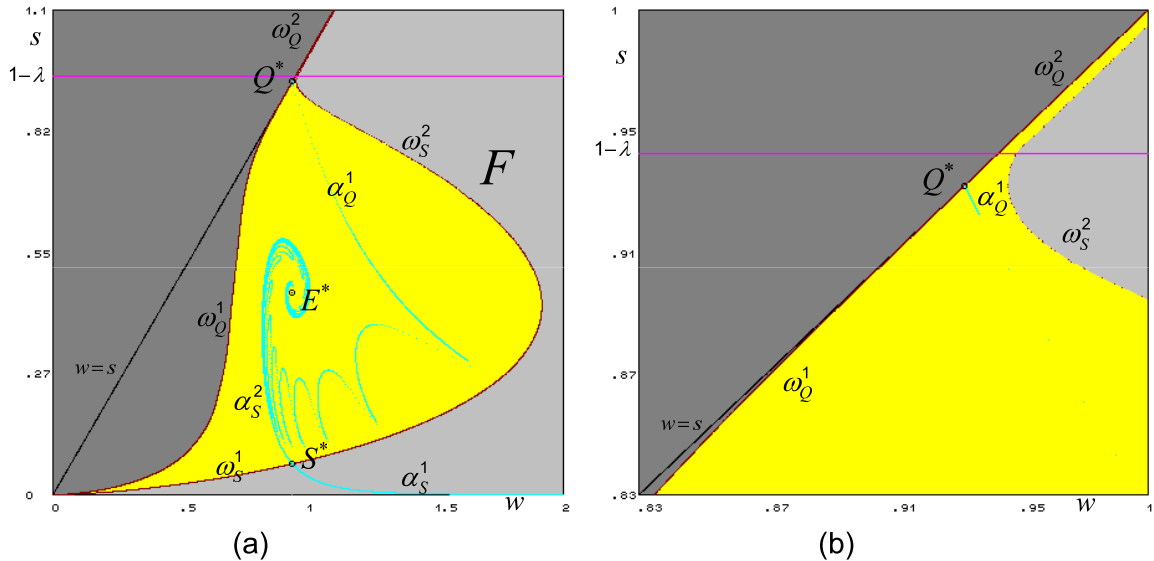
Figure 5: Disappearance of Heteroclinic Points

and S^* belong to the frontier of the set of bounded trajectories (converging to E^*), and a branch of the stable manifold of Q^* separates the basin of E^*, $\mathcal{B}(E^*)$, from the un-feasible trajectories and the one of S^* separates $\mathcal{B}(E^*)$ from the divergent trajectories. Moreover, we can observe that there exists a stripe of points at which the constraint is binding (that is, above $1 - \lambda$) that give rise to trajectories converging to the stable equilibrium E^* (see the enlargement in Figure 5.(b). As a consequence of the occurred heteroclinic bifurcation, the saddle Q^* belongs to the basin of attraction of E^* and then a heteroclinic connection exists between Q^* and E^*, made up by a branch of the unstable manifold of Q^*. The steady state is still locally determinate, but in a small neighborhood of Q^* the equilibrium is indeterminate, since there exist infinitely many paths converging to E^*. Comparing Figures 3.(a) and 5.(a), the effect of the occurred heteroclinic bifurcation can be appreciated.

A second qualitative change in the boundary of $\mathcal{B}(E^*)$ occurs when the parameter θ is further increased and it is still due to a heteroclinic bifurcation involving the two saddles points Q^* and S^*. Indeed, as we can observe in Figure 6, at a certain parameter value,
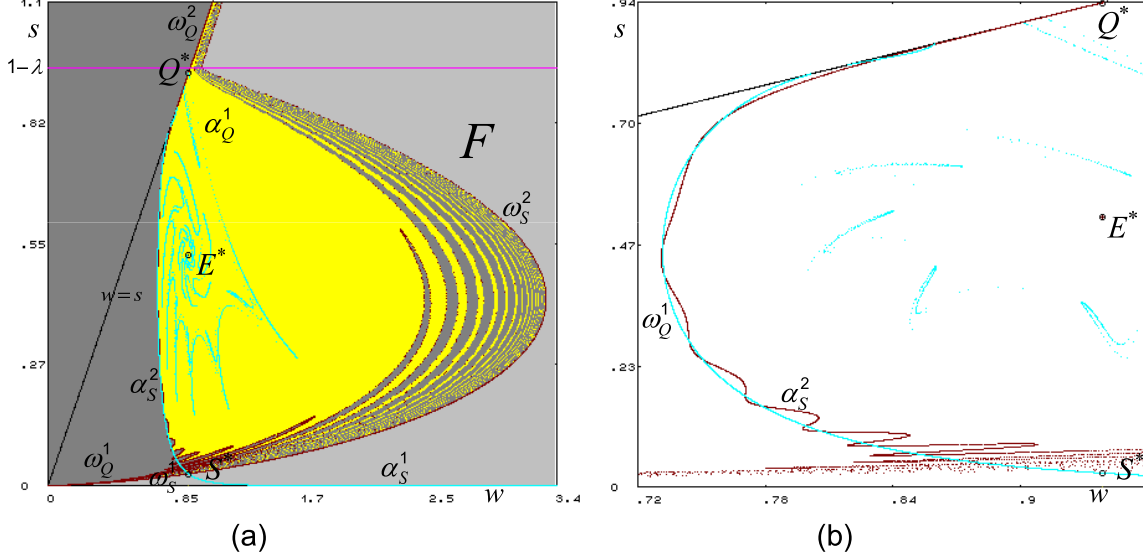
Figure 6: Emergence of Heteroclinic Points from S^* to Q^*

