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A NEW MOBILITY INDEX FOR TRANSITION MATRICES

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A new Mobility Index for Transition Matrices

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Abstract

This work treats the construction of a mobility index able to grasp the *prevailing direction* in the dynamics ruled by a given transition matrix. The states of the matrix are based on an ordered economic variable, such as firm size, income or ratings, for which the future state can be better or worse than the current one. We propose here a whole family of *directional indices*, evaluated as a function of the transition matrix, and defined so that their absolute value measures the intensity of mobility, and their sign (+/-) represents the prevailing direction towards improvement/worsening in the dynamics under study.

Introduction

For a wide range of economic phenomena the analysis of mobility plays a fundamental role. Observed and estimated transition matrices represent the basic tool through whom the evolution of statistical units, measured for example in terms of firm size, incomes, ratings of firms or states, is described. At the same time there is the need of summarizing the degree of mobility in the analyzed dynamics, by means of an index evaluated on the transition matrix. Past economic literature has been enriched by some important contributions which introduce a set of mobility indices: among others Bartholomew (1973), Shorrocks (1978), Sommers and Conlinsk (1979), Geweke et al (1986), Parker and Rougier (2001), Alcalde-Unzu et al (2006). Such indices permit to grasp the overall degree of mobility in the sample under study, because they provide a value in $[0,1]$ which is near 1 (resp. 0) when the degree of mobility is high (resp. low).

In the case of an ordered economic variable a crucial aspect of the dynamics is

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not only the intensity of mobility but also its prevailing direction. Same degrees of mobility can in fact assume a far different economic sense in correspondence of the prevailing direction of the movements; as for example downsizing versus upsizing of firms, or worsening versus improvement in incomes or ratings. Indices produced until now make feasible the measurement of the intensity but miss that of the prevailing direction. In this paper we propose and analyze a family of indices which aims to summarize, with the intensity, this particular feature of the dynamics under study.

The paper is organized as follows: Sect. 1 provides a brief description of the mobility indices proposed in the past literature, showing that every index is related to a different concept of mobility; Sect. 2 introduces the family of indices based on the prevailing direction and shows that each of such indices is a function equipped with the properties of *boundedness*, *perfect mobility* and *weak immobility*; Sect. 3 contains an analysis of the behavior of the directional index, obtained drawing 20000 transition matrices at random and observing the density distribution of the index values; lastly Sect. 4 contains our proposals for assigning a role in the mobility measure to the starting distribution of individuals among the states and to the magnitude of the jumps from one state to the other.

1 Mobility indices in literature

Let $\mathscr{P}$ be the set of transition matrices:

$$\mathscr{P} = \{P \in \mathbb{R}^{k\times k} | p_{ij} \geq 0, \sum_{j=1}^{k} p_{ij} = 1, \forall i = 1,\dots,k\}$$

A mobility index can be defined as a function $I : \mathscr{P} \to \mathbb{R}$ chosen in order to provide a suitable and synthetic description of the mobility. In literature many authors propose different choices for the function I (among others we recall Shorrocks (1978), Sommers and Conlinsk (1979), Bartholomew (1973), Geweke et al (1986), Parker and Rougier (2001) and Alcalde-Unzu et al (2006)). Here it is a list of some indices existing in literature:

- Bartholomew's index $I_b(P) = \frac{k}{k-1} \sum \pi_i (1 - p_{ii})$.
- Trace index $I_{tr}(P) = \frac{k - trP}{k-1}$.
- Determinant index $I_{det}(P) = 1 - |\det P|$.
- Second eigenvalue index $I_2(P) = 1 - |\lambda_2|$.

- Eigenvalues index $I_e(P) = \frac{k-\sum_i |\lambda_i|}{k-1}$.
- Index of predictability $I_p(P) = \frac{k}{k-1}\left(\sum_{i,j} p_{ij}^2 - 1\right)$.
- In Alcalde-Unzu et al (2006) a whole family of indices is introduced, according to the following definition:

$$I_{\omega,\alpha,\nu}(P) = \frac{1}{Z}\sum_i \omega_i \left[\sum_j |p_{ij} - \delta_{ij}|^{\alpha} \nu(|j-i|)\right]^{1/\alpha}$$

  which, for every choice of ω_i, ν and α, measures the weighted distance from the identity matrix (δ_{ij} is the Kronoecker's delta), with Z as a normalizing constant.

Every choice of I provides the measure of a particular feature of the transition matrix under study. As an example we consider the two matrices

$$P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

Evaluating I_{tr}, I_{det}, I_p and I_2 we obtain the following measures:

- $I_{tr}(P) = 0.7$ and $I_{tr}(Q) = 0.6$;
- $I_{det}(P) = 0.91$ and $I_{det}(Q) = 0.93$;
- $I_p(P) = 0.81$ and $I_p(Q) = 0.96$;
- $I_2(P) = 0.7$ and $I_2(Q) = 0.3$.

