UNIVERSITÀ CATTOLICA DEL SACRO CUORE

Dipartimento di Economia e Finanza

Working Paper Series

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Working Paper n. 21

December 2014

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del Sacro Cuore

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Rational Overconfidence and Social Security

by

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Abstract: Is an assumption of bounded rationality needed to explain Social Security and other mandatory pension plans? In this contribution we argue that when rational agents hold inconsistent expectations such programs may be justified.

Two of the features that distinguish Social Security and many other state mandated pension plans around the world are that (i) a minimum level of savings for retirement is imposed on most citizens and (ii) individuals cannot freely decide how their contributions are invested. Here, a rationale for these two features, based on rational overconfidence, is proposed. Rational overconfidence is present when equally informed agents hold diverse confident, rational beliefs. The fact that beliefs are diverse means that all of them cannot be correct, hence seen as a collective agents do not act optimally.

In the face of rational overconfidence, Pareto efficiency is no longer the natural criterion for comparing policies and we suggest ex-post welfare optimality in stead. This criterion makes amends for the possible inconsistencies of agents' beliefs.

Our results on social security are based on a methodology that places itself strictly between the traditional neoclassical approach and that championed by behavioral economics. This methodology does not deviate from the neoclassical assumption of rationality but only broadens it and can therefore readily be applied to many public policy issues.

JEL classification number: D01, D02, D63, D81, D84, H55
Key words: Subjective Expectations, Rational Beliefs, Ex-post Welfare Optimality, Social Security, Rational Overconfidence, Portfolio Choice.

*This work has benefitted from discussions with Ken Arrow, Angiola Cottini, Carlo Devillanova, Peter Hammond, Mordecai Kurz, Maurizio Motolese, Hiro Nakata, Chang-Ho Yoon, Ho-Mou Wu and in particular Tiancheng Sun. It was presented at the EEA 2008 (Milan), SIE 2008(Perugia) and INEX 2012 (Paris) meetings as well as at seminars at Korea University, Stanford University, Universitá Cattolica, University of Essex and Humboldt University. I thank members of the audiences for useful comments. Sections 1 to 4 of this paper rather closely follows the corresponding sections in Discussion Paper no 0916 (with the same title) from the Institute of Economic Research at Korea University (2009), while section 5 is entirely new.

The King's Dilemma:
In the neighboring kingdom the bad king has announced that his daughter, the princess will be married to the one she likes the most among any suitable suitor that shows up next Sunday. Those who are turned down by her are fed to the dragon in the moat. The good king has three sons, Wit, Looks, and Charm who are the only ones who would be considered suitable suitors. Each of them is convinced that the princess will choose him, if he is allowed to go, no matter whether the other two princes show up or not and is sufficiently egoistic not to refrain from going out of concern for his brothers. If the good king is Paretian, he should allow all of them to go, knowing that with certainty one of them will be married, while the two others will serve as dinner for the dragon (after all, much worse than being a bachelor prince). If he is concerned with ex-post efficiency he should let exactly one of them go. Compared to letting all of them go, this decision would make two of the three princes subjectively worse off and the one allowed to go not subjectively better off, and yet seen as a collective they are objectively better off.[1]

1 Introduction

1.1 Basic Ingredients

Many developed nations have either government mandated pensions plans or social security plans, which guarantee a minimum income for those who reach retirement. In this study we argue that to explain such policies, which override individual freedom of choice, there is no need to resort to an assumption of individual irrationality. Rather, we use an approach to thinking about economic policy, introduced in Nielsen(2003), which has as its main building blocks the concepts of rational overconfidence (defined in Nielsen, 2008) and a strengthened version of ex-post welfare optimality (Starr, 1973, Harris, 1978, Hammond, 1981).

The starting point of our analysis of government mandated savings plans and Social Security is an assumption of diverse rational beliefs (Kurz, 1994) in the population. Rational overconfidence is present when agents rationally use distinct subjective models (or beliefs) for interpreting data that is publicly available in the form of a stationary stochastic model, called the empirical distribution. In the words of Varian(1985 and 1992) agents in our model have different opinions but not different information. Since each of these subjective models, if it were correct, would be superior in terms of forecasting power relative to the stationary model, we say that these models/rational beliefs are confident. Since these confident rational beliefs are diverse, at most one of them is correct, thus collectively there is rational overconfidence. This does however not mean that anyone can single out any individual belief as being incorrect.

In the presence of rational overconfidence the concept of ex-post welfare optimality rather than the Pareto criterion (which is based on individual subjective beliefs) becomes relevant when comparing different policies. When individuals take actions based on diverse beliefs, there must be some degree of inefficiency present, as seen from the point of view of society.[2] Knowing that such an inefficiency is present does not necessarily imply knowing how to remedy it. However, we think that there is in many instances of real life a commonly agreed upon core of knowledge, which may form a basis for a commonly agreed upon assessment of the consequences of various policies.[3] The central assumption of rational belief theory is precisely that such a core of objective knowledge

[1] *The King's Dilemma* is taken from Nielsen(2006) and Nielsen (2009a).See also Nielsen(2003).

[2] *The King's Dilemma* illustrates our point: The sum of individual expectations about rewards (= 3 marriages) is greater than the objective sum (= 1 marriage).

[3] In *The King's Dilemma* this knowledge, shared by the king and the princes, is about the rules of the contest for the princess. Any belief (about the princess' preferences) held by the King that embodies the known facts will lead to the same set of ex-post

exists (in the form of a stationary stochastic model) and (hence) that all individual beliefs are compatible with this knowledge. Our approach then provides a framework that integrates the notion of commonly agreed upon information (as for example exposed in the rational beliefs approach to modeling expectations) and the notion of ex-post welfare optimality. The result is a tool for policy analysis that may be used in situations where diversity of beliefs plays an important role.

The importance and prevalence of diverse beliefs have been demonstrated in f.i. Branch (2004), Chavas (2000), Evans and Gulamani (1984), Frankel and Froot (1987), Ito (1990), Kurz (2001), Kurz and Motolese (2011) and Souleles (2004) and there is now a large literature devoted to the implications of such diversity, both empirically and in terms of policy (we shall refer to some contributions below). The above studies as well as many others reveal the shortcomings of the rational expectations approach to assessing the potential benefits of social security and other plans for mandated saving for retirement. At the other extreme is the position that individuals are irrational (in fact more irrational than the decision maker), as witnessed in the behavioral economics literature.[4] Our position is strictly between these two and an important (if not the most important) contribution of the theory of rational beliefs is to make room for such a position. As we shall see, rational beliefs theory provides a tractable model consistent with the claim that *societal irrationality* may be present concurrently with *individual rationality* also in the context of expectations.[5]

When making investments/savings decisions for their retirement, otherwise identical agents who are employing rational beliefs may end up pursuing very different savings strategies for their retirements: some save (invest) too much, while others too little from the point of view of society. By mandating a minimum level of savings, the government makes sure that no one is saving too little (these savings may be required to be invested in safe assets or a relatively safe portfolio, to avoid, on average, suboptimal returns). However, since we assume, realistically, that it is not possible to put a cap on savings, there may still be some who invest too much.

There is disagreement among economists about whether or not undersavings is a serious problem in high-income societies. The reasons for this disagreement is twofold: firstly it is not clear how to define adequacy of savings, secondly there are data issues. According to calibrations by Scholz et al. (2006) only about 15 % of Americans in the $1931-41$ cohort save too little while more save too much. These authors do not provide data on how many households are effectively bound by the minimum savings level imposed by social security rules (and possibly other mandatory savings programs). Engen et al. (2004) report adequate or even oversavings for most of the population in the same cohort as used by Scholz et. al (2006), but that the bottom 25 % are undersaving and rely heavily on social security.[6] Bernheim and Rangel (2005) cite evidence suggesting the prevalence of undersavings, in particular that consumption tends to fall around the time of retirement, indicating that many people have been unsuccessful in achieving consumption smoothing over their life time.

The commonly accepted empirical distribution offers a natural, neutral way of calculating the expected welfare of a mandatory savings program (and of alternatives to it). With the government mandatory minimum savings policy, that we shall interpret as social security, every individual agent agrees that society is strictly better off and in that sense society is *objectively* better off, and yet every individual agent is either subjectively worse off or subjectively not better off. Many economist would argue that in such a situation there could not be voter backing for a mandatory minimum savings policy. Despite this, according to a 2004 poll by The

efficient decisions, namely that exactly one prince should be allowed to go.

[4]See f.i. Bernheim and Rangel(2005 and 2007)

[5]The Pareto inferior Nash equilibrium in the Prisoner's Dilemma Game is the classical example of individual rationality paired with societal irrationality.

[6]We note that concurrent under- and oversavings is a general feature of the models used here.

Wall Street Journal, a majority of Americans at that point thought that there should be something like Social Security. Interestingly, 46% of the individuals polled expressed opposition to letting individuals invest part of their social security savings in stocks and bonds, while only 38% stated that they would *not* take advantage of this opportunity, should it become available. The theory we propose can be seen as explaining the opinions of some of those polled without resorting to assumptions about some sort of irrationality.

Our contribution has two aspects to it. Firstly, it is normative in proposing that in some situations (like in *The King's Dilemma*) the Pareto criterion should be substituted by an ex-post criterion for efficiency. Secondly, it is positive in suggesting that the prevalence of diverse rational beliefs explains observed suboptimal savings decision in the economy at large. We then propose as a rationale for the government mandated savings for pensions we observe that they, in the ex-post sense, offer an improvement in welfare. From a strictly theoretical point of view, our contribution is to highlight that markets may fail to work properly due to an inconsistency of expectations. In particular, we present an example (Section 3) with trivially complete markets, an absence of all other traditional sources of market failures and with ex-ante identical and rational individuals, and yet, where restricting the actions of these individuals leads to an objective increase in overall welfare.