heteroclinic points from S^* to Q^* appear, associated with the contact of the branch α_Q^1 of the unstable manifold of S^* with the branch ω_S^2 of the stable manifold of Q^*.

Even the phase-space of Figure 5.(a) allows us to obtain interesting comments on the determinacy of the two saddle points. Due to the existence of heteroclinic points from S^* and Q^*, we observe that in the neighborhood of the locally determinate steady state S^* there exist a path converging to Q^* (see Figure 6.(b)), besides the one converging to S^* and those converging to E^*. Then we can conclude that in the neighborhood of S^* the equilibrium is not determinate, since there exist equilibrium paths leading to different determinate steady states and to the indeterminate one. Moreover, comparing Figures 5.(a) and 6.(a), we may observe that in the latter case the equilibrium path leading to Q^* may be such that $\omega_0 > \omega^*$ even if $s_0 < 1 - \lambda$.

As a consequence of the bifurcation just described we obtain that, when the heteroclinic points disappear, the saddle S^* no longer belongs to the boundary of the basin
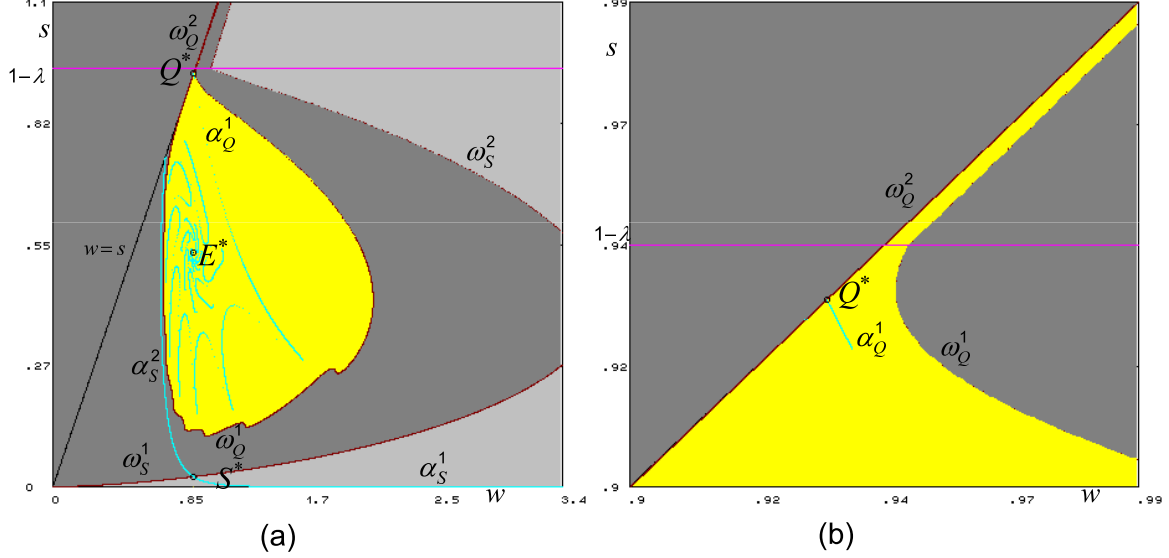
23

Figure 7: Disappearance of Heteroclinic Points

of attraction of E^*, the stable manifold of S^* separating unfeasible and divergent tra-
jectories (see Figure 7). Moreover the basin $\mathcal{B}(E^*)$ is reduced in size, being bounded
by the stable manifold of Q^*, and the feasible bounded trajectories are separated by
the feasible divergent ones. Due to the occurred heteroclinic bifurcation, now in a
small neighborhood of S^* the equilibrium is determined, since there exists only a path
converging to the steady state S^* (this is in contrast with previous analyzed situations).