If we consider I_{tr} and I_2, P has an higher degree of mobility than Q. On the other hand the two indices I_{det} and I_p support the opposite case. This contradiction can be solved by remarking that different indices do not measure the same kind of mobility: for example I_p measures, by definition, the higher or lower predictability of the future state j given the current state i (see Parker and Rougier (2001, p. 64)), whereas I_2, being a function of the second eigenvalue, can be related to the rate of convergence to the equilibrium distribution.
At the same time it is evident that none of the above indices is able to grasp the prevailing direction of the mobility. (The same happens in the classical mechanics where the scalar of a force gives information on its intensity but not on its direction). Our choice is then to propose an index able to seize the prevailing direction of mobility and at the same time to measure its intensity.

2 A new family of mobility indices

2.1 A first formulation of the directional index

Consider now a variable X with ordered states $\{1,\ldots,k\}$. Given the matrix $P \in \mathscr{P}$ and fixed the current state i, there are only two possible directions: moving towards left (the arrival state is $j = 1,\ldots,i-1$), or moving towards right (the arrival state is $j = i+1,\ldots,k$). Movements towards left correspond to downsizing of firms or incomes or more generally to a worsening for other economic variables which can be ordered. On the other hand movements towards right describe an upsizing or improvement of the economic situation.
Then we introduce a new r.v. $S_t = X_{t+1} - X_t$: S_t, conditioned to the current state $X_t = i$, assumes values $s \in \{1-i,\ldots,k-i\}$, with probability

$$\mathbb{P}[S_t = s|X_t = i] = \mathbb{P}[X_{t+1} = s+i|X_t = i] = p_{i,s+i}$$

Consequently the expected value of $S_t|X_t$ can be written as

$$\mathbb{E}[S_t|X_t] = \sum_{s=1-i}^{k-i} s p_{i,s+i} = \sum_{j=1}^{k} (j-i) p_{ij}$$

and it represents the mean distance covered by the individuals starting from i and, at the same time, the prevailing direction. Indeed, the quantity $s = j - i$ measures the "jump" from i to j, and it has negative value if $j < i$ (downsizing) and positive value if $j > i$ (upsizing). The case $j = i$ is neutral. As a consequence, the expected value $\mathbb{E}[S_t|X_t]$ assumes positive/negative sign if the prevailing direction is towards right/left. Then we can consider the above expected value as an index through which it becomes possible to summarize intensity and direction of mobility.

2.2 A more refined formulation for measuring the direction

We want now to empower the information content of the index:

- to guarantee the possibility to assign different weights to jumps characterized by same magnitude but different starting positions;
- to introduce the possibility to model trough a not linear function the different magnitudes of the jumps.

With this aim we will draw on and empower the suggestions of Alcalde-Unzu et al (2006).
The first aspect has an immediate motivation. Considering as an example the size

of firms, we observe that the transition from the $(k-1)$-th to the k-th class, containing the largest firms, is usually not easy as the transition from the first to the second class. More generally, jumps of the same magnitude might require different effort according with the starting state, and then they have different influence on the whole mobility. Then we introduce the weights ω_i to assign variable importance to the starting state.
The second aspect is again related to the effort needed for moving among classes: introducing in the mobility measure a function $v(|j-i|)$ we assume that the work required for going, for example, from state i to state $i+3$ is not necessarily three times bigger than the effort for going from i to $i+1$.
On such basis we propose now a family of mobility indices defined as follows:

$$I^{\omega,v}(P) = \sum_i \omega_i \sum_j p_{ij} sign(j-i) v(|j-i|) \tag{1}$$

where ω_i is the weight corresponding to the starting state i, such that $\omega_i > 0$ and $\sum_{i=1}^k \omega_i = 1$, v is a suitable function of the distance $|i-j|$, and the function *sign* is defined as follows:

$$sign(x) = \begin{cases} -1 \text{ if } x < 0 \\ +1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$

The main feature of the new index here proposed consists in its capability for measuring the prevailing direction in the dynamics, because it is, by definition, equipped with the sign, negative if the prevailing direction is towards left, positive in the opposite case.

2.3 Properties of $I^{\omega,v}$

In literature many requirements have been introduced in order to obtain a well-defined index and a suitable description of the mobility (Shorrocks (1978), Geweke et al (1986)). We consider here the following properties: *monotonicity*, *immobility*, *perfect mobility* and *boundedness*, because they represent some basic requirements about the nature of the function $I : \mathscr{P} \to \mathbb{R}$. The following proposition proves the validity of such properties for the whole family of directional mobility indices. We require a positive a monotonically not decreasing function v and, without loss of generality, we suppose $v(0) = 0$. The monotonicity of v is legitimately imposed to give a heavier weight to the larger jumps. For sake of shortness, and coherently with the remarks in the previous sections, from now on we will indicate the term $sign(j-i)v(|j-i|)$ with $v(j-i)$.

Proposition 2.1 *1. MONOTONICITY*
An index I is said monotone *if $P \prec Q$ implies $I(P) < I(Q)$, where "$\prec$"*

means that the degree of mobility of P is lower than Q's. The index $I^{\omega,v}$ is monotone by definition.

