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Determinacy and sunspots in a nonlinear monetary model

Alessandra Cornaro* Anna Agliari†

Abstract

In this paper we analyze a basic sticky price model with monopolistic competition and price stickiness à la Calvo. Starting by the relations describing a general economic equilibrium model (see Woodford in *Interest and Prices, Foundations of a Theory of Monetary Policy*, The MIT Press, 2003), as it results from the optimizing behavior of the private agents, we provide a nonlinear model for the monetary policy analysis. This kind of model is a candidate for the existence of multiple equilibria, with a dependence of exogenous sunspots. We explore the stability of such a model combined with interest rate rules in order to investigate the determinacy of the model and we find, for some policy and elasticity parameters, the conditions under which it is possible.

Keywords: Monetary policy; Nonlinear; Determinacy; Sunspots

1 Introduction

The study of monetary policy models with the adoption of interest rate rules in order to set a desirable interest rate for the central bank has developed a prosperous literature on the implications of implementing different specification for the interest rate (see Taylor [16]). These rules consider the interest rate as the policy instrument that drives and direct the central bank.

Such models are known to exhibit indeterminacy which implies the existence of many equilibrium paths leading to a steady-state. We recall that a model is said to be determinate if there is a unique *rational expectation equilibrium* (REE) and indeterminate if there are multiple nonexplosive solutions (see, among others, Evans and McGough [9], [10] and Benhabib and Farmer [1]).

Models with indeterminacy are excellent candidates for the existence of sunspot equilibria (see Cass and Shell [5]). Sunspot equilibria can be constructed by randomizing over multiple equilibria of general equilibrium model.

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The introduction of exogenous shocks in models of dynamic economies (not based on fundamentals), in presence of indeterminacy, may be consistent with the equilibrium. In fact, when the sunspot shocks follow a stochastic process that is consistent with the expectations of the agents, the equilibrium conditions can be satisfied.

Each indeterminate steady-state is linked to a continuum of sunspot equilibria and, for this reason, the presence of indeterminacy is not desirable, because the particular equilibrium on which agents ultimately coordinate may not display wanted properties.

In this paper, we consider a basic sticky price model combined with the hypothesis of monopolistically competition à la Dixit-Stiglitz, where the firms face a price stickiness of Calvo type [4], [8]**:** therefore**,** we deal with a framework, often referred as to Newkeynesian, consistent with the optimizing behavior by private agents and incorporates nominal rigidities.

The relations resulting by such a model, lead to a IS-AS model to be closed with a specification for the nominal interest rate. We use, in a first case, a linear policy feedback rule that is an AR(1) and, secondly, a Taylor-rule depending on current inflation and output gap, following Bullard and Mitra [3].

Normally this kind of model is loglinearized in order to study the fluctuations around the steady-state.

The core of this work is to go beyond the loglinearization and to provide a nonlinear model for studying local and global stability.

Since the rewriting of the model in the nonlinear version is not that evident due to the mathematical shape of the equations, not easily treatable without passing through the loglinear approximation, we introduce new variables consistent with the analytical and economical framework.

Another step is to reduce the temporal horizon, assuming myopic agents.

Acting in this way, we get a nonlinear model that is the starting point for the determinacy analysis, under different policy rules.

Our main results support the findings of Bullard and Mitra: when the model is closed with a rule that does not respond to the endogenous variables, namely inflation and output gap: it exhibits indeterminacy.

The model in which the policy rule depends upon the current aggregates can generate the possibility of finding conditions that ensure a unique path leading to the equilibrium and we are going to see how the policy parameters and the elasticities derived form the particular type of utility functions implemented in the model play an important role and these results differs from the linear case.

This paper is organized as follows. Section 2, presents the analytical framework and the general equilibrium model that gives rise to the IS-AS structure, explains the approach implemented to get a nonlinear model and the related problems and briefly introduces the classification of the policy rules using for our investigation.

Section 3 contains some general technical aspects related to indeterminacy and sunspots and furthermore it discusses in the details the methods and the procedure used to analyze the local determinacy.

Section 4 concludes providing a summary of the main features and results.

2 The model

2.1 Analytical setup

We analyze an economy à la Woodford [15], of cashless type, where households supply labor and purchase good for consumption and firms hire labor and produce and sell differentiated products in a monopolistically competitive goods markets drawn from Dixit and Stiglitz [8].

Following Calvo [4], each firm sets the price of the good it produces but not all the firms are able to reset their price in each period.

From the optimizing behavior of households and firms, where the first ones maximize the expected value of utility and firms maximize profits, we obtain the relations that are the starting point for our investigation.

In the Woodford's model, these relations are implicit: in our characterization, we specify the functions involved in the model by using utility function of C.R.R.A. type[1] and linear production function as in Walsh [13].

Acting in this way, we deal with the components of a general economic equilibrium model:

$$Y_t^{-\sigma} = \beta(1+i_t)E_t\left(\frac{P_t}{P_{t+1}}Y_{t+1}^{-\sigma}\right) \tag{1}$$

$$\frac{p_t^*}{P_t} = \left(\frac{\theta}{\theta-1}\right)\frac{E_t\sum\limits_{T=t}^{\infty}(\alpha\beta)^{T-t}\,Y_T^{1-\sigma}\varphi_T\left(\dfrac{P_T}{P_t}\right)^{\theta}}{E_t\sum\limits_{T=t}^{\infty}(\alpha\beta)^{T-t}\,Y_T^{1-\sigma}\left(\dfrac{P_T}{P_t}\right)^{\theta-1}} \tag{2}$$

$$P_t^{1-\theta} = (1-\alpha)(p_t^*)^{1-\theta} + \alpha P_{t-1}^{1-\theta} \tag{3}$$

and they provide (1) the equilibrium condition to determine output, using the GDP identity, (2) the price set by firms able to adjust their price and (3) the aggregate price level à la Calvo.