Existence of Homoclinic Bifurcation of a Saddle Point

Let $\lambda = 0.124$ and $\beta = 0.25$. We still proceed increasing the value of θ. For small values
of θ ($\theta < 0.65$), there exists a unique fixed point, a saddle S^* whose stable manifold
separates feasible and unfeasible trajectories. As in the previous example, generic
feasible trajectories are all divergent. As θ increases, a saddle-node bifurcation occurs
and two further fixed points appear, a saddle Q^* and an unstable node E^*, both located
in the "old" unfeasible region. This means that, as a consequence of the saddle-node
bifurcation, the unfeasible set becomes a disconnected region (with connected closure)
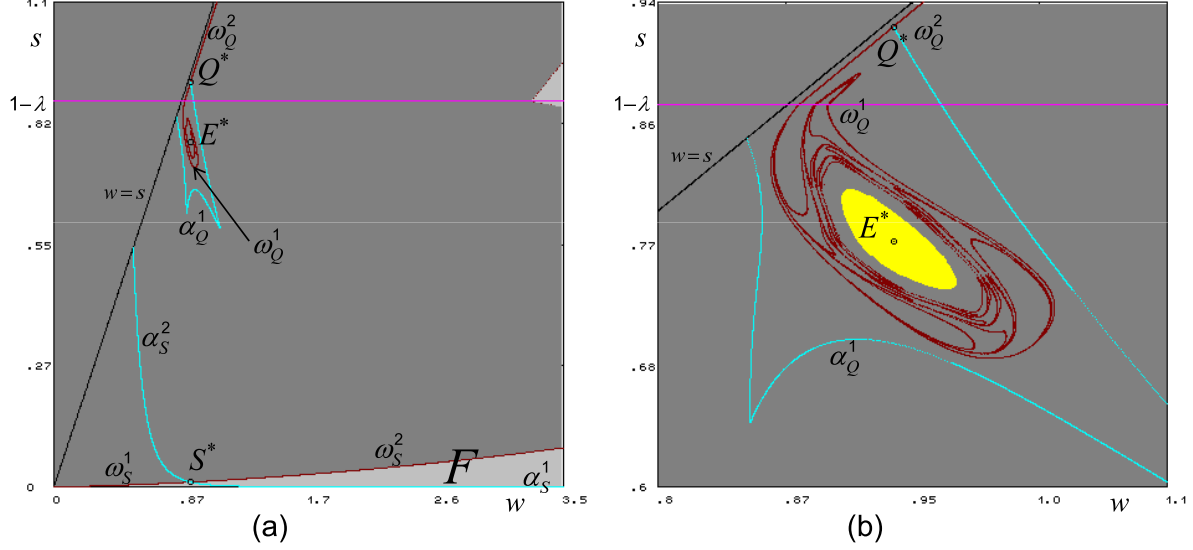
24

Figure 8: Subcritical Neimark-Sacker Bifurcation Causing the Appearance of a Repelling Closed Curve.