Proof: the definition of monotonicity is introduced in Shorrocks (1978, p. 1015) with the aim to give "a quasi-ordering over the set $\mathscr{P}$". This definition is not practically applicable because we can not know a priori if $P \prec Q$ or viceversa. To solve this drawback we propose a different point of view, considering the index as a tool to define the order in $\mathscr{P}$. Then, we define the order $P \prec Q$ if $I(P) < I(Q)$, and the monotonicity is met by definition. In this case we define $P \prec Q$, in the sense of the prevailing direction, if $I^{\omega,v}(P) < I^{\omega,v}(Q)$, for a given choice of ω_i and v.

2. *BOUNDEDNESS*
For every choice of ω_i and v, and for every $P \in \mathscr{P}$ we have

$$m_1 \leq I^{\omega,v}(P) \leq m_2$$

where $m_1 < 0$ and $m_2 > 0$ are constants, not depending on P, defined by:

$$m_1 = \sum_{i=1}^{k} \omega_i v(i-1)$$

$$m_2 = \sum_{i=1}^{k} \omega_i v(k-i)$$

Proof: historically an index I is required to assume values in the set $[0,1]$. In the case of directional indices, we need to consider a function I assuming also negative values, because the sign is the primary representation of the prevailing direction. We can prove that every index in the family defined in Eq. 1 assumes values in the closed and bounded interval $[m_1, m_2]$. Let P_- and P_+ be two matrices in $\mathscr{P}$ defined as follows:

$$P_- = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix} \text{ and } P_+ = \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \qquad (2)$$

then $I^{\omega,v}(P_+) = m_2$ and $I^{\omega,v}(P_-) = m_1$. Consider $P \in \mathscr{P}$ such that $P \neq P_+$: consequently there exists at least one $l \in \{1,\dots,k\}$ such that the l-th row of P is $(p_{l1},\dots,p_{lk-1},p_{lk}) \neq (0,\dots,0,1)$. To prove that $I(P) < I(P_+)$ it is enough to prove that the l-th term $\omega_l \sum_{j=1}^{k} p_{lj} sign(j-l) v(|j-l|)$ in the sum for $I(P)$ is smallest than the corresponding term $\omega_l v(k-l)$ in the sum for $I(P_+)$. In fact, thanks to the monotonicity of v, we have:

$$\sum_{j=1}^{k} p_{lj} sign(j-l) v(|j-l|) = -\sum_{j=1}^{l-1} p_{lj} v(l-j) + \sum_{j=l+1}^{k} p_{lj} v(j-l)$$

$$< \sum_{j=l+1}^{k} p_{lj} v(j-l) \leq \sum_{j=l+1}^{k} p_{lj} v(k-l) < v(k-l) \sum_{j=1}^{k} p_{lj} = v(k-l)$$

Analogously, if $P \neq P_-$:

$$\sum_{j=1}^{k} p_{lj} sign(j-l) v(|j-l|) = -\sum_{j=1}^{l-1} p_{lj} v(l-j) + \sum_{j=l+1}^{k} p_{lj} v(j-l)$$

$$> -\sum_{j=1}^{l-1} p_{lj} v(l-j) \geq \sum_{j=1}^{l-1} p_{lj} v(l-1) < v(l-1) \sum_{j=1}^{k} p_{lj} = v(l-1)$$

Consequently $-m_1 < I(P) < +m_2$ for every matrix $P \neq P_+, P_-$.

3. *PERFECT MOBILITY*
*The index $I^{\omega,v}$ is said strongly perfect mobile because it satisfies $I^{\omega,v}(P) = m_1$ if and only $P = P_-$ (*perfect downsizing*) and $I_*^{\omega,v}(P) = m_2$ if and only $P = P_+$ (*perfect upsizing*).*

   Proof: boundedness and strong perfect mobility have been proved at the same time.

4. *IMMOBILITY*
An index I satisfies the (strong) immobility when $I(P) = 0$ if (and only if) the degree of mobility of the matrix P is equal to zero. In the case of directional indices we can prove that:

   (a) if $\omega_i = \frac{1}{k}$, for every i, and P is a symmetric matrix, then $I^{\omega,v}(P) = 0$.

   (b) for every choice of ω, if P is a matrix such that for every $i = 1, \ldots, k$ and for every $l = i-1, \ldots, k-i$ it holds $p_{ii-l} = p_{ii+l}$, then $I^{\omega,v}(P) = 0$.