1.2 Other explanations offered for the existence of Social Security

Gordon and Varian (1988) and Gotardi and Kubler (2011) among others propose that social security is a remedy against the impossibility of private insurance contracts between unborn and living generations. Their analysis is however confined to the case of only two generations living concurrently. It is an open question whether their results would also be quantitatively important in a context where there are at each date many generations of agents who can trade insurance contracts with each other.

Others, like Feldstein (1985 and 2005), hold that some agents are myopic and fail to make adequate provisions for the future. Feldstein (2005) at the same time proposes that agents be allotted individual social security accounts and be allowed to invest the savings in these accounts in portfolios of stocks and other assets. It seems puzzling though, that an agent who is deemed unable to plan for the optimal stream of consumption over his own lifetime at the same time is deemed capable of planning an investment strategy for the stock market, over the same horizon. Note also that an explanation based on myopia implicitly is an explanation based on an assumption that the government is less myopic than the public.

An alternative, positive explanation for the transfer to future old (currently voters) from future young (currently non-voters) entailed in some social security programs is political power. This idea is formalized in Tabellini (2005), but the results obtained hinge on the assumption that the social security system embodies a substantial transfer from more to less well-off agents, something that according to Feldstein (2005) is not the case for the American Social Security program.

The so-called "Samaritan's dilemma" may provide a reason for why people are required to have a minimum savings level for retirement (but not for why this level is increasing with income or for why oversavings is observed). The explanation is based on the unwillingness of many societies to ignore abject poverty. This means that if the disposable income of a retired person is too low, this person will receive support from the government. Knowing this, some young agents may decide not to save up for retirement. If the government's threat to punish such persons by withholding support is not credible, an alternative may be to require everyone to have a minimal savings level. Note however that this explanation assumes, like we do here, that individuals are willing to forfeit personal welfare to the benefit of other members of society.

While we explore a particular possible explanation for government mandated pension plans, this should not be seen as a rejection of all other explanations - many factors probably play a role in justifying the institutions we observe. However, we do wish to make a methodological point: In our model, agents are rational, but many are wrong. The government's authority is not based on being better informed, more rational (less myopic) than anyone else, but on being able to impose a savings level that *everyone* agrees is *on average* better i.e. in this sense objectively improving.

1.3 Outline

The results obtained in this study are of a qualitative nature and we have opted for simplicity in demonstrating the intuition behind them.[7] We first present, in Section 2 a review of rational beliefs theory and of rational overconfidence (based on Nielsen, 2008) as well as a brief discussion of ex-post welfare optimality. In Sections 3 and 4 respectively, two extended examples are presented, each of them emphasizing one of two aspects of the typical social security arrangement, namely that individuals are required to have a minimum saving for retirement and that there are restrictions on how these savings can be invested. In the first example there is (private) production, but no economic interaction between ex-ante identical individuals. In particular there are no types of externalities that effect individuals, no types of strategic behavior and no benefits from opening new markets. Even then, putting a lower limit on individual savings (which does not imply any explicit or implicit transfers between individuals) objectively improves societal welfare. This example thus serves to isolate the logic of our argument and in particular shows that this logic is theoretically independent of any other motive for market intervention, that has traditionally been studied by economists. Our second example considers an exchange economy, hence with economic interaction, and serves to illustrate that putting limits on how savings are invested may improve objective societal welfare. In Section 5 we illustrate the logic of our argument in the context of a traditional OLG growth model (Diamond, 1965). Here the challenge is to design a social security policy that, in an objective sense, does not lower welfare for any particular type or generation of agents. As it turns out, for this policy social security pensions are only partially financed by contributions, the rest being paid by taxes on wages, which happens to be in line with how the American social security system was designed. We conclude in Section 6.

2 Rational Overconfidence and Ex-post Welfare Optimality

2.1 Rational Beliefs

We shall assume that, at each date $t = 1, 2, 3, \dots$, there is a continuum of agents, indexed by $i \in [0, 1]$ who live for two periods (overlapping or non-overlapping). To explain the idea of rational beliefs which originates with Kurz(1994), we use some particulars of the first of the two examples presented below.[8] In this example any individual agent can at any date, t choose to invest in a riskless asset with exogenous gross return, $r > 0$. The only uncertainty is about the future common initial wealth level of individuals (i.e. about the general wealth level of society excluding savings), W_2. An agent i, with a utility function u, strictly increasing and strictly concave, at date t, solves :

$$\max_{S \in [0, W_1]} E_{it} u(W_1 - S, W_{2t+1} + Sr)$$

[7] For a more formal presentation in the context of the choice of exchange rate regime in a stochastic exchange economy, see Nielsen (2003) and Nielsen (2009a).

[8] See also Appendix 7.1.

where E_{it} denotes his expectation over second period initial wealth, W_{2t+1} and W_1 is the non-stochastic wealth that he starts with.

Empirical analysis is concerned with averages which traditionally has been thought only to make sense when the underlying stochastic process is stationary.[9] It turns out, however, that under a weaker assumption, that of weak asymptotic mean stationarity (WAMS, also called stability) empirical analysis is still possible, although less knowledge can be derived from it. In the model used here, the stochastic sequence, $\{W_{2t}\}_{t=1}^{\infty}$ is assumed to be stable, meaning that there is an empirical distribution, $\overline{\mu}$ for it, based on the limit frequency of finite events.[10] To simplify, suppose that W_{2t} can take only two values, $W^L < W^H$. Based on many (past) observations it is known that the frequency of W^L is $\overline{P}_L$, so that $\overline{P}_H = 1 - \overline{P}_L$ is the frequency of W^H. It is also known that the empirical distribution for other events is the product of these frequencies. For instance, the frequency of W^L followed by W^H is $\overline{P}^L \overline{P}^H$. These frequencies are averages and not necessarily the true probabilities. However, if it were also known that $\{W_{2t}\}_{t=1}^{\infty}$ is stationary, the knowledge about these frequencies would mean that at all dates t, the true probability that $W_{2t} = W^L$ would be $\overline{P}_L$. Stability is a weaker assumption than stationarity, since with it we can only conclude that *on average* the probability of W^L is $\overline{P}_L$. Specifically, if at a fraction $Q \in (0, 1)$ of all dates $t \in \{1, 2, 3, \ldots\}$, the probability is P_L^0 while at the remaining fraction $1 - Q$ of dates the probability is P_L^1, where $QP_L^0 + (1 - Q)P_L^1 = \overline{P}_L$ (and if the dates where P_L^0 is the true probability are sufficiently dispersed) then, for all realizations of $\{W_{2t}\}_{t=1}^{\infty}$, the frequency of W^L will be equal to the observed one, namely $\overline{P}_L$.[11] Thus believing that there are such two one-period probabilities P^0 and P^1, each in force at different dates, rather than one, $\overline{P}$ at all dates, is consistent with the empirical observations, in the sense that if this were really the case, we would observe the frequencies we actually do observe. Such a belief is therefore called a rational belief. Formally, it is a deterministic sequence of probabilities $\{P_t\}_{t=1}^{\infty}$ taking values in $\{P_0, P_1)$ with the just stated properties.[12]

We shall assume that agents are members of dynasties such that each rational belief, in the format of the sequence, $\{P_{it}\}_{t=1}^{\infty}$, is not only that of the individual agent, i, but also that of the entire dynasty, which we also index by $i \in [0, 1]$. Each dynasty is assumed to have a different such sequence, but at every date a fraction Q of the agents in $[0, 1]$ use P^0 while the rest, a fraction $1 - Q$, use P^1.[13] Thus beliefs are diverse at all dates. Note that this construction embodies the idea of Muth (1961), who did *not* assume that all agents have correct beliefs, that is what is now commonly referred to as rational expectations, but only that the *average belief* is equal to the "prediction of the theory" (Muth, 1961, p. 316).

Remark 1 *Endogenous Uncertainty*
If at any date t all agents either use P^0 or P^1, beliefs would not be diverse and the argument presented here in favor of mandatory savings would not carry through. In this case beliefs are correlated, in other words the distribution of beliefs varies over time. In many other studies using rational beliefs (see f.i. Nielsen, 1996 and Kurz and Motolese, 2001) changes in the distribution of beliefs over time is of central importance, giving rise to so called *endogenous uncertainty.* With this types of uncertainty, whenever agents interact, they implicitly or explicitly form expectations about future beliefs of other agents (akin to the Keynes' *Beauty Contest* story).

[9] To take an example, to say that the average population on earth over the last 2 millennia has been X millions does not provide much information, since that population has been highly non-stationary.

[10] By the limit frequency of the set A is meant the limit of $\frac{1}{T}\#\{t \leq T : w_{2t} \in A\}$ as the number, T of observations tends to infinity. See Appendix 7.1.

[11] Almost all realizations, to be precise.

[12] The probability measure just outlined, which is called a SIDS belief, was introduced in Nielsen (1996).

[13] A full specification of this construction is provided in Nielsen (2008). The results obtained here would remain also in the case of complete diversity at all dates i.e. if we assumed that the distribution of rational beliefs is such that at any date every agent holds a different belief about the next period (while the distribution of beliefs is constant over time).