due to the existence on its boundary of the two fixed points and of the branch ω_Q^1 of the stable manifold of Q^* connecting the saddle with E^*. This situation is shown in Figure 8.(a), where E^* is turned in repelling focus. The stable manifold of S^* still separates feasible and unfeasible trajectories and the generic trajectory is divergent. It is worth to observe that the saddle Q^* belongs to the region where the borrowing constraint is not binding (case not considered in the previous example). Bounded trajectories emerge as consequence of a subcritical Neimark-Sacker of the unstable fixed point E^* occurring at $\theta \approx 0.85$. As shown in Figure 8.(b), where a smaller portion of the state place has been represented, after the occurrence of such a bifurcation the basin of attraction of the stable fixed point E^* (yellow points) is bounded by a repelling closed curve Γ_u, appeared at the bifurcation value. The feasible bounded trajectories are separated from the feasible divergent one and the branch ω_Q^1 of the stable manifold of the Q^* comes from Γ_u (i.e., admits Γ_u as α-limit set). At the parameter constellation of Figure 8.(a) there exists a heteroclinic connection between the saddle Q^* and the unstable focus E^*. Both the steady states are locally determinate. But in the neighborhood of E^*,
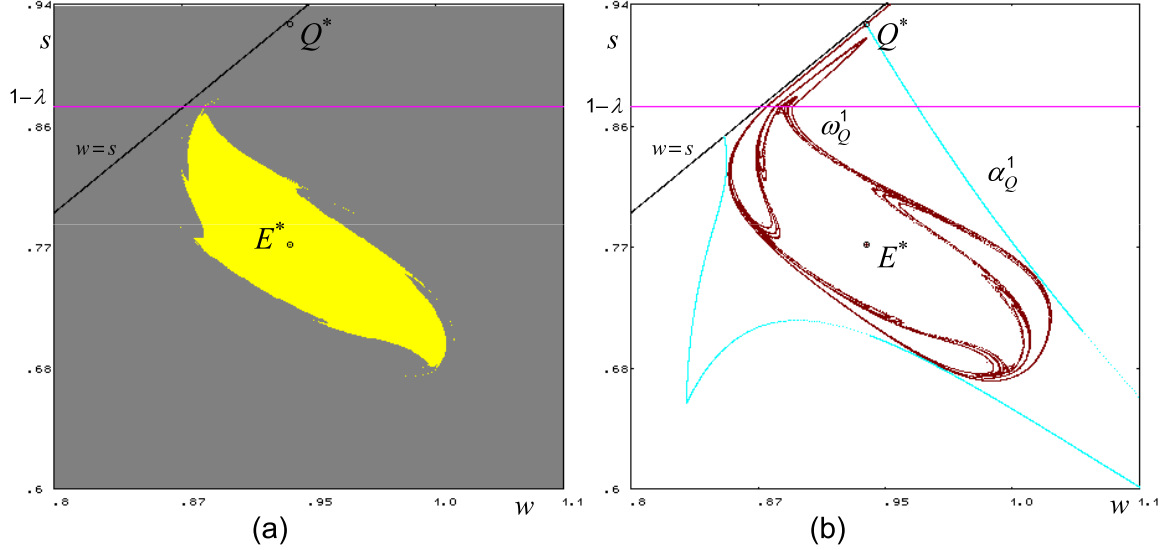
Figure 9: Emergence of a Repelling Cycles

the equilibrium is indeterminate since, besides E^*, there exists also a path converging to Q^*. Furthermore, before converging to Q^*, such a path fluctuates around the steady state E^*. In Figure 8.(b) the stationary equilibrium E^* is locally indeterminate and the infinitely many equilibrium paths reaching it belong to a quite small set, bounded by the repelling closed curve. No heteroclinic connection exists between the three steady states. Then in small neighborhoods of the two locally determinate steady states, the perfect foresight equilibrium is determinate as well.

As the parameter θ is further increased we observe that the curve Γ_u becomes more and more irregular as shown in Figure 9.(a), loosing its smooth property. This fact can be explained by a progressive appearance on it of many different repelling and saddles cycles, preparing the appearance of a chaotic repellor. Indeed if we look at the stable and unstable manifold of the saddle point Q^*, represented in Figure 9.(b), we observe that the branch ω_Q^1 and α_Q^1 are very closed each other, suggesting that a homoclinic bifurcation is close to occur. The phase-space shown in Figure 10 is obtained at a θ value belonging to the parameter range where the homoclinic tangle
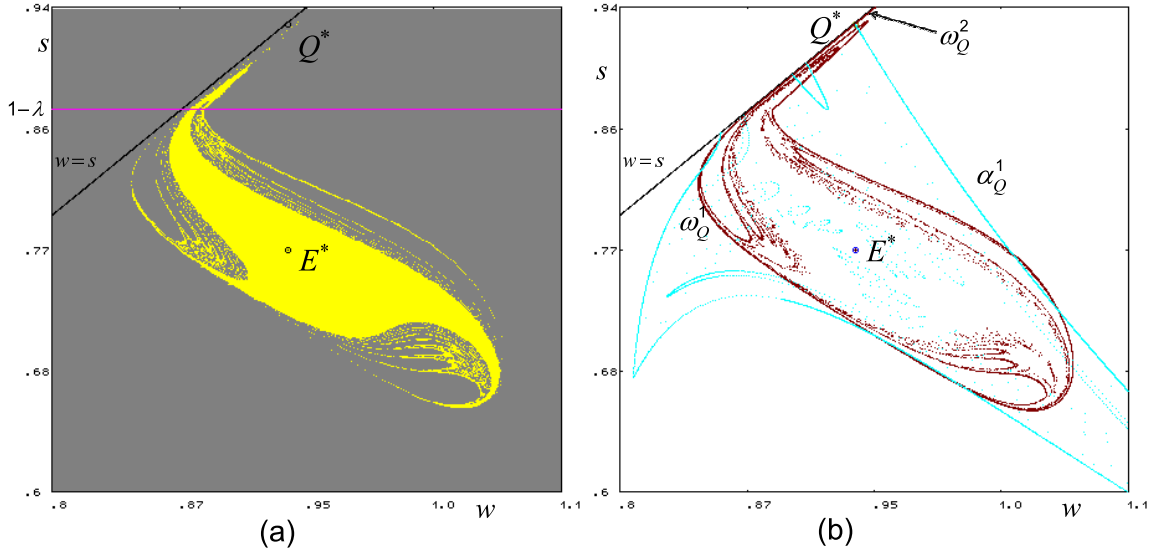
26