They form (shape) a forward-looking NewKeynesian IS-AS model, that it has to be closed with a specification of the interest rate.

Normally this model, also known as "New Phillips curve" model, is obtained as a loglinearization of the previous relations, in order to study small fluctuations around the steady-state. This process allows us to get a linear model where the inflation and the output gap[2] are the state variables.

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[1] The constant relative risk aversion (C.R.R.A.) utility function is given by $U(C) = \dfrac{C_t^{1-\sigma}}{1-\sigma}$ for $\sigma > 0, \sigma \neq 1$. The parameter σ is the intertemporal substitution elasticity between consumption in any two periods, i.e., it measures the willingness to substitute consumption between different periods. Note also that the coefficient of relative risk aversion $\dfrac{-CU''(C)}{U'(C)} = \sigma$.

[2] The output gap is the difference between potential GDP and actual GDP.

2.2 Keeping nonlinearity

In this section we are going to show the way we use to keep the nonlinearity in the equations involved in this framework, for studying the determinacy of the economic system.

Without passing through the loglinearization, since the relations are not easily treatable by a mathematical point of view, it's necessary to introduce new measures expressing the quantities of the state variables corresponding to inflation and output gap.

In particular, we consider $\Pi_{t+1} = \dfrac{P_{t+1}}{P_t}$ as the gross inflation rate and the variable $g_{t+1} = \dfrac{Y_{t+1}}{Y_f}$, in order to introduce the output gap, as the ratio of the output from its value in flexible price $Y_f = \dfrac{A^{\frac{1+\eta}{\sigma+\eta}}}{\mu^{\frac{1}{\sigma+\eta}}\chi^{\frac{1}{\sigma+\eta}}}$.[3].

So far, the relation (1) resulting from the maximization problem of the household can be rearranged, dividing by $Y_f^{-\sigma}$ and rewritten as

$$g_t^{-\sigma} = \beta(1+i_t)E_t\left(\frac{g_{t+1}^{-\sigma}}{\Pi_{t+1}}\right)$$

or better

$$E_t\left(\frac{g_{t+1}^{-\sigma}}{\Pi_{t+1}}\right) = \frac{1}{\beta g_t^{\sigma}(1+i_t)}. \tag{1}$$

The other relation we are interested in is the specification of Calvo Price (3); let $L_t = \dfrac{p_t^*}{P_t}$ be the relative price chosen by all firms that adjust their price in period t, considering the gross inflation rate, the relation becomes:

$$L_t = \left(\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(\frac{1}{\Pi_t}\right)^{1-\theta}\right)^{\dfrac{1}{1-\theta}}. \tag{2}$$

Finally, the equation (2), obtained from the maximization problem faced by the firms, placing $\left(\dfrac{\theta}{\theta-1}\right) = \mu > 1$ as the markup, can be rearranged in the following way:

$$\left[E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\,Y_T^{1-\sigma}\left(\frac{P_T}{P_t}\right)^{\theta-1}\right]L_t = \mu\left[E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\,Y_T^{1-\sigma}\varphi_T\left(\frac{P_T}{P_t}\right)^{\theta}\right]. \tag{3}$$

With reference to this relation, it's not possible to deduce the inflation because, considering the gross inflation rate, we deal with an infinite sum of a products and it's not that evident how to simplify the equation.

[3]For the expression of the output in flexible price in the nonlinear case see Cornaro [7].

In order to go over this problem, we have to make some further assumptions to turn (3) in a more treatable shape.

Therefore, we consider one step forward looking agents.

This means that we assume myopic households and firms, hence they are not able to maximize facing a infinite-horizon.

This hypothesis may be quite realistic for firms, if we assume that they are not able to get all informations they need to compute marginal cost considering an infinite-horizon, and in particular for households, in fact they may find more difficulties with respect to the firms to obtain the information they need because they have less instruments at their disposal.

Introducing such an assumption, we may rewrite equation (3) as follows :

$$\left\{Y_t^{1-\sigma} + \alpha\beta E_t \left[Y_{t+1}^{1-\sigma}\left(\frac{P_{t+1}}{P_t}\right)^{\theta-1}\right]\right\} L_t = \mu \left\{Y_t^{1-\sigma}\varphi_t + \alpha\beta E_t \left[Y_{t+1}^{1-\sigma}\varphi_{t+1}\left(\frac{P_{t+1}}{P_t}\right)^{\theta}\right]\right\} \tag{4}$$

and we obtain

$$L_t Y_t^{1-\sigma} - \mu\varphi_t Y_t^{1-\sigma} = \alpha\beta E_t \left[Y_{t+1}^{1-\sigma}\Pi_{t+1}^{\theta}\left(\mu\varphi_{t+1} - \frac{L_t}{\Pi_{t+1}}\right)\right].$$

Now, dividing by $Y_f^{1-\sigma}$ and using the output gap relation we state

$$g_t^{1-\sigma}\left\{L_t - \mu\varphi_t\right\} = \alpha\beta E_t \left\{g_{t+1}^{1-\sigma}\Pi_{t+1}^{\theta}\left(\mu\varphi_{t+1} - \frac{L_t}{\Pi_{t+1}}\right)\right\}. \tag{5}$$

Substituting in (5) the real marginal cost (as computed in [7]) we get:

$$\left(L_t - g_t^{\sigma+\eta}\right) g_t^{1-\sigma} = \alpha\beta E_t \left[g_{t+1}^{1-\sigma}\Pi_{t+1}^{\theta}\left(g_{t+1}^{\sigma+\eta} - \frac{L_t}{\Pi_{t+1}}\right)\right]. \tag{6}$$

In this way, we have deduced an equation that expresses the relation between the output gap and the gross inflation rate.