These expectations about the future distribution of beliefs may then be diverse, giving rise to a further layer of inefficiency, see Nielsen (2003) and Nielsen (2009a) ■

Remark 2 *Alternative to dynasties*
In stead of assuming that short lived agents inherit the beliefs of the infinite lived dynasty they belong to, we could assume that beliefs, defined on infinite sequences, are at each date randomly assigned to agents. For every single young agent the relevant part of any belief assigned to him is its prediction for the periods where he is alive (one period in our model). Because there is to each agent ample data available on the past performance of the economy, it does make sense to assume that he or she can form a rational belief for the entire future. Because we assume a continuum of agents, we may assume that the distribution of belief is constant over time (see Nielsen, 2007 for more on this construction) ■

2.2 Rational Beliefs and Rational Overconfidence

Holding a non-stationary rational belief means being more confident about one's ability to predict the future than when holding the stationary rational belief. This is most easily seen by imaging the following situation: We are being told that at the beginning of time Nature has picked for any date t with probability Q the probability vector (P^0_L, P^0_H) for W_{2t} and with probability $1-Q$ the probability vector (P^1_L, P^1_H). If an agent thinks that he does not know the choice of Nature, the probability he should use in forecasting W_{2t+1} is $\overline{P} = (\overline{P}_L, \overline{P}_H)$, the compound probability for the two states. If, in contrast, at date 0 the agent is being told the whole sequence of Nature's choices, that is for every date in the future, whether P^0 or P^1 is the true distribution, this extra information allows him to make a more precise forecast at every date. An agent holding a non-stationary rational belief in effect believes that he knows the entire sequence of Nature's choice of probability vectors and in this sense believes his forecasts to be more precise (having a smaller support) than that based on the stationary, empirical distribution[14]. We therefore say that his belief exhibits rational confidence. Since we assumed that every agent (dynasty) has a different sequence of one-period beliefs, at most one of them (we do not know who, if any) has the correct timing and hence the correct belief. Thus looked at collectively rational overconfidence is present: the vast majority of agents (or more precisely, dynasties) hold a belief which is rational, confident *and* not correct. We summarize with two definitions:[15]

Definition 1 *Confident Belief*
A rational belief is said to exhibit confidence if it is more precise (has a smaller support) than the empirical distribution ■

Definition 2 *Rational Overconfidence*
We say that rational overconfidence is present when there are two non-negligible groups of agents, with all members holding non-stationary rational beliefs, such that the belief held by any member of one group is held by no member of the other group ■

Note that if all agents hold the same confident rational belief we do not say that overconfidence is present. Since this belief may be correct, there is no objective justification for using the prefix "over".

Remark 3 *Entropy and confidence*

[14]Statistical analysis of the past performance of this belief may reveal that it did not perform well. However, there is no way to turn this observation into an assessment of how the particular belief will perform in the future. In other words there is no logically compelling reason to abandon or modify the belief as far as the future is concerned.

[15]For the formal versions of these definitions, see Nielsen(2008).

For any two probability vectors $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ which are not identical, we have $-\sum_{i=1}^{n} p_i \ln p_i < -\sum_{i=1}^{n} p_i \ln q_i$. This means that for $k = 0, 1$, $-(P_L^k \ln P_L^k + P_H^k \ln P_H^k) < -(P_L^k \ln \overline{P}_L + P_H^k \ln \overline{P}_H)$. On the left hand side of this inequality, we have the entropy of the probability (P_L^k, P_H^k). Multiplying by the frequency weights Q and $1-Q$ and summing over the k's we get

$$- \big(Q(P_L^0 \ln P_L^0 + P_H^0 \ln P_H^0) + (1-Q)(P_L^1 \ln P_L^1 + P_H^1 \ln P_H^1)\big) <$$
$$- \big(Q(P_L^0 \ln \overline{P}_L + P_H^0 \ln \overline{P}_H) + (1-Q)(P_L^1 \ln \overline{P}_L + P_H^1 \ln \overline{P}_H)\big) = \overline{P}_L \ln \overline{P}_L + \overline{P}_H \ln \overline{P}_H$$

On the right hand side of this inequality, we find the entropy of $(\overline{P}_L, \overline{P}_H)$, while on the left hand side we find the *average* entropy (over time) of the non-stationary rational belief. If we interpret lower average entropy as meaning more informative, this is another way of stating that a non-stationary rational belief is exhibiting more confidence ■

The concept of overconfidence originates from outside the economics literature, Oskamp(1982) and Svensson (1981) are early examples.[16] It plays a prominent role in the behavioral economics literature where it is invariably linked to some sort of irrationality.[17] In contrast, here overconfidence is not the outcome of irrationality but of the fact that it is, in most situations, possible to interpret the available data in many different ways.

2.3 Ex-post Welfare Optimality

In a situation where decisions are decentralized (to the agent level) and where expectations are inconsistent, these decisions cannot all be optimal. If decisions are, on the other hand, centralized, the decision maker would supposedly base them on one belief. Formally, the concept of Pareto efficiency and improvements (typically) can be associated with the aggregate

$$\int_0^1 w_i E_i U_i(C_i) \mathrm{d}i$$

where the integration is over agents (indexed by i), w_i refers to some weight given to agent i, while E_i is the expectation based on his belief, U_i is his utility and C_i is his consumption. From the point of view of ex-post efficiency, the maximization should however be of

$$\int_0^1 w_i E U_i(C_i) \mathrm{d}i = E \left\{ \int_0^1 w_i U_i(C_i) \mathrm{d}i \right\}$$

where E refers to expectation according to some specific belief. In Section 3 we shall provide some arguments why using expectations based on the stationary distribution is a natural choice. However, it is important to keep in mind, that no one can claim that this belief is correct, it is only one out of many possible rational beliefs.

Remark 4 *The ex-post concept*
Dreze (1970) was probably the first to note that, in the presence of uncertainty, for a Pareto optimal allocation the marginal rate of substitution for incomes in two different states may not be the same for all consumers unless also the subjective probabilities they attach to those states are the same. In Starr's (1973) reformulation of this observation, an allocations that is Pareto efficient also if the stochastic state is known at the date before

[16]Svensson (1981) observed a group of drivers of which more than half thought they were above the median in terms of driving skills. There are countless similar studies (where agents are asked to rank themselves in terms of competence, likability etc,). Many of the studies share the problem, that the subjects may interpret the question differently - f.i., what does it mean to be a "skilled driver"?. In our model every individual (dynasty) holds that he is better at forecasting than anyone else.

[17]See f.i. Odean (1998) and Daniel, Hirschleifer and Subrahmanyam (1998).

its realization, is said to be ex-post Pareto optimal (for the given state).[18] Universal ex-post efficiency (Harris, 1978) then means that given any realization of the future state, the allocation of present and future resources is Pareto undominated. These definitions only make sense in a stochastic exchange economy like the one we study in Section 4 (or for very particular cases like the so called intratemporal production economies), see Harris (1978).[19] In Sandmo(1983) "ex-post" refers to the social welfare function being defined on realized utility levels. Hammond(1981) defines *ex-post welfare optimality* (called Allais Optimality by Mirrlees, 1974) as the outcome of the decision maker maximizing the expected value of a Bergsonian welfare function, where the expectation is based on one set of "social probabilities" (not necessarily that of any agent). Harris (1978) defines expected ex-post efficiency, as a stochastic allocation that is Pareto optimal if all agents use some given probability in forming expected utility. The problem with the definitions of Hammond (1981) and Harris (1978) is that they provide no suggestions for what probabilities should be used in forming expectations. We place ourselves closest to the approach of Hammond(1981) but impose that the expectation (probability measure) used must have certain objective properties, a requirement which is especially important for production economies. Note that, in contrast with Hammond (1981), we do not consider the possibility of ignoring individual's tolerances towards risks, i.e. when measuring welfare we use the individuals' preferences over sure consumption. What makes the ex-post approach relevant to us is that it ensures consistency of expectations, although we do not necessarily agree with its motivation. So while we shall stick to the term "ex-post", we note that "consistent expectations welfare optimality" would have been a, from our point of view, more appropriate terminology ■

The literature on ex-post efficiency or ex-post welfare optimality did not have a great impact on economic policy research. However recently some economist have returned to critically assessing the Pareto criterion in the context of inconsistent subjective beliefs. Brunnermeier, Simsek and Xiong (2012) argue that whenever there is a dead-weight loss associated with bets based on such inconsistent beliefs, markets should be regulated. Their leading examples is similar to *The King's Dilemma* and our Example 5. Gilboa, Samuelson and Schmeidler (2014) argue that a change in contingent allocations should only be considered a Pareto improvement if it is also such when *some* probability beliefs is used by all agents which is akin to the ideas of Hammond (1981) and Harris (1978). None of these contributions discuss the nature of the beliefs being used by individuals.

3 Example 1: Savings and Economy-wide Wealth

A young agent expecting to have a high income in the future (as middle aged) will save up less for retirement, expecting to make up for the shortfall later. If the state of the macroeconomy is good, typically individual incomes are high. Thus someone who expects the macroeconomy to perform well, can also expect to earn a high income and so will save less for retirement. In our model the macroeconomy is simply the initial wealth level that everyone of the identical agents have in the second, final period of their lives.[20]

[18] The usage of "ex-post" may cause confusion. Hammond(1981) calls an allocation *intratemporal efficient* if, given any realized state, it is Pareto efficient, that is, given the realized state, there is no room for a Pareto improvement within the period. This efficiency criterion, which Dreze (1970) calls Pareto optimality ex-post, is implied by all the other efficiency concepts we mention here.

[19] Our production economy satisfies what Starr (1973) calls weak independence in production but not intratemporality.

[20] Our model has two periods only and for this reason there is dynamic completeness of markets. If we assumed that the returns on savings for retirement are stochastic and beliefs about it differ our basic insights would not be altered, i.e. the ex-post optimal policy would, as in our model, be to require agents to have a minimum savings level.