Figure 10: Occurrence of a Homoclinic Bifurcation

develops. In particular, in Figure 10.(a) the chaotic repellor existing on the boundary of the basin of attraction of the stable focus E^* is clearly evident, while in Figure 10.(b) the transversal crossing of the branches ω_Q^1 and α_Q^1 is shown. We also observe that there are points of the unstable manifold of Q^* converging to E^*. The occurrence of the homoclinic bifurcation causes the appearance of equilibrium paths converging to E^* in proximity of Q^*, still locally determinate. Then in a small neighborhood of Q^* the equilibrium is indeterminate. Furthermore in such neighborhoods there may exist periodic points belonging to the existing infinitely many cycles of any period as well as points belonging to a chaotic repellor, due to the persistent homoclinic tangle.

As a consequence of the homoclinic bifurcation just described we obtain the disappearance of the repelling closed curve Γ_u, the boundary of the set of bounded trajectories being given by the stable manifold of the saddle Q^*. This fact is illustrated in Figure 11, where the fixed point E^*, after been turned in stable node, has lost its stability trough a flip bifurcation. Now the generic bounded trajectories converge to a cycle of period 2, whose basin of attraction is represented in yellow. The enlargement of Figure
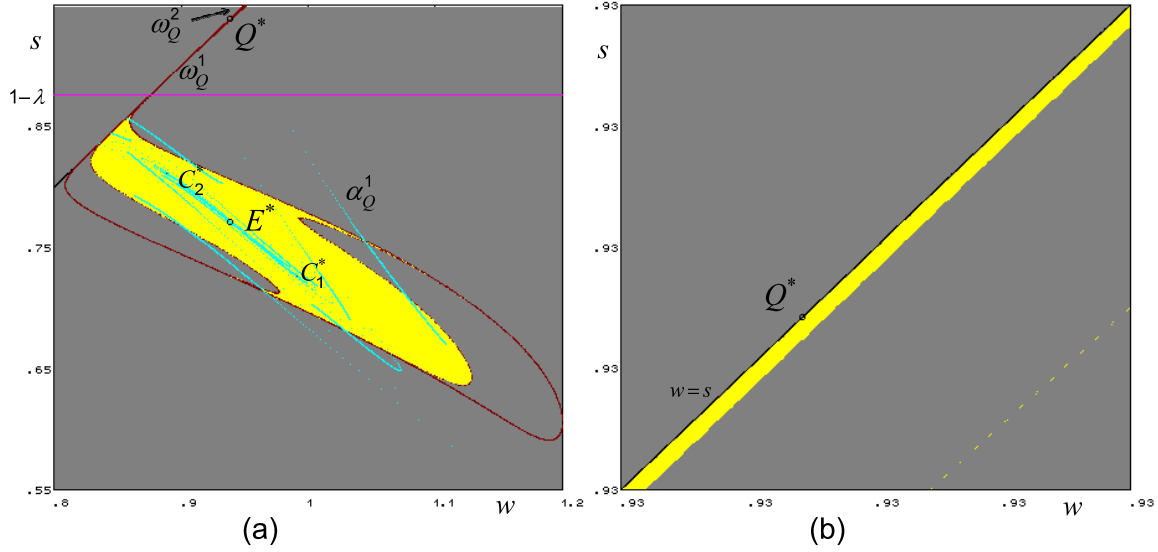
27

Figure 11: Disappearance of the Repelling Closed Curve Γ_u

11.(b) shows a very narrow stripe of points converging to the 2-cycle and belonging to the region where the constraint is binding and allows us to conclude that the closed repelling curve is disappeared (even if at such a parameter constellation the homoclinic tangle is not yet closed).