Summing up, the model is composed by the following relations:

$$E_t\left(\frac{g_{t+1}^{-\sigma}}{\Pi_{t+1}}\right) = \frac{1}{\beta g_t^{\sigma}(1+i_t)}$$

$$L_t = \left(\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(\frac{1}{\Pi_t}\right)^{1-\theta}\right)^{\frac{1}{1-\theta}}$$

and

$$\left(L_t - g_t^{\sigma+\eta}\right) g_t^{1-\sigma} = \alpha\beta E_t \left[g_{t+1}^{1-\sigma}\Pi_{t+1}^{\theta}\left(g_{t+1}^{\sigma+\eta} - \frac{L_t}{\Pi_{t+1}}\right)\right]$$

Obviously it has to be closed with a specification of a rule for the interest rate.

2.3 Interest rate specifications

We aim to explore the possibility of existence of determinacy in the model under two different types of specifications for the interest rate, in order to see how the region and the nature of a model's determinacy depend critically on the specification of the policy rule.

Firstly, we suppose a linear monetary policy feedback rule, represented by:

$$PR_1 : \quad i_t = \rho i_{t-1} + v_t \tag{7}$$

where $0 \leqslant \rho < 1$ is a correlation parameter and the interest rate is an exogenous $AR(1)$ process with innovation v_t (i.i.d. noise).

The second specification is a Taylor rule of the type:

$$PR_2 : \quad i_t = \varphi_g g_t + \varphi_\Pi \Pi_t. \tag{8}$$

We assume throughout that φ_g and φ_Π are non-negative, with at least one of them strictly positive.

This rule assume that current data on inflation and output gap are available to the policymakers when the interest rates are set.

Therefore, monetary policy authorities act in response to inflation and size of the output gap.

3 Analysis of local determinacy

3.1 Technical aspects

Accordingly to our goal of studying the determinacy of the model, we briefly introduce some technical notions.

In general, it is possible to analyze a model by writing the reduced form equation as a discrete difference equation where the exogenous noise terms are the errors in the agents forecasts of variable that are non-predetermined, i.e. whose expectation can differ from the actual realization.

The errors play an important role for establishing if a model is locally determinate or not: if the nonexplosive requirement of a REE pins down the forecast errors, then the model is determinate.

Otherwise, if the errors are not pinned down, multiple equilibria exist and they are called sunspots. In this case extrinsic fluctuations in agents' expectations can be captured by the forecast errors and they are consistent (compatible) with the hypothesis of rationality.

By an analytical point of view, these conditions have been introduces by Blanchard and Khan [2].

The condition for a determinate solution is that there are exactly as many as non-predetermined (i. e. forward-looking) variables as explosive eigenvalues of the Jacobian matrix associated to the model evaluated at the steady-state.

This means that if the dimension of the unstable manifold is equal to the number of non-predetermined variables, then the model is determinate, otherwise if the dimension of the unstable manifold is less than the number of non-predetermined variables, sunspots equilibria are possible.

Coming back to our framework, we have obtained a two-equations model in terms of output gap, gross inflation rate and the nominal interest rate and we can rewrite it as follow:

$$\frac{g_{t+1}^{-\sigma}}{\Pi_{t+1}} = \frac{1}{\beta g_t^{\sigma}(1+i_t)} + \varepsilon_{t+1} \tag{9}$$

$$\left(L_t - g_t^{\sigma+\eta}\right) g_t^{1-\sigma} = \alpha\beta \left[g_{t+1}^{1-\sigma} \Pi_{t+1}^{\theta} \left(g_{t+1}^{\sigma+\eta} - \frac{L_t}{\Pi_{t+1}} \right) \right] + \varkappa_{t+1} \tag{10}$$

where

$$\varepsilon_{t+1} = E_t \left(\frac{g_{t+1}^{-\sigma}}{\Pi_{t+1}} \right) - \frac{g_{t+1}^{-\sigma}}{\Pi_{t+1}}$$

and

$$\varkappa_{t+1} = \alpha\beta \left\{ E_t \left[g_{t+1}^{1-\sigma} \Pi_{t+1}^{\theta} \left(g_{t+1}^{\sigma+\eta} - \frac{L_t}{\Pi_{t+1}} \right) \right] - g_{t+1}^{1-\sigma} \Pi_{t+1}^{\theta} \left(g_{t+1}^{\sigma+\eta} - \frac{L_t}{\Pi_{t+1}} \right) \right\}$$

are the stochastic errors or "shocks to expectations".

We start considering the case of *Perfect Foresight Dynamics.* Letting the stochastic errors ε_{t+1} and $\varkappa_{t+1}$ be equal to 0;

In this case equation (9) becomes

$$\Pi_{t+1} = \beta(1+i_t)\frac{g_t^{\sigma}}{g_{t+1}^{\sigma}} = \frac{\beta(1+i_t)g_t^{\sigma}}{g_{t+1}^{\sigma}} \tag{11}$$

and equation (10) can be rewritten as:

$$\left(L_t - g_t^{\sigma+\eta}\right) g_t^{1-\sigma} = \alpha\beta \left\{ g_{t+1}^{1-\sigma} \Pi_{t+1}^{\theta} \left(g_{t+1}^{\sigma+\eta} - \frac{L_t}{\Pi_{t+1}} \right) \right\}$$

Rearranging the terms

$$L_t \left(g_t^{1-\sigma} + \alpha\beta g_{t+1}^{1-\sigma} \Pi_{t+1}^{\theta-1} \right) = g_t^{1+\eta} + \alpha\beta g_{t+1}^{1+\eta} \Pi_{t+1}^{\theta} \tag{12}$$

Furthermore we have the specification for L_t :

$$L_t = \left(\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha} \Pi_t^{\theta-1} \right)^{\frac{1}{1-\theta}} . \tag{13}$$

Now, it's possible to compute the equilibrium conditions and to show that there is a unique steady-state.