   Proof: consider a symmetric matrix $P \in \mathscr{P}$, having chosen $\omega_i \equiv \frac{1}{k}$, for every $i = 1, \ldots, k$. Then the two terms

$$\omega_i p_{ij} sign(j-i) v(|j-i|)$$

and

$$\omega_j p_{ji} sign(i-j) v(|i-j|)$$

cancel out for every i and j, and $I^{\omega,v}(P) = 0$, proving *(a)*.
Analogously, if $p_{ii-l} = p_{ii+l}$ for every $i = 1,\dots,k$ and $l = i-1,\dots,k-i$ (which means $p_{ii\pm l} = 0$ if the correspondent $p_{ii\mp l}$ is not defined) then the two terms

$$\omega_i p_{ii-l} sign(-l) v(l)$$

and

$$\omega_i p_{ii+l} sign(+l) v(l)$$

cancel out and $I(P) = 0$ again, proving *(b)*.
We note that the two cases described in *(a)* and *(b)* are actually related to the absence of mobility, in the sense of prevailing direction. Indeed, in both the cases, there is a sort of symmetry which makes positive contributes to the mobility to be balanced by the negative ones (an explicit example will be provided in the following section). Nevertheless we remark that *(a)* and *(b)* do not cover the whole set of matrices with null value of the index. In this sense, only the weak version of immobility is proved.

2.3.1 Normalization of $I^{\omega,v}$

The last proposition proves that the function $I^{\omega,v}$ assumes values in the closed and bounded interval $[m_1, m_2]$, where $m_1 < 0$ and $m_2 > 0$ and both depend on k, v and $\{\omega_i\}_i$. A further transformation of the index is required, to obtain a function with values in $[-1,+1]$ which makes easier the comparison among different matrices. The naive idea consists in calculating the linear transformation $I \in [m_1, m_2] \to I' \in [-1,+1]$ given by the formula $I' = \frac{2}{m_1+m_2} I + \frac{m_1-m_2}{m_1+m_2}$. Unfortunately this choice has a relevant drawback when the interval $[m_1, m_2]$ is not symmetric respect to 0, that is when $m_2 \neq -m_1$. Indeed it happens that matrices with the original index value $I^{\omega,v}(M) < 0$ result to have positive normalized value $I'(M)$ and viceversa.
To solve such a drawback we propose a different transformation to normalize the index, according with the following definition:

$$I' = \begin{cases} -\frac{1}{m_1} I \text{ if } I < 0 \\ \frac{1}{m_2} I \text{ if } I > 0 \end{cases} \tag{3}$$

From now on we will indicate with $I^{\omega,v}$ (or, shortly, with I) the index defined in Eq. 1, and with I' its normalized version.

3 The distribution of the mobility

To analyze some mathematical properties and the expected behavior of the directional index, we perform the following trial: assuming that, given k, transition

matrices are uniformly distributed in $\mathscr{P}$, we randomly draw 20000 matrices and we evaluate the corresponding index value. On this basis we obtain the observed density distribution of I (built up using 100 histograms), which provides some information on its expected behavior, when evaluated on a specific matrix.
In the following we will consider mainly matrices with 3, 6 and 9 states, since these three values represent typical numbers of classes used to analyze empirical datasets (among others see Frydman et al (1985), Fougere and Kamionka (2003), Geweke et al (1986) and Frydman and Kadam (2004)).
For sake of simplicity we set $v(j-i) = sign(j-i)|j-i|$. Different choices of v will be treated in Sect. 4.2.

3.1 The directional index as a mixture

Any index in the family (1) can be split as a sum of k terms, each one describing the mobility due to individuals starting from state i, for $i = 1,\dots,k$. In fact we can write

$$I^{\omega,v}(P) = \sum_{i=1}^{k} \omega_i I_i(P)$$

where $I_i(P) = \sum_{j=1}^{k} p_{ij} v(j-i)$. In consequence of that, the index is actually a mixture of different terms I_i. Then we start the analysis drawing 20000 matrices and evaluating the term $I_i(P)$ for a fixed i.

Fig. 1 displays the observed distribution of I_1, I_2 and I_3, that is the contribution to the index value given by individuals starting from the first, second and third state, for k =3, 6 and 9. To have a deeper insight we consider as an example the term I_1, varying k, and we evaluate skewness and kurtosis, showed in Fig. 2.

In the following we list some evident features of the distributions.

- The distribution's shape is strongly symmetric and does not vary with i, whereas it results to be shifted and centered in its mean value $\frac{1}{k}\sum_j v(j-i)$ (see Sect. 3.3). The skewness is, as expected, around zero.
- The same distribution is triangular for small k's, and tends to be closer and closer to a Normal distribution when k increases. In fact the kurtosis grows with k and tends to three.
- Every term I_i is a function of P assuming values among its minimum, given by $v(1-i)$, and its maximum, equal to $v(k-i)$.
- The variance of I_i grows with k, as a consequence of its increasing range $v(k-i) - v(i-1)$.