As the parameter θ is further increased the periodic point of the attracting cycle of period 2 move more and more towards the boundary of their basin of attraction, until they reach it. Then a final bifurcation takes place, after which no bounded trajectories exist, unless a set of zero measure that contains the three steady states all locally determinate.

5   Summary and Conclusions

The main message of the paper is the demonstration of the possibility of endogenous fluctuations due to a self-fulfilling expectation in an economy with a) constant return to scale technology, b) sufficiently high elasticity of substitution between capital and labor inputs and c) gross substitutability between the current period leisure and the next period consumption. A fully neoclassical growth model with capital accumulation is modified to include imperfect investor protection and minimum capital investment requirements. In this setting we explore the global properties of the two-dimensional dynamical system generated by the model. Without imposing any ad-hoc non-linearities, we get a straightforward route to self-sustained oscillations. The story implied by the model is: when capital stock is low, agents increase their labor supply because they expect high deposit rate. This leads in increase of labor supply and saving. As a result, credit market imperfection weakens and portion of individuals who start a new investment projects increases. This implies high next period capital shock and high deposit rate. When capital stock is high then, agents do decrease their labor effort because their expectations about next period deposit rate is sluggish. Low labor supply causes low output and low savings, which translates into tight credit market, fewer number of investment projects and low next period capital stock.

The investigation in this paper has shed some additional light on the occurrence of heteroclinic and homoclinic connections under the assumption of perfect foresight. In particular, the equilibrium is globally indeterminate (even if the steady states are all locally determinate) when multiple steady states exist. As a result one can chose the initial value of the control variable in order to obtain an equilibrium converging to any steady state if agents do not deviate from the optimal trajectory once the initial condition is agreed upon. If agents are free to choose between different trajectories then the local determinacy of the steady state is not sufficient in order to understand if the perfect foresight is determinate.

We have stressed this fact through a numerical example, where, due to the nonlinearity of the model, we have shown that the simple analysis of the local determinacy of the steady states may be not sufficient, and some times misleading, in order to understand if the perfect foresight equilibrium is determinate, even when we restrict the choice of the control variable to a small neighborhood of a steady state. Indeed, the existence of some heteroclinic connection between a saddle steady state (locally determinate)

and a different stationary equilibrium (either locally determinate or indeterminate) is associated with global indeterminacy, since in any neighborhood of one determinate steady state there exist also bounded equilibrium paths converging to the second one. Furthermore, the possible occurrence of homoclinic bifurcations involving two saddle steady states causes noticeable qualitative changes in the dynamical behavior of the perfect foresight equilibrium, that may be related to global indeterminacy. Indeed, as it is well known from the theory of dynamical systems, in the parameter range in which the associated homoclinic tangle develops, infinitely many cycles of any period and a chaotic repellor exist, so that the equilibrium may fluctuate even far from the steady states.

The existence of multiple equilibria around the determinate steady states implies that the "animal spirit" can be a driving force of business cycle fluctuations and the initial conditions can have limited effect on the eventual fate of the economy. In other words, similar economies may end up with different accumulation patterns on the sole grounds of different expectations. This paper can explain why, some countries but not others, can fall into poverty and how differences in the initial expectation may be responsible for drastic differences in the long-run standard of living.

6 Appendix

Proof of Lemma 1: By definition (7), it follows that

$$S\left[W(k)\right] \equiv \frac{W(k)}{k}. \tag{33}$$

When Assumption 1 is satisfied then the right hand side of (33) is strictly decreasing. Since W is strictly increasing function, claim of the lemma follows from (33).

QED.

Proof of Lemma 2: It follows from Assumption 1 that

$$\frac{kW'(k)}{W(k)} < 1 \quad \Rightarrow \quad \frac{RW^*(R)W'\left[RW^*(R)\right]}{W\left[RW^*(R)\right]} < 1. \tag{34}$$

Since, by definition, $W^*(R) \equiv W\left[RW^*(R)\right]$, it follows from (34) that $RW'\left[RW^*(R)\right] < 1$. Direct differentiation of the following identity, $W^*(R) \equiv W\left[RW^*(R)\right]$, with the last inequality implies the claim of the lemma.

QED.

Proof of Lemma 3: Since

$$D = W'\xi_1 \quad \text{and} \quad \xi_1 = -\frac{u'\left(\frac{s}{w}\right)}{Rw^2 f''(\xi)}\left(1 + e_{u'}\left(\frac{s}{w}\right)\right) > 0. \tag{35}$$

(35) with monotonicity property of W implies the claim of the lemma.