3.1 The model

There is an infinite sequence of generations each with a life span of two periods.[21]At any date t any agent i, choose the level of savings, S, resulting in the consumption $C^i_{1t} = W_1 - S$ in the first period and the stochastic consumption $C^i_{2t+1} = \tilde{W} + Sr$, where $\tilde{W}$ is stochastic and r is the exogenous return on investments. Thus when holding the belief P^k with $k \in \{0, 1\}$, he solves the following problem:

$$\max_{S\in[0,W_1]} \sum_{j\in\{H,L\}} u(W_1 - S, W^j + Sr)P^k_j \tag{1}$$

Let S^k be the solution (independent of i) to this problem with resulting stochastic consumption of dynasty i equal to $\{(C^i_{1t}, C^i_{2t+1})\}^\infty_{t=1}$ (note that no one knows the true associated distribution, since it depends on the unknown, true probability distribution of $\{W_t\}^\infty_{t=1}$). Let $\overline{S}$, assumed > 0, solve (1) after replacing P^k with $\overline{P}$ and let $\{(\overline{C}_{1t}, \overline{C}_{2t+1})\}^\infty_{t=1}$ be the resulting stochastic consumption stream. The allocation, where each agent consumes this stream, is referred to as $\overline{C}$. The government imposes a minimal savings level, S^* for all individuals (it would be difficult if not impossible to put a cap on savings, since it is typically possible to hide wealth). With this required minimum savings level, the problem of an agent, who uses the belief $P^k, k = 0, 1$ becomes:

$$\max_{S\in[0,W_1-S^*]} \sum_{j\in\{H,L\}} u(W_1 - (S^* + S), W^j + (S^* + S)r)P^k_j \tag{2}$$

with solution $\tilde{S}^k$. Then if $S^* \geq S^k$, $\tilde{S}^k = 0$, else $\tilde{S}^k = S^k - S^*$ and the effective savings level of an agent with belief $P^k, k = 0, 1$ is $\max(S^*, S^k)$.

3.2 The government's problem

Let $\overline{E}$ denote expectations using the empirical distribution. Since this distribution is stationary, expectations are the same at all dates so we do not use a subscript t here. There are three ways of arriving at the proposed objective function of the government. One is, for any t, to find the expected value of the (unweighted) sum of agents' utility, using $\overline{E}$:

$$\overline{E}\{Qu[W_1 - \max(S^*, S^0), W_{2t+1} + \max(S^*, S^0)r] + (1 - Q)u[W_1 - \max(S^*, S^1), W_{2t+1} + \max(S^*, S^1)r]\} =$$

$$\begin{aligned}\sum_{j=L,H} \{&Qu[W_1 - \max(S^*, S^0), W^j + \max(S^*, S^0)r] \\ &+(1 - Q)u[W_1 - \max(S^*, S^1), W^j + \max(S^*, S^1)r]\}\, \overline{P}_j\end{aligned} \tag{3}$$

Another is to take the average (over time) of the expected (unweighted) sum of individual utility using an expectation, E_t, based on *any* rational belief about $\{W_{2t}\}^\infty_{t=1}$:

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E_t\{&Qu[W_1 - \max(S^*, S^0), W_{2t+1} + \max(S^*, S^0)r] + \\ &(1 - Q)u[W_1 - \max(S^*, S^1), W_{2t+1} + \max(S^*, S^1)r]\}\end{aligned}$$

which, since we assumed that $\{W_{2t}\}^\infty_{t=1}$ is stable, is equal to (3).[22] Notice that this equality in particular holds if, at any date, we use the true expectation, which is however not known by anyone. In other words, it can be ascertained that (3) is the true average expected value, over time, of the average utility, over agents.

[21]For conceptual clarity, these generations could, as a though experiment, be assumed to be non-overlapping, making it physically impossible for them to interact. This would in particular rule out ex-ante Pareto improvements like in Gordon and Varian (1988).

[22]See Appendix 7.1, equation (26).

Notice also, that the equality holds for any of the rational beliefs being used by the individual agents of this economy. Thus, like in *The King's Dilemma*, which of the individual agents' beliefs is used, is immaterial for the decision being made.

The third way is to consider the average (over time) of the unweighted sum of individual *realized* utility:

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\{Qu[W_1-\max(S^*,S^0),W_{2t+1}+\max(S^*,S^0)r]$$
$$(1-Q)u[W_1-\max(S^*,S^1),W_{2t+1}+\max(S^*,S^1)r]\}$$

which, again by stability, with probability 1, is equal to (3).[23]

In the first approach we do not compare the welfare of different generations, but use the stationary probability $\overline{P}$, which may be incorrect and which every single agent (by assumption) holds to be incorrect at any date t. In the second and third approaches, we give equal weight to all generations (and all agents) when constructing the objective. Note that in the special case of stationarity where $E_t = \overline{E}$ for all t (something that *could* be the case), the equivalence of the first and second approach is trivial, while the equivalence of the the second and third approach is a consequence of Birkhoff's Ergodic Theorem. Note also that, in the special case we are considering, using the stationary measure at every date amounts to using, at every date, the *average* of all agents' beliefs.

The reason for the equivalence of the three ways of constructing an objective function for the government is that the distribution of actions is constant over time (which follows from the fact that the distribution of beliefs is constant), which because of stability means that the average (over time and across agents) realized utility is equal to the expected average (across agents) utility when taking expectations using the empirical measure. It is noticeable that what beliefs agents have does not influence the objective of the government - in particular, the government need have no knowledge about the beliefs of agents other than that the distribution of beliefs is constant.[24]

Obviously, with the objective function (3), it is then optimal for the government to set $S^* = \overline{S}$. This means that all agents in the economy save up at least $\overline{S}$ while others suboptimally save up more. Note that in many periods some agents are able to state ex-post that they would have been better off without government intervention. Note also that in this economy each agent (viewed as a dynasty) would hold that the policy imposes an inferior consumption plan on himself, but that at the same time, it represents an objective improvement for the average population. Accepting the objectively improving policy in this sense implies a sacrifice of subjective individual expected utility.

The important point of our approach is that the expected average utility in (3) is *objective*. In other words, every agent in the economy agrees that (3) expresses the average utility, over agents and over dates, that will be observed for (almost) all realizations of the sequence $\{W_{2t}\}_{t=1}^{\infty}$. The policy solution is thus not arbitrary, but neither is it based on any superior knowledge possessed by any authority. Rather, it is based on what is commonly agreed on by all agents, namely the empirical distribution of $\{W_{2t}\}_{t=1}^{\infty}$. From these consideration a general principle suggests itself: *That any policy should respect the common content of diverse opinions.* Since diverse opinions are rooted in the same reality, such a common content is likely to be present in many situations.

To elucidate the conclusion we have reached, it is useful to decompose into two the properties of the allocation resulting from the policy solution. Firstly, every agent agrees on the welfare consequences of the policy. Secondly,

[23] Again see Appendix 1 for details.

[24] This assumption can be weakened. Even in the case where the distribution does vary over time, under the assumption of Structural Independence (see Nielsen, 2007), we retain the format of the objective.

there is no further feasible policy that according to the beliefs of all agents lead to an improvement in terms of expected welfare.

Definition 3 *Universal Agreement about Welfare Consequences*
We say that there is *universal agreement about welfare consequence* of a stochastic allocation if, for this allocation, the expected welfare achieved (for given welfare weights) is the same independently of which agent's belief is being used when taking expectations ■

Secondly, we propose a criterion for objective welfare improvements as well as a notion of objective welfare optimality:

Definition 4 *Objective Welfare Improvement and Objective Welfare Optimality*
For given welfare weights, we say that a feasible allocation is objectively welfare improving over some given allocation if, according to all agents' beliefs, the weighted expected utility of it is higher than that of the other, given allocation. If there is no objective welfare improvement for a feasible allocation, this allocation is said to be objectively welfare optimal ■
This notion of an improvement embodies unanimity in evaluation. Obviously, weaker notions of the degree of approval could be adopted - one could for instance require the unanimity to hold among a set of agents of measure 1. To explore the meaning of these two definitions, notice that the consumption stream, $\{C_{1t}^{ij}, C_{2t+1}^{ij}\}_{i\in[0,1],t=1,2,...}$ where $C_{1t}^{ij} = C_{1t}^{j}$ and $C_{2t+1}^{ij} = C_{2t+1}^{j}, \forall i$ is objectively welfare optimal (since, according to (the belief of) agent j, it achieves the highest possible welfare for the given weights). However, there is no universal agreement about the welfare consequences of this consumption stream and this allocation is not objectively welfare improving over the laissez-faire allocation.

We then suggest that the government should seek to achieve an allocation about which there is universal agreement about consequences and which is objectively ex-post welfare optimal. Such an allocation might then be said to be *objectively socially optimal*. The consumption stream $\overline{C}$ (where each agent invests $\overline{S}$) is such an allocation, however, in our model it cannot be achieved since the government cannot prevent agents from saving up too much. We note that this allocation is expected ex-post efficient (in the sense of Harris, 1978) only when using the stationary distribution, but not when using the individual rational beliefs of agents (or any other rational belief that a decision maker may hold). We summarize the main conclusions of the analysis in the following

Proposition 1
If the decision maker weights all agents and generations equally, independently of the beliefs it uses, as long as this belief is rational, the optimal minimum savings level is $S^* = \overline{S}$. Moreover, all agents in the economy agree that if the objective is to maximize average, over agents and over time, expected utility, S^* should be set equal to $\overline{S}$. This savings level will induce a consumption stream that is objectively welfare improving over any other consumption stream, that can be attained with a minimum savings level (including the laissez faire consumption stream where $S^* = 0$), and for which there is universal agreement about consequences. At the same time, compared to the laissez faire allocation all agents (viewed as dynasties) are subjectively worse off under this policy (so that the laissez faire allocation strictly Pareto dominates the allocation achieved under the minimum savings rule) and no member of any dynasty is subjectively better off ■

We already noted that if all agents hold the same non-stationary, rational belief, we cannot say that over-confidence is present, in particular, we cannot say that there is an inefficiency present. But what if agents in a non-negligible set hold one non-stationary, rational belief, while in another non-negligible set agents hold a different non-stationary, rational belief? In that case there is rational overconfidence and if everyone saves according to their own beliefs, an inefficiency is objectively present. None the less setting a minimum savings requirement may not objectively improve on the welfare of the laissez-faire distribution.

We assumed that all agents have the same initial wealth level, the only difference among them being what beliefs they hold. If we instead assume that different groups of agents have different stochastic wealth levels (in terms of the empirical distribution of W_2) then there are also different objectively optimal savings levels, say $\overline{S}_K > \overline{S}_{K-1} > \ldots > \overline{S}_1$ for the K different groups. If these wealth levels are observable, the government will impose wealth dependent minimum savings rates. In fact, for the case of Social Security in the US, the mandatory contribution is increasing in income.