From (11) we have that

$$\Pi = \beta(1+i^*) = 1,$$

it means that the nominal interest rate in equilibrium is[4]

$$i^* = \frac{1}{\beta} - 1 \tag{14}$$

and

$$\Pi^* = 1.$$

The condition (12) becomes

$$Lg^{1-\sigma}\left(1 + \alpha\beta\Pi^{\theta-1}\right) = g^{1+\eta}\left(1 + \alpha\beta\Pi^{\theta}\right)$$

and manipulating the previous equation we have

$$g^{\sigma+\eta} = L$$

but

$$L^* = \left(\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha} \right)^{\frac{1}{1-\theta}} = 1$$

hence

$$g^* = L^* = 1$$

[4] We consider a Taylor rule of the form:

$$i_t = \Phi\left(\frac{\Pi_t}{\Pi_t^*}; \nu_t\right)$$

where $\Pi_t = \dfrac{P_t}{P_{t-1}}$ is the gross inflation rate, Π_t^* is a target rate, ν_t is an exogenous shift and $\Phi(\cdot;\nu_t)$ is an increasing function for each value of ν. We consider equilibria near a zero-inflation steady-state.

With the assumption that $\Phi(1,0) = \beta^{-1} - 1$, $\Pi_t^* = 1$ and $\nu_t = 0$ at all times in such a steady-state equilibrium.

3.2 Determinacy under AR process

In order to illustrate our methodology, we recall here the relations componing the model: (11), (12) and (13), that can be rewritten as follow:

$$\Pi_{t+1} = \frac{\beta(1+i_t)g_t^{\sigma}}{g_{t+1}^{\sigma}} \tag{15}$$

$$F(L_t, g_t, g_{t+1}) = L_t g_t^{1-\sigma} - g_t^{\eta+1} - \alpha\beta^{\theta+1}(1+i_t)^{\theta} g_t^{\theta\sigma} g_{t+1}^{\eta-\theta\sigma+1} + L_t\alpha\beta^{\theta}(1+i_t)^{\theta-1} g_t^{\theta\sigma-\sigma} g_{t+1}^{1-\theta\sigma} = 0 \tag{16}$$

and

$$L_t = \left(\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\Pi_t^{\theta-1}\right)^{\frac{1}{1-\theta}}. \tag{17}$$

We close the model with

$$PR_1: \quad i_t = (1-\beta)i_{t-1} + v_t \qquad \text{with } v_t \sim \mathcal{N}(1-\beta, \sigma).$$

At the steady-state:

$$i^* = (1-\beta)i^* + 1 - \beta$$

and we get:

$$i^* = \frac{1}{\beta} - 1$$

Now, since the provided model is not explicitable, in order to obtain the Jacobian matrix and study the local stability of the system, we have to make use of the Implicit Function Theorem, neglecting the error term.

The Jacobian matrix has to be compute in the following way:

$$J = \begin{bmatrix} \frac{\partial g_{t+1}}{\partial g_t} & \frac{\partial g_{t+1}}{\partial \Pi_t} & \frac{\partial g_{t+1}}{\partial i_{t-1}} \\ \frac{\partial \Pi_{t+1}}{\partial g_t} & \frac{\partial \Pi_{t+1}}{\partial \Pi_t} & \frac{\partial \Pi_{t+1}}{\partial i_{t-1}} \\ \frac{\partial i_t}{\partial g_t} & \frac{\partial i_t}{\partial g_t} & \frac{\partial i_t}{\partial i_{t-1}} \end{bmatrix}$$

Making the calculations we obtain:

$$J = \begin{bmatrix} \frac{-\sigma-\eta-\sigma\alpha\beta}{\eta\alpha\beta} & \frac{1+\alpha\beta}{(1-\alpha)\eta\beta} & -\frac{\alpha\beta^3(2\theta+1)}{\eta\alpha\beta} \\ \frac{(\alpha\beta+1)(\sigma+\eta)\sigma}{\eta\beta\alpha} & -\frac{\sigma(1+\alpha\beta)}{(1-\alpha)\eta\beta} & \beta(1-\beta) \\ 0 & 0 & 1-\beta \end{bmatrix}$$

We can observe that J is a block matrix of the type:

$$J = \begin{bmatrix} J_1 & J_2 \\ 0 & 1-\beta \end{bmatrix}$$

For this reason we immediately note that an eigenvalue $\lambda_1 = 1 - \beta$ and $0 < \lambda_1 < 1$ since $0 < \beta < 1$.

Now we can limit our analysis to the partition matrix J_1 :

$$J_1 = \eta\beta \begin{bmatrix} \dfrac{-\sigma - \eta - \sigma\alpha\beta}{\alpha} & \dfrac{1+\alpha\beta}{(1-\alpha)} \\ \frac{(\alpha\beta+1)(\sigma+\eta)\sigma}{\alpha} & -\dfrac{\sigma(1+\alpha\beta)}{(1-\alpha)} \end{bmatrix}$$

whose eigenvalues will be denoted by λ_2 and λ_3.