QED.

Proof of Proposition 1: Property of u' with (20) implies that $H(0) = H(1) = 0$ and $H(w^*) = -\infty$ when $w^* > 1$ and $H(w^*) = \infty$ when $w^* \leq 1$. If $w^* < 1 - \lambda < 1$ then it follows from existence of at most two critical points of H on $(0, w^*)$ that there can exits either one or three steady states s_i^* solving $H(s) = \rho(Rw^*)$ and thus satisfying $s_i^* < 1 - \lambda$. Situation is similar when $w^* \geq 1 - \lambda$ and $H(1 - \lambda) \geq \rho(Rw^*)$. When a unique steady state exists then either $s^* < \underline{s}^c$ or $\overline{s}^c < s^* < 1 - \lambda$, and when multiple steady states exist then $s_1^* < \underline{s}^c < s_2^* < \overline{s}^c < s_3^* < 1 - \lambda$.

If $w^* \geq 1 - \lambda$ and $H(1 - \lambda) < \rho(Rw^*)$ then steady state saving solves $u'\left(\frac{s}{w^*}\right) = \rho(Rw^*) > H(1 - \lambda) = u'\left(\frac{1-\lambda}{w^*}\right)$ and thus $s^{**} = w^* \cdot (u')^{-1}\left[\rho(Rw^*)\right] > 1 - \lambda$ always exists. Existence of other steady states below $1 - \lambda$ depends on shape of H as well as on value of w^*.

Proof of Proposition 2: In order to show the invertibility of the map M, we show that the system of equations

$$
\begin{cases}
x = W\left[\xi(w,s)\right] \\[2mm]
y = u\dfrac{Rs}{\xi(w,s)}
\end{cases}
\tag{36}
$$

has a unique solution with respect to (w,s) for any pair $(x,y) \in \mathbb{R}_+^2$. Equation (36) implies

$$
\begin{cases}
x = W\left[\dfrac{x}{y}Rs\right] \\[2mm]
\xi(w,s) = \dfrac{x}{y}Rs.
\end{cases}
\tag{37}
$$

Since the function W is monotonically increasing with $W(0) = 0$ and $W(\infty) = \infty$ it follows that the first equation of the system (37) admits a unique solution $\widehat{s}$. This with the second equation of the system implies

$$
\xi(w,\widehat{s}) = \dfrac{x}{y}R\widehat{s}.
\tag{38}
$$

Based on (21), equation (38) can be rewritten as

$$
\dfrac{\widehat{s}}{w}u'\left(\dfrac{\widehat{s}}{w}\right) =
\begin{cases}
\dfrac{\lambda}{1-\widehat{s}}R\widehat{s}f'\left(\dfrac{x}{y}R\widehat{s}\right) & \text{if } \widehat{s} \in (0, 1-\lambda) \\[4mm]
R\widehat{s}f'\left(\dfrac{x}{y}R\widehat{s}\right) & \text{if } \widehat{s} \in (0, 1-\lambda).
\end{cases}
\tag{39}
$$

Since the left hand side of (39) is a continuous and strictly decreasing function (defined in $(\widehat{s}, +\infty)$ and ranging in $(0, +\infty)$) in w and the right hand side is a finite constant, it follows that (39) admits a unique solution $\widehat{w}$. This implies invertibility of the map M.

QED.

References

Aghion, Philippe; & Bolton, Patrick; (1997): "A Theory of Trickle-Down Growth and Development", *Review of Economic Studies*, 64, 151–72.

Azariadis, Costas; & Drazen, Allan; (1990): "Threshold Externalities in Economic Development", *Quarterly Journal of Economics*, 105, 501–26.

Bai-Lin Hao; (1989): *Elementary Symbolic Dynamics and Chaos in Dissipative Systems*, World Scientific, Singapore.

Banerjee, Abhijit, V.; & Newman, Andrew, F.; (1993): "Occupational Choice and the Process of Development", *Journal of Political Economy*, 101, 274–98.

Barinci, Jean-Paul; & Cheron, Arnaud; (2001): "Sunspots and the Business Cycle in a Finance Constrained Economy", *Journal of Economic Theory*, 97, 30–49.

Benhabib, Jess; & Eusepi, Stefano; (2005): "The Design of Monetary and Fiscal Policy: A Global Perspective", *Journal of Economic Theory*, 123, 40–73.