Remark 5 *Pecuniary externalities in production economies*
In *The King's Dilemma* one may argue that there is a negative externality in the sense that the probability of a given prince being chosen by the princess decreases when another prince is allowed to go. However, *subjectively* there is no such negative externality. Note also, that in the model of savings and social security just discussed there is, in any sense of the concept, no negative externality from an individual's investment decision and in this sense the case for imposing *social rationality* has been made more difficult.

Here we briefly consider the possibility of a pecuniary externality in a production economy.[25] Suppose individual agents do not consume the product they produce, but are selling it on a market with a downward sloping demand curve and that the return on their investment is stochastic, leading to a problem of the following format, for an agent with belief k:

$$\max_{I\in[0,W_1]} \sum_{j\in\{H,L\}} u(W_1 - I, Ir^i_j p)P^k_j \tag{4}$$

with solution $I^k, k = 0,1$ where $r^i_j, j \in \{L,H\}$ is the return of i's investment in the two states, taking values in $\{r_H, r_L\}$, and p is the price at which the product is being sold. Notice that we now assume idiosyncratic shocks rather than economy-wide ones (we return to idiosyncratic shocks in Section 5). As before, at every date a fraction Q of agents use the belief (P^0_L, P^0_H) and a fraction $1-Q$ use (P^1_L, P^1_H) where $Q(P^0_L, P^0_H) + (1 - Q)(P^1_L, P^1_H) = (\overline{P}_L, \overline{P}_H)$ - the stationary probability at any date. Assume that at every date among both groups of agents (those who use (P^0_L, P^0_H) and those who use (P^1_L, P^1_H)), a fraction $\overline{P}_L$ of the agents experience a low return, r_L. These assumptions together imply that if the price is constant, aggregate output is constant over time, in turn implying that the price will be constant as well, so justifying the format (4). Let $\overline{p}$ be the market clearing price when all agents use the stationary probability and hence invest $\overline{I}$. In an equilibrium we may have either over- or underinvestments, i.e. either $I^* \equiv QI^0 + (1-Q)I^1 > \overline{I}$ or $< \overline{I}$. In the case of overinvestments which leads to overproduction in the aggregate, the equilibrium price, p^* is less than $\overline{p}$ and a negative pecuniary externality is present.[26] This case may be interpreted as a situation with "excess entry". It has been argued that empirically there is excess-entry, i.e. too many start-ups, in the sense that the failure rate of such new businesses is very high (see f.i. Wu and Knot, 2006 and the references therein).
In either case, we argue that from the point of view of society, and despite the fact that the allocation is (loosely speaking, since we do not consider the demand side) Pareto efficient, the market is not functioning optimally because individuals are not investing $\overline{I}$ ■

[25] See Nielsen(2009b) for a related model developed in more detail.

[26] Thus, we use "pecuniary externality" not in the traditional sense (see f.i. Loong and Zeckhauser, 1982), but relate it to an ex-post concept of efficiency.

4 Example 2: Portfolio Choice

So far we have considered a rationalization of mandatory savings programs. A remarkable aspect of the quite recent debate about Social Security in the USA (during the first year of the second term of president Bush, jr.) is that both of the main camps (those who wanted to fix it by making more funds available to it and those who wanted to reform it) agreed that there should be some limits to choices. Firstly, there were the, at least as stated, fundamental agreement that there should be some sort of mandatory savings for retirement. Secondly, there were also an agreement that there should be restrictions on how these savings are placed (to avoid what the then president called ”frivolous investments”). As the program works now, the running surplus is placed in government bonds, while reformers have suggested that individuals should have the option to place a certain fraction of their contributions for retirement in a limited array of portfolios of relatively ”safe” assets, the income from these individual accounts then being used to pay out pensions on retirement. Evidence of suboptimal portfolio choices for individual retirement plans, see f.i. Agnew (2002), Choi, Laibson and Madrian (2005), Munnell (1999) and Palma and Sunden (2004), should however cause some skepticism about these proposals. We here provide a formal justification, in the context of rational overconfidence, for such doubts.

The model is a stochastic exchange economy, with two trees, A and B in fixed supply of 1 each. The two stochastic sequences $\{\Omega_{At}\}_{t=1}^{\infty}$ and $\{\Omega_{Bt}\}_{t=1}^{\infty}$ with state space $\Re_+$ are the returns of these two trees and we shall assume that they are both known to be independently distributed over time and mutually independent, and furthermore that they are *empirically* known to be mutually independent with the same i.i.d., one-period distribution, $\overline{P}$ with associated expectation $\overline{E}$.[27]

With an overlapping generations structure young agents buy the asset and sell it to the then young agents on retirement. To simplify the analysis we shall assume that agents only consume when old. As young they have a certain endowment, e of the single commodity and their only problem then is how to invest this endowment in a portfolio of the two trees.

As before all agents know the empirical distribution of the dividends of the two trees. Assuming risk aversion they would, if the two distributions were believed by all to be the empirical distribution, invest $e/2$ in each of the trees. Since this investment decision results in the consumption $e + (\Omega_{At} + \Omega_{Bt})$ for all old agents (i.e. perfect risk sharing), this is also the investment that the government would like them to make.

Knowledge of the empirical distribution does not rule out that agents believe that the true unknown distribution of $\{\Omega_{At}\}_{t=1}^{\infty}$ and $\{\Omega_{Bt}\}_{t=1}^{\infty}$ differ. We shall assume that each agent may have one of two different one-period beliefs (in terms of distributions), P_0 and P_1 about $\Omega_c, c = A, B$ such that there is a $0 < Q < 1$ with $QP_0 + (1-Q)P_1 = \overline{P}$. At every date t there is a measure Q^2 of agents who use the distribution $P_0 \times P_0$ on $\Re_+^2$, a measure $Q(1-Q)$ who use $P_0 \times P_1$, a measure $(1-Q)Q$ who use $P_1 \times P_0$, and a measure $(1-Q)^2$ who use $P_1 \times P_1$. Thus, like in Example 1, the distribution of beliefs is constant over time.

Let p_{ct} be the price of the commodity in terms of tree $c, c = A, B$. The problem, at date t, of an agent of type (z_A, z_B), $z_c \in \{0, 1\}$ (meaning that he holds belief (P_{z_A}, P_{z_B}) with associated expectation E^{Z_A, Z_B}) now is:

$$Max_{q\in[0,1]} E^{z_A,z_B} u \left[\left(q p_{At} [\frac{1}{p_{At+1}} + \Omega_{At+1}] + (1-q) p_{Bt} [\frac{1}{p_{Bt+1}} + \Omega_{Bt+1}] \right) e \right] \tag{5}$$

where u is a strictly increasing, strictly concave utility function. Since, in terms of the distribution of beliefs, this economy does not change between dates, it makes sense to concentrate on those equilibria where p_{ct} is constant $= p, c = A, B$, and where consequently the fraction of total real endowments invested in each tree, π is constant and equal to 1/2 (owing to the symmetry of beliefs). In that case the problem of an agent of type

[27]It is possible that $\{\Omega_{At}\}$ and $\{\Omega_{Bt}\}$ are independent, but their empirical distribution is not. See Nielsen(2008) for details.

(z_A, z_B) becomes

$$Max_{q\in[0,1]}E^{z_A,z_B}u\left[\left(qp[\frac{1}{p}+\Omega_{At+1}]+(1-q)p[\frac{1}{p}+\Omega_{Bt+1}]\right)e\right] \tag{6}$$

and in equilibrium, $\pi ep = 1$ i.e. $p = \frac{2}{e}$ (demand equal to supply on the markets for the trees). It follows that the problem of the agent is:

$$Max_{q\in[0,1]}E^{z_A,z_B}u\left[\left(q[1+\frac{2\Omega_{At+1}}{e}]+(1-q)[1+\frac{2\Omega_{Bt+1}}{e}]\right)e\right] \tag{7}$$

with solution π_{z_A,z_B} where $\pi_{0,1} = 1-\pi_{1,0}$ (due to symmetry), $\pi_{0,0} = \pi_{1,1} = \frac{1}{2}$ (due to risk aversion), consequently $Q^2\pi_{0,0} + (1-Q)^2\pi_{1,1} + Q(1-Q)\pi_{0,1} + (1-Q)Q\pi_{1,0} = 1/2$ as was postulated before. At every date t the sum (or average) of agents' utilities is:

$$\begin{aligned}\overline{u}(\Omega_{At},\Omega_{Bt}) \equiv \left[(1-Q)^2+Q^2\right]u(e+\Omega_{At}+\Omega_{Bt}) + Q(1-Q)u(e+2\{\pi_{01}\Omega_{At}+(1-\pi_{01})\Omega_{Bt}\}) \\ +Q(1-Q)u(e+2\{\pi_{10}\Omega_{At}+(1-\pi_{10})\Omega_{Bt}\})\end{aligned} \tag{8}$$

To find the average (over time) sum (across agents) of realized utility, by stability, we take expectations in (8) with respect to the stationary distribution $\overline{P}$. We shall use U^D to refer to this value:

$$U^D = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\{\overline{u}(\Omega_{At},\Omega_{Bt})\} = \overline{E}\{\overline{u}(\Omega_{At},\Omega_{Bt})\} =$$

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}E^P\{\overline{u}(\Omega_{At},\Omega_{Bt})\}$$

(where the second equality holds almost surely) for any rational belief, P about $\{(\Omega_{At},\Omega_{Bt})\}_{t=1}^{\infty}$, hence expectation E^P. Here the last equality follows from stability and the three expressions have the same interpretations as government criterion functions as in Section 3.

The optimal portfolio, using $\overline{P}$, is $\overline{\pi} = \frac{1}{2}$ with resulting realized sum (across agents) of utility, $u(e+\Omega_{At}+\Omega_{Bt})$ and resulting average (across agents and time) realized utility

$$\overline{U} = \overline{E}u(e+\Omega_{At}+\Omega_{Bt}) \tag{9}$$

which, by strict concavity of u, is strictly largerr than U^D.