In order to determine the location of the eigenvalues of J_1 with respect to the unit circle, we shall make use the triangle of stability ([12]) given by the following conditions:

$$\begin{aligned} \mathbf{1} - \mathbf{tr} + \mathbf{det} &> \mathbf{0} \\ \mathbf{1} + \mathbf{tr} + \mathbf{det} &> \mathbf{0} \\ \mathbf{1} - \mathbf{det} &> \mathbf{0} \end{aligned}$$

We start computing the trace and the determinant of the Jacobian Matrix:

$$Tr(J) = -\frac{\sigma + \eta - \alpha\eta + \alpha\sigma\beta}{(1-\alpha)\,\alpha\beta\eta}$$

$$Det(J) = -\frac{\sigma(1+\alpha\beta)}{(1-\alpha)\,\beta\eta} < 0.$$

We can immediately observe that the determinant is always smaller than 0, then we can't find complex eigenvalues.

Furthermore, we have

$$\mathbf{1} - \mathbf{tr} + \mathbf{det} : \frac{(\sigma+\eta)(\alpha\beta+1)}{\eta\beta\alpha} > 0$$

$$\mathbf{1} + \mathbf{tr} + \mathbf{det} : -\frac{\sigma(\alpha+1)(\alpha\beta+1) + \eta(1-\alpha)(1-\alpha\beta)}{(1-\alpha)\,\alpha\beta\eta} < 0$$

$$\mathbf{1} - \mathbf{det} : \frac{\eta\beta(1-\alpha) + \sigma(\alpha\beta+1)}{(1-\alpha)\,\beta\eta} > 0.$$

For the model's determinacy, following Blanchard-Kahn, we need that the number of eigenvalues outside the unit circle are equal to the non-predetermined variables of the model. In our case we have three state variables, two of them are non-predetermined,namely the output gap and the inflation rate. From the performed analysis we can concluding that $\lambda_1 = 1 - \beta < 1$, $\lambda_2 > 1$ and $\lambda_3 < 1$, then the equilibrium point is a *saddle* and *locally indeterminate* in the original model.

3.3 Determinacy under Taylor rule

For studying another case, we combine the relations (15), (16), (17) with

$$PR_2: \quad i_t = \varphi_g g_t + \varphi_\Pi \Pi_t.$$

Considering the equilibrium condition for the nominal interest rate, (14), with this type of specification we find that:

$$\beta = \frac{1}{\varphi_\Pi + \varphi_g + 1}. \tag{18}$$

We can write now the Jacobian Matrix, that, in this particular case, is of dimension 2:

$$J = \begin{bmatrix} -\dfrac{\sigma + \eta + \alpha\sigma\beta + \alpha\beta^2\varphi_g}{\alpha\beta\eta} & \dfrac{1 + \alpha\beta - (1-\alpha)\beta^2\varphi_\Pi}{\beta\eta(1-\alpha)} \\ \beta\varphi_g + \sigma + \sigma\left(\dfrac{\sigma + \eta + \alpha\sigma\beta + \alpha\beta^2\varphi_g}{\alpha\beta\eta}\right) & \beta\varphi_\Pi - \sigma\dfrac{(1 + \alpha\beta - (1-\alpha)\beta^2\varphi_\Pi)}{\beta\eta(1-\alpha)} \end{bmatrix}$$

In order to determine the location of the eigenvalues of J, using the triangle of stability, we compute the determinant and the trace of matrix J and we get

$$Det(J) = -\frac{1}{\eta\beta\alpha\,(1-\alpha)}(\alpha(\alpha\beta+1)(\sigma+\beta\varphi_g) + \beta\varphi_\Pi(\sigma+\eta)(1-\alpha))$$

and

$$Tr(J) = -\frac{(\sigma + \eta - \alpha\eta + \alpha\sigma\beta + \beta^2\alpha(\alpha-1)(\varphi_\Pi(\sigma+\eta) - \varphi_g))}{\alpha\beta\eta\,(1-\alpha)}.$$

It's easy to see that $Det(J) < 0$, hence the eigenvalues are real and $\mathbf{1 - det > 0.}$

The other conditions of the triangle of stability are given by:

$$\mathbf{1 - tr + det}:$$

$$-\frac{(\sigma+\eta)(1-\alpha)(\alpha\beta+1)(\beta\varphi_\Pi - 1) + \varphi_g\alpha\beta(2\alpha\beta - \beta + 1)}{\alpha\beta\eta\,(1-\alpha)} = A \tag{19}$$

$$\mathbf{1 + tr + det}:$$

$$-\frac{(\sigma+\eta)(\beta\varphi_\Pi\,(1-\alpha)\,(1-\alpha\beta) + \alpha^2\beta + 1) + \alpha(\beta+1)(\sigma - \eta + \beta\varphi_g)}{\alpha\beta\eta\,(1-\alpha)} = B. \tag{20}$$

In order to get determinacy, according to the Blanchard-Kahn conditions, (19) and (20) have to be negative.

Since we have state that φ_Π and φ_g are both non-negative, we can consider the parameter space Ω, where $\varphi_\Pi > 0$ and $\varphi_g > 0$.