Benhabib, Jess; Schmitt-Grohé, Stephanie; & Uribe, Martín; (2001): "The Perils of Taylor Rules", *Journal of Economic Theory*, 96, 40–69.

Cazzavillan, Guido; Lloyd-Braga, Teresa; & Pintus, Patrick, A.; (1998): "Multiple Steady States and Endogenous Fluctuations with Increasing Returns to Scale in Production", *Journal of Economic Theory*, 80, 60–107.

Chiappori, P., A., & Guesnerie, R.; (1991): "Sunspot Equilibria in Sequential Markets Model." In Werner Hildenbrand and Hugo Sonnenschein, eds., *Handbook of Mathematical Economics*, Vol. 4. Amsterdam: North-Holland, pp. 1683–1762.

Christiano, Lawrence, J.; & Harrison, Sharon, G.; (1999): "Chaos, Sunspots and Automatic Stabilizers", *Journal of Monetary Economics*, 44, 3–31.

Freeman, Scott; (1996): "Equilibrium Income Inequality Among Identical Agents", *Journal of Political Economy*, 104, 10473–64.

Galor, Oded; & Zeira, Joseph; (1993): "Income Distribution and Macroeconomics", *Review of Economic Studies*, 60, 35–52.

GLAVAN, BOGDAN; (2008): "Coordination Economics, Poverty Traps, and the Market Process: A New Case for Industrial Policy?", *Independent Review*, 13, 225–43.

GOMIS-PORQUERAS, PERE: & HARO, ÀLEX; (1998): "Global bifurcations, Credit Rationing and Recurrent Hyperinflations", *Journal of Economic Dynamics and Control*, 31, 473–91.

GRANDMONT, JEAN-MICHEL; (1998): "Introduction to Market Psychology and Nonlinear Endogenous Business Cycles", *Journal of Economic Theory*, 80, 1–13.

GRANDMONT, JEAN-MICHEL; PINTUS, PATRICK; & DE VILDER, ROBIN; (1998): "Capital-Labor Substitution and Competitive Nonlinear Endogenous Business Cycles", *Journal of Economic Theory*, 80, 14–59.

GUCKENHEIMER, JOHN; & HOLMES, PHILIP; (1983): *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York.

KIKUCHI, TOMOO; & VACHADZE, GEORGE; (2009): "Self-fulfilling Expectations in an OLG Model with Credit Market Imperfection", *Manuscript*.

KUZNETSOV, YURI, A.; (2003): *Elements of Applied Bifurcation Theory*, Springer-Verlag, New York.

MATSUYAMA, KIMINORI; (1991): "Increasing Returns, Industrialization, and Indeterminacy of Equilibrium", *Quarterly Journal of Economics*, 106, 617–50.

MATSUYAMA, KIMINORI; (2000): "Endogenous Inequality", *Review of Economic Studies*, 67, 743–59.

MATSUYAMA, KIMINORI; (2004): "Financial Market Globalization, Symmetry-Breaking and Endogenous Inequality of Nations", *Econometrica*, 72, 853–84.

MATSUYAMA, KIMINORI; (2006): *Poverty Traps*, New Palgrave: A Dictionary of Economics, 2nd edition, Macmillan.

MATSUYAMA, KIMINORI; (2007): "Credit Traps and Credit Cycles", *American Economic Review*, 97, 503–516.

MIRA, CHRISTIAN; (1987): *Chaotic Dynamics From the One-Dimensional Endomorphism to the Two-Dimensional Difeomorphism*, World Scientific, Singapore.

PINTUS, PATRICK; SANDS, DUNCAN; & DE VILDER, ROBIN; (2000): "On the Transition from Local Regular to Global Irregular Fluctuations", *Journal of Economic Dynamics and Control*, 24, 247–72.

REICHLIN, PIETRO; (1986): "Equilibrium Cycles in an Overlapping Generations Economy with Production", *Journal of Economic Theory*, 40, 89–102.

SAIDI, AURÉLIEN; (2008): "Balnced-Budget Rule, Distortionary Taxes and Aggregate Instability: A Comment", *EconomiX Workings Papers*, 2008–44.

SLOBODYAN, SERGEY; (2005): "Indeterminacy, Sunspots, and Development Traps", *Journal of Economic Dynamics and Control*, 29, 159–85.

WIGGINS, STEPHEN; (1988): *Global Bifurcations and Chaos, Analytical Methods*, Springer-Verlag, New York.

WOODFORD, MICHAEL; (1986): "Stationary Sunspot Equilibria in a Finance Constrained Economy", *Journal of Economic Theory*, 40, 128–37.