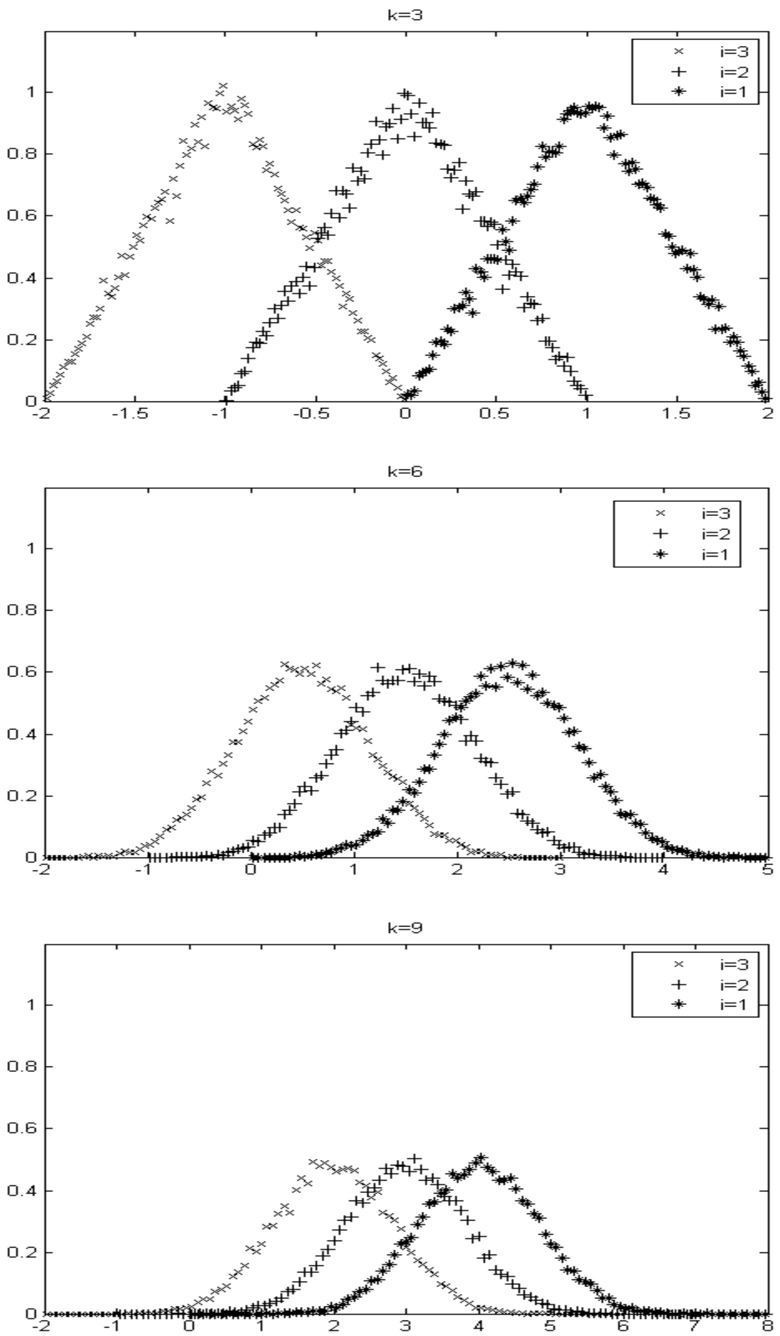


Figure 1: Contribution to the mobility of the first three states, for $k = 3, 6, 9$.

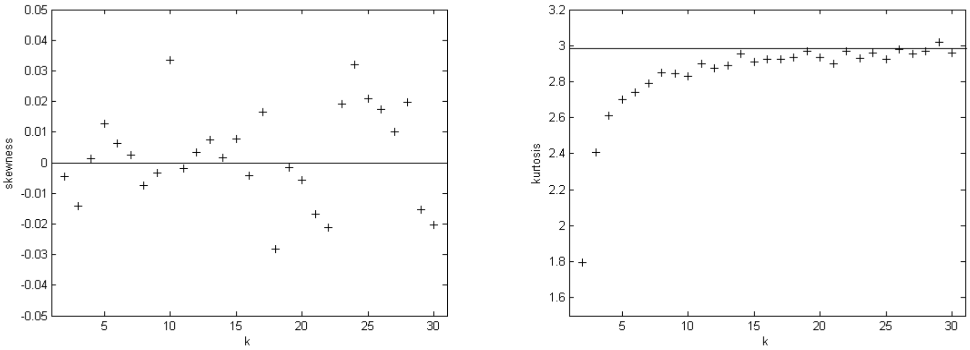


Figure 2: Skewness and kurtosis of I_1, w.r.o. k.

The knowledge of the distribution of every I_i is useful to deduce the shape of the distribution of I. Furthermore, for large k's; we can say that such distribution is approximately a mixture of Normal distributions.

3.2 The normalized directional index compared with other mobility indices

We propose here the comparison of the directional index with other indices, in particular with I_{tr}, I_2 and I_{det}. From now on we will use the normalized index I', given by Eq. 3, because it makes easier the comparison with the indices and different values of k. Fig. 3 displays the observed distribution of I', again based on 20000 draws. The weights ω_i are equal to $1/k$ and $v(j-i) = sign(j-i)|j-i|$ as before.