Substituting in (19) and (20) the equilibrium condition (18), we get the condition:

$$\begin{aligned} B = & -\eta(\alpha-1)^2 - \sigma(\alpha+1)^2 - \varphi_g(\eta(1-\alpha)(2-\alpha) + \sigma(\alpha+2)(\alpha+1) + 2\alpha) + \\ & -\varphi_g^2((1-\alpha)\eta + \sigma + \alpha\sigma + \alpha) - \varphi_\Pi\left(\sigma\left(\alpha + 2\alpha^2 + 3\right) + \eta(1-\alpha)(3-2\alpha)\right) + \\ & -2\varphi_\Pi^2(\eta(1-\alpha) + \sigma) - \varphi_\Pi\varphi_g(3\eta(1-\alpha) + \sigma(\alpha+3) + \alpha). \end{aligned} \tag{21}$$

and:

$$\begin{aligned} A = & (1-\alpha)(\alpha+1)(\sigma+\eta) + \varphi_\Pi(\sigma+\eta)(1-\alpha) - \varphi_g\left(2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta)\right) + \\ & -\varphi_g^2(\alpha - (\sigma+\eta)(1-\alpha)) - \varphi_g\varphi_\Pi(\alpha - (\sigma+\eta)(1-\alpha)) \end{aligned} \tag{22}$$

Proposition 1 *The quantity B in (21) is always negative in the parameter space Ω.*

Indeed, as we can easily observe, looking at B as a polynomial in φ_g and φ_Π, the coefficients are all negative.

Proposition 2 *The quantity A in (22) is positive in Ω if $\sigma + \eta > \dfrac{\alpha}{1-\alpha}$ (The proof is given in* Appendix A*)*

Proposition 3 *The quantity (22) is negative in Ω if*

$$\sigma + \eta < \frac{\alpha}{1-\alpha}$$

and

$$\varphi_\Pi > \frac{\left(-\varphi_g^2(\alpha - (\sigma+\eta)(1-\alpha)) - \varphi_g\left(2\alpha^2 - (\sigma+\eta)(\alpha+2)(1-\alpha)\right) + (\sigma+\eta)(\alpha+1)(1-\alpha)\right)}{(\sigma+\eta)(1-\alpha) - \varphi_g(\alpha - (\sigma+\eta)(1-\alpha))}$$

(see Appendix B *for more details).*

Resuming, with the implementation of contemporaneous data in the interest rate rule, the equilibrium point in Perfect Foresight Dynamics is a *saddle* and then we have *local indeterminacy* in the original model if $\sigma + \eta > \dfrac{\alpha}{1-\alpha}$; otherwise, if $\sigma + \eta < \dfrac{\alpha}{1-\alpha}$ and

$$\varphi_\Pi > \frac{\left(-\varphi_g^2(\alpha - (\sigma+\eta)(1-\alpha)) - \varphi_g\left(2\alpha^2 - (\sigma+\eta)(\alpha+2)(1-\alpha)\right) + (\sigma+\eta)(\alpha+1)(1-\alpha)\right)}{(\sigma+\eta)(1-\alpha) - \varphi_g(\alpha - (\sigma+\eta)(1-\alpha))}$$

the equilibrium is an *unstable node* and for this reason we have the *local determinacy* of the equilibrium in the original model.

We provide here some examples in order to better analyze this second case and to visualize the region where the condition (22) is negative.

If we consider, following our hypothesis on the parameters:

$$\alpha = 0.7$$

$$(\sigma + \eta) < 2.3333$$

$$\sigma = 0.9$$

$$\eta = 1.2$$

we get the hyperbola:

$$0.63y + 0.721\,x - 0.07xy - 0.07x^2 + 1.071 = 0$$

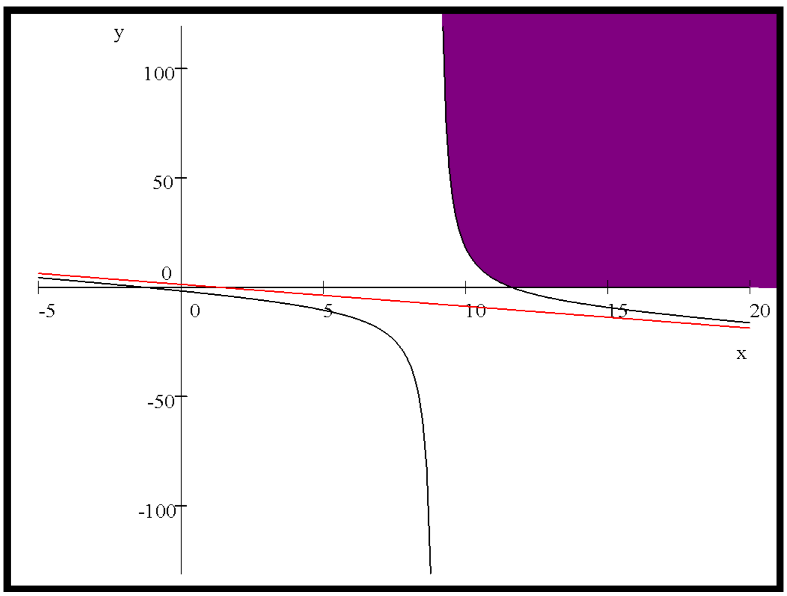

Figure 1: A first example

In a different case, when

$$\alpha = 0.7$$
$$(\sigma + \eta) < 2.3333$$
$$\sigma = 0.3$$
$$\eta = 0.4$$

we get:

$$0.21y - 0.413\,x - 0.49yx - 0.49x^2 + 0.357 = 0$$

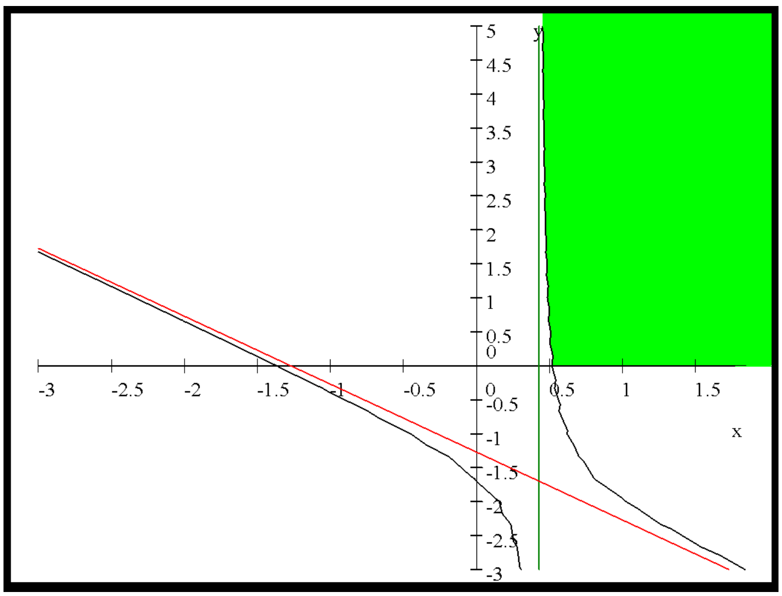

Figure 2: A second example

Resuming, with the implementation of contemporaneous data in the interest rate rule, the equilibrium point in Perfect Foresight Dynamics is a *saddle* and then we have *locally indeterminacy* in the original model if $\sigma + \eta > \dfrac{\alpha}{1-\alpha}$; otherwise, if $\sigma + \eta < \dfrac{\alpha}{1-\alpha}$ and

$$\varphi_\Pi > \frac{\left(-\varphi_g^2\left(\alpha - (\sigma+\eta)(1-\alpha)\right) - \varphi_g\left(2\alpha^2 - (\sigma+\eta)(\alpha+2)(1-\alpha)\right) + (\sigma+\eta)(\alpha+1)(1-\alpha)\right)}{(\sigma+\eta)(1-\alpha) - \varphi_g\left(\alpha - (\sigma+\eta)(1-\alpha)\right)}$$

the equilibrium is an *unstable node* and for this reason we have the *local determinacy* of the equilibrium in the original model.

4 Conclusion

In this work we have explored a monetary policy model, derived from a sticky price context with Calvo price type and we have developed this framework building a nonlinear model, compatible with the general economic equilibrium with optimizing agents.

We have studied the determinacy of the model, under two different monetary policy rules, following Blanchard [2], Evans and McGough [9], Cazzavillan [6] and Grandmont [11].

Closing the model with an AR(1) interest-rate rule, we have found that the equilibrium is locally indeterminate. Then we conclude that an exogenous policy rule, therefore a rule that doesn't respond to the endogenous variables, introduces the possibility of multiple equilibria. The same result is obtained in the log-linear case investigated by Bullard and Mitra and Walsh.

On the other end, if we implement a policy rule as the Taylor rule, where the central bank react to current inflation values and output deviation, we can find some conditions in order to ensure a unique equilibrium.

In our nonlinear specification, the condition to obtain a unique and determinate equilibrium depends on the parameters σ and η that are, namely, the intertemporal substitution elasticity between consumption and the elasticity of the labour disutility. Such parameters have to satisfy a certain bound with respect to the parameter α that is the degree of nominal rigidity of prices. Hence, it's worth to stress that the utility functions chosen in the optimization problem influence the determinacy of the model, entering the conditions to ensure a unique path leading to the equilibrium. Evidently, even the policy parameters appearing in the policy rule surely play an important role in order to achieve a determinate equilibrium.

With the same policy rule, in the log-linear case, the condition necessary to guarantee that the economy has a unique stationary equilibrium (see Bullard and Mitra [3]) depends on both the policy parameters referred to the inflation and the output gap and corresponds to the Taylor principle as discussed by Woodford [14], [15]. Differently from our findings, there is no quotation to the elasticity parameters, even if they have employed the same structure of preferences for consumption and labor, described by analogous utility functions.

APPENDICES

Appendix A *: Proof of proposition 3*

With reference to the condition

$$\begin{array}{c} A = (1-\alpha)(\alpha+1)(\sigma+\eta) + \varphi_{\Pi}(\sigma+\eta)(1-\alpha) - \varphi_g\left(2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta)\right) + \\ -\varphi_g^2(\alpha-(\sigma+\eta)(1-\alpha)) - \varphi_g\varphi_{\Pi}(\alpha-(\sigma+\eta)(1-\alpha)) \end{array},$$

it's easy to see that all the coefficients are positive, also the term associated to φ_g.

In fact, it's possible to show that

$$2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta) < \alpha(\alpha-2).$$

Since $\alpha(\alpha-2)$ is a negative quantity, because $0 < \alpha < 1$, we can state that

$$2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta)$$

is clearly negative and that the term associated to φ_g is finally positive. In this way we have shown that all the coefficients of the quantity A.

Appendix B *: Proof of proposition 4*

Starting from

$$\begin{array}{c} A = (1-\alpha)(\alpha+1)(\sigma+\eta) + \varphi_{\Pi}(\sigma+\eta)(1-\alpha) - \varphi_g\left(2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta)\right) + \\ -\varphi_g^2(\alpha-(\sigma+\eta)(1-\alpha)) - \varphi_g\varphi_{\Pi}(\alpha-(\sigma+\eta)(1-\alpha)) = 0 \end{array}$$

we can make explicit φ_{Π} in $A=0$, obtaining

$$\varphi_{\Pi} = \frac{\left(\varphi_g^2(\alpha-(\sigma+\eta)(1-\alpha)) + \varphi_g\left(2\alpha^2 - (\sigma+\eta)(\alpha+2)(1-\alpha)\right) - (\sigma+\eta)(\alpha+1)(1-\alpha)\right)}{(\sigma+\eta)(1-\alpha) - \varphi_g(\alpha-(\sigma+\eta)(1-\alpha))}$$

This is an hyperbola with two asymptotes:

$$\varphi_g = \frac{(\sigma+\eta)(1-\alpha)}{\alpha-(\sigma+\eta)(1-\alpha)} \tag{23}$$

$$\varphi_g = -\varphi_{\Pi} + \frac{(1-\alpha)(\alpha+1)(\sigma+\eta) - 2\alpha^2}{\alpha-(\sigma+\eta)(1-\alpha)} \tag{24}$$

and it cross the axe φ_{Π} in the point $\left[\varphi_{\Pi} = -(\alpha+1), \varphi_g = 0\right]$.