Note that both U^D and $\overline{U}$ are objective quantities, i.e. (i) these are the average utilities that will be realized in this economy for the two cases of no intervention and intervention respectively and (ii) all agents agree this is so. We summarize in the following proposition:

Proposition 2
With the welfare weights being equal for all agents, the policy to require all agents to invest half of their savings in each tree, $\pi_{z_A,z_B} = \frac{1}{2}$ for all (z_A, z_B) leads to an allocation which is objectively welfare optimal and about which there is universal agreement about welfare consequences. This allocation is strictly Pareto dominated by the allocation resulting from the laissez faire policy ■

Remark 6 *Distribution of beliefs*
Notice that because this is a stochastic exchange economy, the policy leads to the unique expected ex-post efficient allocation (in the sense of Harris, 1978) independently of the rational belief being used.[28] There is, in other words, a strong case for arguing that, irrespectively of the distribution of beliefs, everyone of the identical (in terms of preferences) and strictly risk averse agent should consume $e + (\Omega_{At} + \Omega_{Bt})$ which is supported by the portfolio $(\frac{1}{2}, \frac{1}{2})$. This observation is akin to Proposition 3.2 in Harris (1978) ■

[28]In fact for any belief that is in accordance with the fundamental facts about the stochastic process.

5 Social Security in the Context of an OLG Growth Model

We consider a discrete time OLG model where agents sell their labor inelastically in both of the two periods of their lives. There is idiosyncratic uncertainty about the productivity of individuals in the final period of their lives, which these individuals have to take into account when deciding on their level of savings. Since the distribution of forecasts about individual productivity is constant over time as is the aggregate level of productivity, all macro variables are deterministic.

5.1 Model fundamentals

There is a continuum of agents all having the CRRA utility function $u(C_1, C_2) = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta}, \theta \in (0,1), \rho > 0$. A quantity Y is produced of the single consumption good (which can also be used as capital) as a function of capital and effective units of labor supplied: $Y = F(K, L) = K^{\alpha}L^{1-\alpha}$. The population of young (and old) workers grows at a constant rate: $L_{t+1}^y = (1+n)L_t^y, t \geq 1$ as does, A_t, the average effectiveness of labor at date t: $A_{t+1} = (1+g)A_t, t \geq 1$ with $A_1 > 0$ given. There are two novelties that distinguish our model from the classical one. Firstly, we assume that the labor force consists of both young and old workers: $L_t = L_t^y + \delta L_t^o$ where $\delta \in (0,1)$ captures the idea that old workers do not work the whole period (i.e. they go into retirement), alternatively that they only work part time.[29] The other novelty is that we assume that each young agent faces an idiosyncratic risk regarding his growth rate of productivity as old which can either be g_L or $g_H > g_L$. However, we assume no aggregate uncertainty, i.e. in any period a proportion $\overline{\pi} \in (0,1)$ are high productivity while the rest is low productivity, where

$$\overline{\pi}g_H + (1-\overline{\pi})g_L = g$$

a relationship which is known by everyone in the economy. We shall also assume independence, over time for each generation and that this independence is common knowledge. These two assumptions together imply that the savings decisions of young workers depend on their forecasts about their future productivity. Workers who are optimistic about their earnings prospective will, as we shall see, save up less, while others will save up more. With the number of old at date 1 being $L_1^o > 0$, and with the relation $L_t^y = (1+n)L_t^o, t \geq 1$, we define the effective young respectively old labor supply at date $t \geq 1$ as

$$L_t^{ye} = A_t L_t^y = (1+n)^t(1+g)^{t-1}A_1L_1^o \text{ and } L_t^{oe} = A_t\delta L_t^o = \delta(1+n)^{t-1}(1+g)^{t-1}A_1L_1^o$$

We then arrive at the overall effective labor supply at date $t \geq 1$:

$$L_t^e = L_t^{ye} + L_t^{oe} = (1+n+\delta)(1+n)^{t-1}(1+g)^{t-1}A_1L_1^o \text{ so that } L_{t+1}^e = (1+n)(1+g)L_t^e \tag{10}$$

We shall in the following use L_t^e to transform all variables into per effective labor units.

5.2 Rational Belief Structure

At all dates a fraction $Q \in (0,1)$ of all agents use the one-period belief $\Pi^1 = (\pi^1, 1-\pi^1)$ to forecast the event of high/low productivity while the remaining fraction $1-Q$ use $\Pi^2 = (\pi^2, 1-\pi^2)$ with $\pi^1 > \pi^2$ - we call the first type "optimists", the second type "pessimists". As explained earlier, in the background is an assumption that dynasties have rational beliefs in the form of infinite sequences $\{\Pi_t\}$ of one-period probabilities.

[29] It would have been more satisfactory to consider a three-period model where agents work when young and middle aged and are in retirement as old, however such a model would bring technical issues into the picture which would distract us from the main message of the model.

Since the distribution of one-period probabilities is constant, with the stationary empirical distribution being the infinite product of $\overline{\Pi} = (\overline{\pi}, 1 - \overline{\pi})$, the rationality condition becomes:

$$Q\Pi^1 + (1 - Q)\Pi^2 = \overline{\Pi} \tag{11}$$

Note that with Q and $\overline{\pi}$ given, π^2 is then determined by:

$$\pi^2(\pi^1) = \frac{\overline{\pi} - Q\pi^1}{1 - Q} \text{ for } \pi^1 \in (\max\{0, \frac{\overline{\pi} - (1 - Q)}{Q}\}, \min\{1, \frac{\overline{\pi}}{Q}\}) \tag{12}$$

With this rational belief structure we have moved from one extreme to the other. In the previous two examples agents were engaged in predicting the outcome of an economy-wide shock, while here there is no aggregate, but only individual, uncertainty. This means that concurrent optimism and pessimism does not automatically imply inconsistency of beliefs. However, we shall assume that any individual agent (or dynasty) i, at any date, uses $\overline{\Pi}$ to forecast the productivity of any other agent. This individual is thus confident about his ability to forecast his own future productivity, but not about his ability to forecast that of others.[30] Beliefs are now inconsistent, but in a less fundamental way than in the two examples. However the assumption does capture what we seem to observe, witness the Wall Street Journal poll, that individuals think think that others are not making the right savings decisions for retirement. Note that if the fraction Q of optimist was fluctuating over time (which is possible in the context of rational beliefs - (11) only needs to hold on average), knowing that at every date, a fraction $\overline{\pi}$ have high productivity, we could say with certainty that some beliefs must be wrong (however not which ones). While an assumption of fluctuating distributions of beliefs would therefore bring us closer to the previous two examples and probably also be more realistic, it would introduce endogenous uncertainty, that is fluctuations in endogenous variables, that would complicate the analysis considerably. Finally, it is also of theoretical interest to see that the general argument carries through also in a context of only idiosyncratic uncertainty.

According to our thinking, people save up too little for retirement because they are optimistic about their own future income, whether that is based on optimistic forecasts about the general state of the economy, as in our first example, or based on expectations of above average growth of personal future income, as in the present model. Their optimism may also be based on a belief that their strategy for investing their savings will yield above average returns, as in our second example. In contrast with the thinking of the bounded rationality approach, it is in none of these cases possible to single out an individual for having unrealistic expectations. All that can be stated is that, from the point of view of the collective, the savings behavior of society is not optimal.

5.3 The problem of an individual agent

As is standard in OLG growth models agents save up by buying capital which they then rent out and sell (we assume zero depreciation) in the following period. We let w_t denote the wage rate per effective unit of labor at date t and r_t the rental rate for capital. When we get to defining a Social Security Policy, government lump sum transfers of the form $V_t = vA_t, v > 0$ will be introduced. The problem of a young worker, living at date t and with subjective probability π of having high productivity next period, then is:

$$\max_{s\in[0,1]} \frac{1}{1-\theta}[(1-s)A_t w_t]^{1-\theta} + \frac{1}{(1-\theta)(1+\rho)} \left\{\pi[\delta(1+g_H)A_t w_{t+1} + (1+r_{t+1})sA_t w_t + vA_{t+1}]^{1-\theta} + \right.$$

[30] Another interpretation is that any agent believes that others are not able to forecast their productivity growth with a precision beyond that implied by the stationary distribution.

$$(1-\pi)[\delta(1+g_L)A_t w_{t+1}+(1+r_{t+1})sA_t w_t+vA_{t+1}]^{1-\theta}\} = \max_{s\in[0,1]}\frac{[A_t w_t]^{1-\theta}}{1-\theta}(1-s)^{1-\theta}+$$
$$\frac{[A_t w_t]^{1-\theta}(1+r_{t+1})^{1-\theta}}{(1-\theta)(1+\rho)}\left\{\pi[\delta(1+g_H)\frac{w_{t+1}}{w_t(1+r_{t+1})}+s+(1+g)\frac{v}{w_t(1+r_{t+1})}]^{1-\theta}+\right.$$
$$\left.(1-\pi)[\delta(1+g_L)\frac{w_{t+1}}{w_t(1+r_{t+1})}+s+(1+g)\frac{v}{w_t(1+r_{t+1})}]^{1-\theta}\right\}$$

The FOC (for an interior solution) is:

$$-(1-s)^{-\theta}+\frac{(1+r_{t+1})^{1-\theta}}{1+\rho}\left\{\pi[\delta(1+g_H)\frac{w_{t+1}}{w_t(1+r_{t+1})}+s+(1+g)\frac{v}{w_t(1+r_{t+1})}]^{-\theta}+\right.$$
$$\left.(1-\pi)[\delta(1+g_L)\frac{w_{t+1}}{w_t(1+r_{t+1})}+s+(1+g)\frac{v}{w_t(1+r_{t+1})}]^{-\theta}\right\}=0 \tag{13}$$