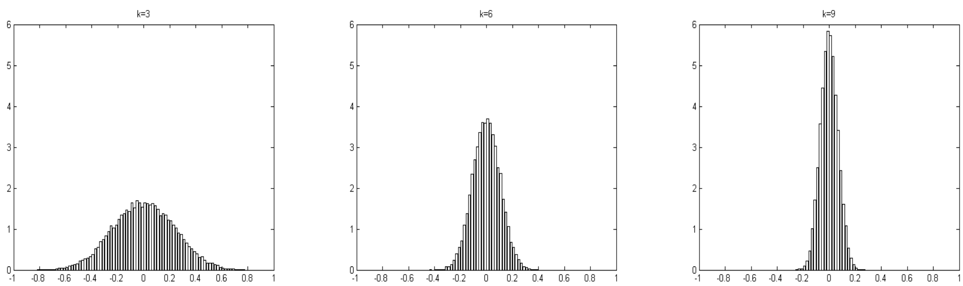


Figure 3: Density distribution of the normalized index I', for k =3, 6 and 9, with $\omega_i \equiv \frac{1}{k}$ and $v(j-i) = sign(j-i)|j-i|$.

We repeat the same experiment for the indices I_{tr}, I_2 and I_{det} (Fig. 4[1], 5, and

[1]Note that the trace index is not normalized, then we normalize it by dividing by its maximum

6). The shape of the resulting distributions reveal a strong asymmetry. The determinant index represents an extreme case because it shows an high concentration around 1.

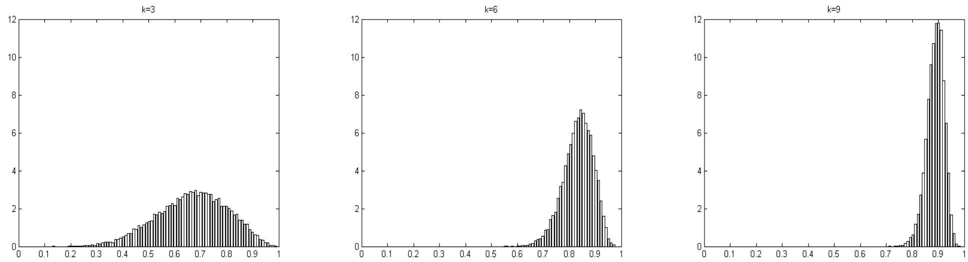


Figure 4: Trace index for $k = 3$, 6 and 9.

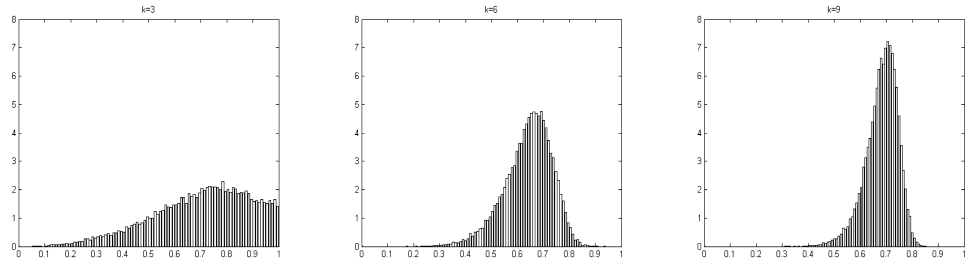


Figure 5: Second eigenvalue index for $k = 3$, 6 and 9.

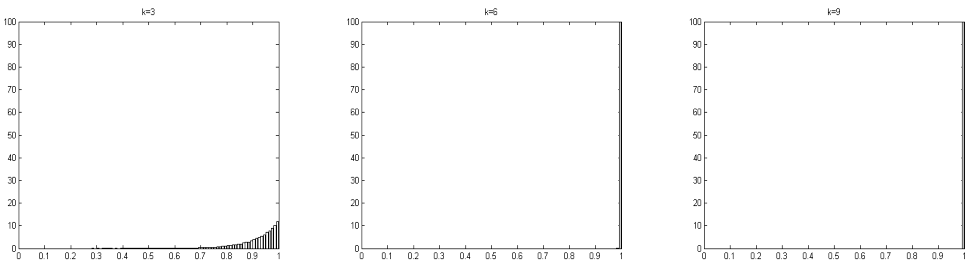


Figure 6: Determinant index for $k = 3$, 6 and 9.

value $\frac{k-1}{k}$.

The decreasing variance of the normalized index with respect to k can be explained by the fact that both the variance and the range in the not-normalized version of the index $I^{\omega,\nu}$ tend to increase with k. This is not the only reason, in fact we note that I_{tr}, I_2 and I_{det} have the same behavior. The decreasing variance can be motivated observing that if the number of states increases, individuals have more chances to move. As a consequence the absolute measures of mobility tend to assume larger values with higher frequency. This claim is supported by the fact that the distribution of I_{tr}, I_2 and I_{det} is shifted (and more concentrated) towards 1 when k increases. From the point of view of the directional indices, the (positive) contribute to the mobility given by individuals moving from i to j, with $i < j$, is, with higher probability, balanced by the (negative) contributes due to movements in the opposite direction. In consequence of that the index assumes with higher probability values near 0. In this sense the index value should be read considering also the number of states k, since $I' = 0.2$, for example, corresponds to a rarer case for $k = 9$ than for $k = 3$.