If $\sigma+\eta < \dfrac{\alpha}{1-\alpha}$ the asymptote (23) is positive. If we compare the intersection of the hyperbola with the axe φ_{Π} and the intercept of the asymptote (24) we have

$$-(\alpha+1) > \frac{(1-\alpha)(\alpha+1)(\sigma+\eta) - 2\alpha^2}{\alpha-(\sigma+\eta)(1-\alpha)}$$

and

$$0 < \alpha (1 - \alpha)$$

that is always true; it means that the intercept of the hyperbola with the axe φ_Π is below the intercept of the asymptote (24) with the axe φ_Π, whose sign is not univocally determined.

The intersections of (23) and (24) $\left[\varphi_g = \frac{(\sigma+\eta)(1-\alpha)}{\alpha-(\sigma+\eta)(1-\alpha)},\ \varphi_\Pi = \frac{((\sigma+\eta)(1-\alpha)-2\alpha)\alpha}{\alpha-(\sigma+\eta)(1-\alpha)}\right]$ are respectively positive and negative and the intersection of the hyperbola with the axe φ_g are:

$$\varphi_{g1,2} = \frac{-\left(2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta)\right) \pm \sqrt{\alpha^2 \left(2\alpha - (1-\alpha)(\sigma+\eta)\right)^2 + 4\alpha (1-\alpha)^2 (\sigma+\eta)}}{(\alpha - (\sigma+\eta)(1-\alpha))}$$

that, since we have stated until now, have to be one negative and one positive.

Since

$$(1-\alpha)(\alpha+1)(\sigma+\eta) - \varphi_g \left(2\alpha^2 - (\alpha+2)(1-\alpha)(\sigma+\eta)\right) + \varphi_\Pi (\sigma+\eta)(1-\alpha)$$
$$-\varphi_g^2 (\alpha - (\sigma+\eta)(1-\alpha)) - \varphi_g \varphi_\Pi (\alpha - (\sigma+\eta)(1-\alpha)) > 0$$

is always true $(0,0)$ we conclude that the condition (22) holds if

$$\varphi_\Pi > \frac{\left(-\varphi_g^2 (\alpha - (\sigma+\eta)(1-\alpha)) - \varphi_g \left(2\alpha^2 - (\sigma+\eta)(\alpha+2)(1-\alpha)\right) + (\sigma+\eta)(\alpha+1)(1-\alpha)\right)}{(\sigma+\eta)(1-\alpha) - \varphi_g (\alpha - (\sigma+\eta)(1-\alpha))}.$$

References

[1] J. Benhabib and R. Farmer, *Indeterminacy and Sunspots in Macroeconomics*, in J. Taylor and M. Woodford, Handbook of Macroeconomics, North-Holland, Amsterdam, 387-448, 1999.

[2] O. Blanchard and C. Kahn, *The Solution of Linear Difference Model under Rational Expectations*, Econometrica 48-5: 1305-1313, 1980.

[3] J. Bullard and K. Mitra, *Learning about Monetary Policy Rules*, Journal of Monetary Economics 49: 1105-1129, 2002.

[4] G. Calvo, *Staggered Prices in a Utility-Maximizing Framework*, Journal of Monetary Economics 12: 383-398, 1983.

[5] D. Cass and K. Shell, *Do Sunspots Matter ?*, Journal of Political Economy, 91: 193-227, 1983.

[6] G. Cazzavillan, T. Lloyd Braga and P. Pintus, *Multiple steady state and endogenous fluctuations with increasing returns to scale in production*, Journal of Economic Theory 80: 60-107, 1988.

[7] A. Cornaro, *The Nonlinearity in Optimal Monetary Policy Models under Sticky Prices,* Contributi di Ricerca in Econometria e Matematica, n. 95, aprile 2008, Università Cattolica del Sacro Cuore, Milano.

[8] A.V. Dixit and J.E. Stiglitz, *Monopolistic Competition and Optimum Product Diversity,* American Economic Review 67: 297-308, 1977.

[9] G. W. Evans, B. McGough, *Monetary policy, indeterminacy and learning*, Journal of Economic Dynamics and Control 29: 1809-1840, 2005.

[10] G. W. Evans, B. McGough, *Stable sunspot solutions in models with predetermined variables*, Journal of Economic Dynamics and Control 29: 601-625, 2005.

[11] J. M. Grandmont, P. Pintus and R, de Vilder, *Capital Labor Substitution and Competitive Nonlinear Endogenous Business Cycles*, Journal of Economic Theory 80: 14-59, 1998.

[12] A. Medio and M. Lines, *Nonlinear Dynamics, A Primer*, Cambridge University Press, 2001.

[13] C. Walsh, *Monetary Theory and Policy*, The MIT Press, 2003.

[14] M. Woodford, The Taylor ruel and optimal monetary policy, American Review Papers and Proceedings 91, 232-237, 2001.

[15] M. Woodford, *Interest and Prices, Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.

[16] J. B.Taylor, *Discretion versus Policy Rules in Practice*, Carnegie-Rochester Conference Series on Public Policy 39: 195-214, 1993.