This defines the continuous savings rate function $s_t = s(r_{t+1}, w_{t+1}, w_t, v, \pi) \in [0,1)$

We write (13) as

$$H(s, r_{t+1}, w_{t+1}, w_t, v, \pi) = 0 \tag{14}$$

where for $s \in (0,1)$, $H'_s < 0$ and $H'_{r_{t+1}} > 0$ so that

$$\frac{\partial s}{\partial r_{t+1}} \geq 0 \text{ with } > \text{ if } s \in (0,1)$$

From the FOC, (13), as $r_{t+1} \to \infty$, $s(r_{t+1}, w_{t+1}, w_t, v, \pi) \to 1$. Also

$$H'_\pi = \frac{(1+r_{t+1})^{1-\theta}}{1+\rho}\left\{[\delta(1+g_H)\frac{w_{t+1}}{w_t(1+r_{t+1})}+s+(1+g)\frac{v}{w_t(1+r_{t+1})}]^{-\theta}-\right.$$
$$\left.[\delta(1+g_L)\frac{w_{t+1}}{w_t(1+r_{t+1})}+s+(1+g)\frac{v}{w_t(1+r_{t+1})}]^{-\theta}\right\}<0$$

so that

$$\frac{\partial s}{\partial \pi} < 0 \text{ for} s \in (0,1)$$

- i.e. optimists save less. Finally, since

$$H'_{w_{t+1}} < 0, \text{ we have } \frac{\partial s}{\partial w_{t+1}} < 0 \text{ and similarly } \frac{\partial s}{\partial v} < 0 \text{ for } s \in (0,1)$$

5.4 Equilibrium, stationarity and efficiency

The aggregate savings rate out of labor income (which, because of (12), can be defined independently of π^2) is

$$\overline{s}(r_{t+1}, w_{t+1}, w_t, v, \pi^1) = qs(r_{t+1}, w_{t+1}, w_t, v, \pi^1) + (1-q)s(r_{t+1}, w_{t+1}, w_t, v, \pi^2(\pi^1)) \tag{15}$$

We first consider the situation where there is no intervention by the government, so that in particular $v = 0$ and total savings at date t is

$$\overline{s}(r_{t+1}, w_{t+1}, w_t, 0, \pi^1)L_t^{ye}w_t = K_{t+1}$$

- the capital stock next period.

Defining $k_t = \frac{K_t}{L_t^e}$, output per effective labor is $y_t = k_t^\alpha$ and, in a competitive equilibrium,

$$r_t = \alpha k_t^{\alpha-1} \text{ and } w_t = (1-\alpha)k_t^\alpha \tag{16}$$

An equilibrium sequence, $\{k_t\}_{t=1}^{\infty}$, with k_1 given, must, since $\frac{L_t^{ye}}{L_t^e} = \frac{1+n}{1+n+\delta}$, then satisfy

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}^e} = \frac{\overline{s}(r_{t+1}, w_{t+1}, w_t, 0, \pi^1) L_t^{ye} w_t}{(1+n)(1+g)L_t^e} = \frac{\overline{s}(r_{t+1}, w_{t+1}, w_t, 0, \pi^1)}{(1+n+\delta)(1+g)} w_t, t \geq 1 \tag{17}$$

with r_{t+1}, w_{t+1} and w_t defined in (16).

We first show that such an equilibrium sequence can be described by $k_{t+1} = \phi(k_t)$ for some function $\phi(\cdot)$. First note that, given $k_t > 0$ (and hence $w_t > 0$), as $k_{t+1} \downarrow 0$, $r_{t+1} \uparrow \infty$ and $w_{t+1} \downarrow 0$ so that $\overline{s}(r_{t+1}, w_{t+1}, w_t, 0, \pi^1) \uparrow 1$. Thus eventually the left hand side of (17) is smaller than the right hand side. Next note that, again given $k_t > 0$, as $k_{t+1} \uparrow \infty$, $r_{t+1} \downarrow 0$ and $w_{t+1} \uparrow \infty$ so that, using (13), $\overline{s}(r_{t+1}, w_{t+1}, w_t, 0, \pi^1) \downarrow 0$, meaning that eventually the left hand side of (17) is larger than the right hand side. Thus, given $k_t \geq 0$, there is, by continuity, always a solution to (17) (for $k_t = 0$, $k_{t+1} = 0$ is the solution). Finally, because

$$\frac{\partial \overline{s}}{\partial r_{t+1}} \frac{dr_{t+1}}{dk_{t+1}} + \frac{\partial \overline{s}}{\partial w_{t+1}} \frac{dw_{t+1}}{dk_{t+1}} < 0$$

the solution is unique, proving that the function $\phi(\cdot)$ is well defined.

We next consider the existence of a positive stationary equilibrium i.e. a $k^* > 0$ s.t. $k^* = \phi(k^*)$ or, using (16) and (17), a solution to

$$\frac{(1+n+\delta)(1+g)]}{1-\alpha} k^{1-\alpha} - \overline{s}(\alpha k^{\alpha-1}, w, w, 0, \pi^1) = 0 \tag{18}$$

- note, from the format of the FOC (13), that this equation does not depend on w. As $k \downarrow 0$, $\overline{s}(\alpha k^{\alpha-1}, w, w, 0, \pi^1) \uparrow 1$ so that the left hand side of (18) is eventually < 0. Note that

$$\phi(k) \leq \frac{(1-\alpha)k^{\alpha}}{(1+n+\delta)(1+g)}, \forall k \tag{19}$$

Defining $\overline{k}$ by

$$\overline{k} = \frac{(1-\alpha)\overline{k}^{\alpha}}{(1+n+\delta)(1+g)} \text{ i.e. } \overline{k} = \left[\frac{1-\alpha}{(1+n+\delta)(1+g)}\right]^{\frac{1}{1-\alpha}}$$

this then means that there is a stationary $k^* < \overline{k}$. The upper bound $\overline{k}$ then also gives us an upper bound on the wage rate and a lower bound on the rental rate in the stationary equilibrium. Because $k^{1-\alpha}$ is increasing, while $\overline{s}(\alpha k^{\alpha-1}, w, w, \pi^1)$ is decreasing in k the positive stationary state, k^* is furthermore unique. Note that as usual there is another steady state, namely $k = 0$. One can also show that the stationary state is stable (see appendix), however this is not of importance in this context. Finally note, that for δ small, $s(\alpha k^{*\alpha-1}, (1-\alpha)k^{*\alpha}, (1-\alpha)k^{*\alpha}, 0, \pi^1) > 0$, which we shall assume in the following.

The golden rule capital level, k_{GR} maximizes $k^{\alpha} + k - (1+n)(1+g)k$, i.e.

$$k_{GR} = \left[\frac{\alpha}{n+g+ng}\right]^{\frac{1}{1-\alpha}}$$

If $\overline{k} \leq k_{GR}$ then $k^* < K_{GR}$ i.e. we do not have dynamic inefficiency. In other words, if

$$\frac{1-\alpha}{(1+n+\delta)(1+g)} < \frac{\alpha}{n+g+ng} \tag{20}$$

which holds for α sufficiently large (when capital's share of the income is sufficiently large) or for n and g small, we do not have dynamic inefficiency. Notice that neither $\overline{k}$ nor k_{GR} depend on the belief structure of the economy and are objective, i.e. agreed upon by all agents in the economy. We shall in the following assume the absence of dynamic inefficiency which is the theoretically interesting and probably also empirically relevant case. As was already shown by Samuelson(1975), in the presence of dynamic inefficiency, there is an unfunded social security program that is Pareto improving, so while our result holds in both cases, it is more relevant when there is no dynamic inefficiency (see Appendix 7.4 for a decomposition of the changes following the introduction of social security).

5.5 Social Security

We now proceed to show that, starting from the steady-state, a partially funded Social Security policy that increases the average *objective* expected utility can be designed. By 'objective', we mean expected utility after replacing π with $\overline{\pi}$, i.e. the average expected utility according to the beliefs of all agents in the economy.[31]

As in our first example, it is agreed by everyone in the economy that this objective utility will be realized in equilibrium, however no-one can claim that this is the expected utility for any particular individual.

With $w^* = (1-\alpha){k^*}^{\alpha}$ and $r^* = \alpha {k^*}^{\alpha-1}$ the (representative) type i agent has a savings rate $s_i^* = s(r^*, w^*, w^*, 0, \pi^i)$ and thus the objective utility is:

$$U^a(r^*, w^*, w^*, s_i^*, v; t) = \frac{[(1-s_i^*)A_t w^*]^{1-\theta}}{1-\theta} + \frac{1}{(1-\theta)(1+\rho)}\left\{\overline{\pi}[\delta(1+g_H)A_t w^* + (1+r^*)s_i^* A_t w^* + vA_{t+1}]^{1-\theta} + (1-\overline{\pi})[\delta(1+g_L)A_t w^* + (1+r^*)s_i^* A_t w^* + vA_{t+1}]^{1-\theta}\right\}$$

Note then that
$$\frac{\partial U^a}{\partial s_i^*}(r^*, w^*, w^*, s_i^*, 0; t) > 0 \text{ for } i = 1 \text{ and } < 0 \text{ for } i = 2$$
i.e., from an objective point of view, in the stationary equilibrium, optimists save too little and pessimists too much.

The policy we consider involves imposing a lower bound, $\underline{s}$ on the savings of all agents, where $s_2^* > \underline{s} > s_1^*$, which forces optimists to save more but does not affect pessimists directly. The increased level of savings increases the capital level in the following period and thus decreases interest rates and increases wage rates. To compensate for these changes, the social security policy is complemented with a proportional tax, τ on wage incomes as well as the lump sum subsidy, $v_i A_t > 0$ to old agents of type i. The government policy $(\underline{s}, \tau, (v_1, v_2))$ is constructed such that the economy immediately moves to a new steady state $\hat{k} > k^*$, implying a new wage rate $\hat{w} = (1-\alpha)\hat{k}^{1-\alpha}$ and a new interest rate $\hat{r} = \alpha\hat{k}^{\alpha-1}$ respectively. Setting $\tau = \frac{\hat{w}-w^*}{\hat{w}}$, workers continue to receive the after tax wage rate w^*. Since $s(\hat{r}, w^*, w^*, v, \pi^1) < s_1^* < \underline{s}$, the effective savings rate of optimists will be $\underline{s}$ while that of pessimists will be $s(\hat{r}, w^*, w^*, v, \pi^2) < s_2^*$.