3.3 The expected value of $I^{\omega,\nu}$

We provide now a formula for the expected value of the not-normalized index $I(P)$. By linearity we have $\mathbb{E}(I(P)) = \sum \omega_i \mathbb{E}(I_i)$, where I_i is a function only of the i-th row of P, indicated with $(p_{i1}, \ldots, p_{ik})$. Rows of P are random vectors in the simplex $\Delta_{k-1} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k | x_i > 0, \sum x_i = 1\}$. As before we suppose that vectors are uniformly distributed in Δ_{k-1}, with p.d.f. $f(P_{(i)}) \equiv (k-1)!$. The expected value of I_i is

$$\mathbb{E}(I_i) = \int_{\Delta_{k-1}} f(P_{(i)}) I(P_{(i)}) dP_{(i)} = (k-1)! \sum_j \nu(j-i) \int_{\Delta_{k-1}} p_{ij} dP_{(i)}$$

Let g_j be defined by $g_j(x_1, \ldots, x_k) = x_j$, then $\int_{\Delta_{k-1}} g_j(x) dx = \frac{1}{k!}$. In fact it is always possible to reorder the coordinates for obtaining $x_j = x_{k-1}$ and rewriting Δ_{k-1} as

$$\Delta_{k-1} = [0,1] \times [0, 1-x_1] \times \ldots \times [0, 1 - \sum_{j=1}^{k-2} x_j]$$

(it is a $k-1$-dimensional space in $\mathbb{R}^k$), then:

$$\int_{\Delta_{k-1}} x_k dx = \int_0^1 \left(\int_0^{1-x_1} \left(\ldots \int_0^{1-\sum_{j=1}^{k-2} x_j} x_{k-1} dx_{k-1} \right) \ldots dx_2 \right) dx_1 =$$

$$= \frac{1}{2} \int_0^1 \left(\int_0^{1-x_1} \left(\ldots \int_0^{1-\sum_{j=1}^{k-3} x_j} \left(1 - \sum_{j=1}^{k-2} x_j \right)^2 dx_{k-2} \right) \ldots dx_2 \right) dx_1 =$$

$$= \frac{1}{2} \cdot \frac{1}{3} \int_0^1 \left(\int_0^{1-x_1} \left(\dots \int_0^{1-\sum_{j=1}^{k-4} x_j} \left(1 - \sum_{j=1}^{k-3} x_j \right)^3 dx_{k-3} \right) \dots dx_2 \right) dx_1 =$$

$$= \dots = \frac{1}{k!}$$

Consequently we have

$$\mathbb{E}(I(P_{(i)})) = \frac{1}{k} \sum_j v(j-i)$$

and

$$\mathbb{E}(I(P)) = \frac{1}{k} \sum_i \omega_i \sum_j v(j-i)$$

Note that when ω_i are symmetric, that is $\omega_1 = \omega_k$, $\omega_2 = \omega_{k-1}$ etc..., the the expected value is equal to zero.

4 The role of $\{\omega_i\}$ and v

4.1 The importance of the weights ω_i

In many empirical applications of transition matrices and mobility indices, we do not have information on the weights to give to the i-th state. We consider now the starting distribution p_0, whose element $p_0(i)$ is the probability of starting from i (or the observed percentage of individuals starting from i). In the previous sections we have decomposed the directional index as a mixture of contributions due to individuals starting from different states; in the light of that we propose to assign to p_0 a role in the degree of mobility, by setting $\omega_i = p_0(i)$, for every $i = 1, \dots, k$. The advantages of this choice are pointed out by the following example: we consider $k = 2$, and the transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

According with P, every individual is forced to move from its starting state. Evaluating different indices on P we obtain:

- $I_B(P)$ is not evaluable, because the stationary distribution π does not exist;
- $I_{det}(P) = I_2(P) = I_e(P) = 0$;
- $I_{tr}(P) = 1$.

Such results are not surprising: I_{det}, $I_2(P)$ and $I_e(P)$ are functions of the eigenvalues, and then they measure the rate of convergence towards the stationary distribution, equal to 0 since the chain ruled by P does not converge. On the other hand I_{tr} reaches the maximum value because it describes the tendency to leave the current state.
None of the mentioned indices depend on the number of individuals starting from every state. On the contrary, from the point of view of the prevailing direction, we claim that:

1. if individuals are uniformly distributed among the two states, movements towards left cancel out with movements towards right, and the whole mobility should be null;

2. if the starting numbers of individuals in the states are not balanced, the mobility value should be influenced by the fact that a larger number of individuals moves from left to right than from right to left, and viceversa.

In the previous example, since $k = 2$, we have $p_0 = (p_0(1), 1 - p_0(1))$. To analyze the influence of p_0 on the mobility, we calculate the directional index $I'(P)$ for every value $p_0(1) \in [0, 1]$. Fig. 7 displays the variation of the index value respect to the percentage of individuals in the first state.

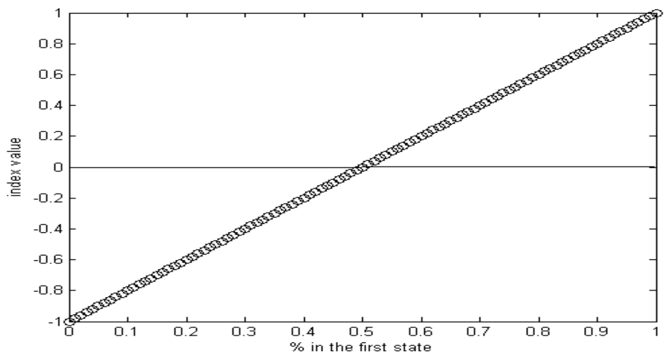


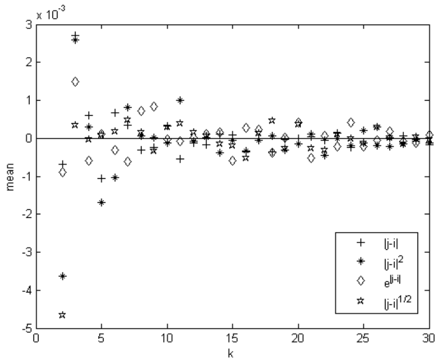
Figure 7: Index value I' varying the weights ω_i, with $k = 2$.

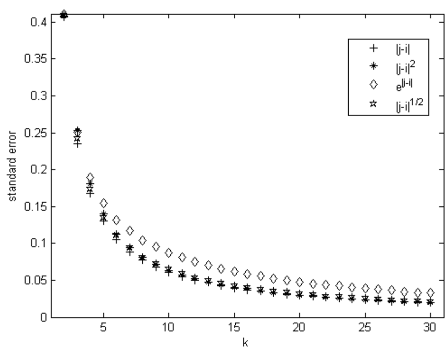
When the sample is equally distributed among the states, positive contributes to the mobility are perfectly balanced by the negative ones, and the index is equal to 0 (as required by point $4a$ of Prop. 2.1). In the extreme case of $p_0(1) = 1$ (or, equivalently, $p_0(1) = 0$), the individuals can provide only positive contributes to the mobility, and the index is equal to 1. In this sense, the directional index allows a deeper insight in the mobility of the sample under study.

4.2 The role of v

The function v in the index has the role to provide a measure for the covered distance, given by $|j-i|$, $i,j=1,\dots,k$. In the previous sections we have used a linear measure of that distance. It means that jumps from a give state i to the state $i+2$ has a double weight in the mobility measure with respect to the jump from i to $i+1$. Such linear measure of the jumps is a suitable choice in many empirical cases, for example when the classes are equally spaced, such as the income subdivided in fractiles (Geweke et al (1986)), or when the variable under study is qualitative, such as the bond ratings (Frydman and Kadam (2004)).
The relevance of the function v become evident with transition matrices based on not equally spaced classes. This is the case of income's matrices where the states are intervals increasing in the length (Champernowne (1953)). In such cases we might need a mobility measure which includes the fact that moving among farther states can require, for example, an exponentially increasing effort.
We conclude by showing the influence of the function v on the distribution of I', still based on 20000 draws. Fig. 8 displays mean and standard error of I', having set $v(j-i)=sign(j-i)|j-i|$, $v(j-i)=sign(j-i)|j-i|^{1/2}$, $v(j-i)=sign(j-i)|j-i|^2$, $v(j-i)=sign(j-i)(e^{|j-i|}-1)$. Weights are still equal to $\frac{1}{k}$.
As expected, the choice of v has not a relevant influence on the mean, which, according with Sect. 3.3, is equal to zero. On the other hand we note that the standard error is higher, particularly if we choose an exponential v.
To conclude we note that we can not propose an objective method for choosing v, as in the case of the weights ω, because such choice is strictly related to the specific dynamics and to the kind of variable we are working with.

Figure 8: Mean and standard error of I' w.r.o. k, for squared, linear, quadratic and exponential v.





Conclusions

In this work we face the problem of defining a mobility index, which is a function of the transition matrix, able to indicate both the intensity of the mobility in the dynamics under study and its prevailing direction. This kind of index represents a useful tool when we treat ordered economic variables such as firms size or income, which can be affected by downsizing or worsening.
We propose a family of new indices which are able, besides providing information on the prevailing direction, to guarantee the possibility of assigning a different role to individuals starting from different states, and increasing weights to larger jumps. All the indices in such family are equipped with the properties of boundedness, strong perfect mobility and weak immobility, and they can be used to reveal an order among transition matrices in the set $\mathscr{P}$.
Further researches will regard an application to data derived from administrative archives, with an analysis of the properties of the sampling distribution of the directional indices.

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