With the average savings rate $\overline{s}^* = qs_1^* + (1-q)s_2^*$ and letting

$$v_i(\hat{w}) = \frac{s_i^*}{\overline{s}^*}(1+n+\delta)(\hat{w} - w^*)$$

the government runs a balanced budget in every period:

$$[qv_1 + (1-q)v_2]A_t L_t^o = (1+n+\delta)A_t L_t^o[\hat{w} - w^*] = [L_t^o(1+n)A_t + \delta A_t L_t^o][\hat{w} - w^*] = L_t^e[\hat{w} - w^*]$$

- the latter being the tax revenue.

[31] However, in contrast to the two examples, it is not the average utility according to any possible rational belief.

We next show that for every $\underline{s}$ larger than but close to s_1^* there is a stationary $\hat{k}$, i.e. fulfilling

$$\hat{k} = \frac{q\underline{s} + (1-q)s(\hat{k}, w^*, w^*, v_2((1-\alpha)\hat{k}^\alpha), \pi^2)}{(1+n+\delta)(1+g)}(1-\alpha)\hat{k}^\alpha \tag{21}$$

First note that for $\hat{k} = k^*$ the RHS of (21) is larger than k^*, while for $\hat{k} = \overline{k}$ we have $>$ in (21). By continuity there is then the desired $\hat{k}$ which is furthermore unique. Since the RHS is increasing in $\underline{s}$, so is $\hat{k}$.

We finally turn to evaluating the change in objective utility for the Social Security policy $(\underline{s}, \tau, (v_1, v_2))$. Letting, with $\Delta k = \hat{k} - k^*$, $\Delta r = \hat{r} - r^* \approx \alpha(1-\alpha){k^*}^{\alpha-1}\Delta k$, $\Delta s_1 = \underline{s} - s_1^* > 0$ and $\Delta s_2 = s(\hat{k}, w^*, w^*, v_2(\hat{w}), \pi^2) - s_2^* < 0$, the change in objective utility for type i can, for small changes in all variables, be approximated as follows:

$$\Delta U_i^a \approx \frac{\partial U^a}{\partial r^*}(r^*, w^*, w^*, s_i^*, 0; t)\Delta r + \frac{\partial U^a}{\partial s^*}(r^*, w^*, w^*, s_i^*, 0; t)\Delta \hat{s}_i + \frac{\partial U^a}{\partial v}(r^*, w^*, w^*, s^*, 0; t)v_i$$

Then note that, letting

$$B_i = \frac{1}{1+\rho}\left\{\overline{\pi}[\delta(1+g_H)A_t w^* + (1+r^*)s_i^* A_t w^*]^{-\theta} + \right.$$
$$\left.(1-\overline{\pi})[\delta(1+g_L)A_t w^* + (1+r^*)s_i^* A_t w^*]^{-\theta}\right\}$$

$$\frac{\partial U^a}{\partial r^*}(r^*, w^*, w^*, s_i^*, 0; t)\Delta r + \frac{\partial U^a}{\partial v}(r^*, w^*, w^*, s^*, 0; t)v_i = B_i[s_i^* A_t w^* \Delta r + A_{t+1}\frac{s_i^*}{\overline{s}^*}(1+n+\delta)(\hat{w} - w^*)] =$$
$$B_i A_t \frac{s_i^*}{\overline{s}^*}[\overline{s}^* w^* \Delta r + (1+n+\delta)(1+g)(\hat{w} - w^*)] \approx B_i A_t \frac{s_i^*}{\overline{s}^*}[(1+n+\delta)(1+g)k^*\alpha(\alpha-1){k^*}^{\alpha-2}\Delta k +$$
$$(1+n+\delta)(1+g)(1-\alpha)\alpha {k^*}^{\alpha-1}\Delta k] = 0, i = 1,2$$

Since $\frac{\partial U^a}{\partial s^*}(r^*, w^*, w^*, s_i^*, 0; t)\Delta \hat{s}_i > 0, i = 1,2$, it follows that $\Delta U_i^a > 0, i = 1,2$ for $\Delta s_1 > 0$ but small, i.e. that the Social Security policy has a strictly positive impact on the objective welfare for both optimists and pessimists. This positive impact is a consequence of an objectively better saving rates for both types, however, since we assumed $k^* < k_{GR}$, the increased capital level will also leave more consumption per capita in the new stationary equilibrium.

We have assumed different transfers to old workers depending on their type, with $v_2 > v_1$. Under an assumption that it is not possible to overstate savings (we retain that it is possible to hide savings), optimists do, for $\underline{s}$ close to s_1^*, not wish to behave as pessimist (by choosing a savings level close to s_2^*), i.e. the subsidy scheme is incentive compatible. We summarize in the following

Proposition 3

The government policy $(\underline{s}, \tau, (v_1, v_2))$ induces a consumption stream about which there is universal agreement about welfare consequences and which is objectively welfare improving ■

Remark 7 *Interest subsidies*

An alternative to the policy we just described is to let the government subsidize savings by offering an interest rate subsidy $r^* - \hat{r}$ financed by the proportional tax $\tau = \frac{\hat{w} - w^*}{\hat{w}}$ as described before (where $\hat{r}, \hat{w}, \hat{k}$ now refers to steady state values with this new policy). The net revenue per unit of effective labor of this policy would be $\hat{w} - w^* - \hat{k}(r^* - \hat{r})$ which would be close to zero for $\hat{k}$ close to k^*. The pessimists would face the same after tax interest and wage rates and so would choose the same savings rate, hence their objective or subjective welfare would not change ■

The Social Security policy that combines a lower bound on savings with lump sum transfers financed by taxes on wages to some degree resembles the actual Social Security policy, which is part pay-as-you-go part defined contributions. Note also that, since the type dependent lump sum transfers are independent of the realized productivity of an old worker, the Social Security policy also provides some insurance to workers, in line with what the Social Security program in the US does. On the other hand, since, at least in the US, contributions to private savings accounts are subsidized via tax reductions, the alternative Social Security that was sketched in Remark 7 also has some resemblance to reality. Obviously, in the context of our model, it would be possible to design a Social Security policy that combines the two policies discussed.

Remark 8 *Undersavings*

It is of some interest to know how aggregate savings and hence the stationary equilibrium capital level is affected by the presence of rational overconfidence, as compared to the case of rational expectations. Making explicit the dependence of ϕ on π^1, we note that if $\phi(k,\pi^1) < \phi(k,\overline{\pi}), \forall k, \forall \pi^1 > \overline{\pi}$ then $k^*(\pi^1) < k^*(\overline{\pi})$, where we have also made explicit the dependence of the stationary equilibrium on π^1.

From (16) we can write the savings rate function $\overline{s}$ as a function of only k_{t+1}, k_t and π^1: $\overline{s}(k_{t+1}, k_t, \pi^1)$.Then from (17) we get:

$$\phi(k_t,\pi^1) \equiv \frac{\overline{s}(\phi(k_t,\pi^1), k_t, \pi^1)}{(1+n+\delta)(1+g)}\alpha k_t^{\alpha}$$

so that

$$\frac{\partial \phi(k_{t+1}\pi^1)}{\partial \pi^1} = \frac{\frac{\partial \overline{s}(k_{t+1},k_t,\pi^1)}{\partial \pi^1}\frac{(1-\alpha)k_t^{\alpha}}{(1+n+\delta)(1+g)}}{1-\frac{\partial \overline{s}(k_{t+1},k_t,\pi^1)}{\partial k_{t+1}}\frac{(1-\alpha)k_t^{\alpha}}{(1+n+\delta)(1+g)}} \tag{22}$$

From (16) we can write the FOC as

$$-(1-s)^{-\theta} + \frac{(1+\alpha k_{t+1}^{\alpha-1})^{1-\theta}}{1+\rho}\Bigg\{\pi[\delta(1+g_H)(1+g)\frac{(1-\alpha)k_{t+1}^{\alpha}}{w_t(1+\alpha k_{t+1}^{\alpha-1})}+s]^{-\theta}+ \tag{23}$$

$$(1-\pi)[\delta(1+g_L)(1+g)\frac{(1-\alpha)k_{t+1}^{\alpha}}{w_t(1+\alpha k_{t+1}^{\alpha-1})}+s]^{-\theta}\Bigg\}=0$$

The partial derivative of the LHS of (23) w.r.t. k_{t+1} is positive so that (for $\overline{s}>0$)

$$\frac{\partial \overline{s}(k_{t+1}, w_t, \pi^1)}{\partial k_{t+1}} < 0$$

From (22) it follows that a sufficient condition for $\frac{\partial \phi(k,\pi^1)}{\partial \pi^1}$ being negative is that $\frac{\partial \overline{s}(k',k,\pi^1)}{\partial \pi^1} < 0$. However, it is easy to see that $\frac{\partial \overline{s}(k',k,\overline{\pi})}{\partial \pi^1} = 0$ so that $\overline{s}$ attains either a local minimum or a local maximum at $\overline{\pi}$.
In Appendix 3 we show that under the following condition:[32]

$$\left[\frac{3\theta+1}{1+\theta}\right]^{-\theta/2} < \frac{(1+r_{t+1})^{1-\theta}}{1+\rho}(1-\pi^i), i=1,2 \tag{24}$$

$s(r_{t+1}, w_{t+1}, w_t, \pi^1)$ is, as a function of π^1, strictly concave. Then $\overline{s}(r_{t+1}, w_{t+1}, w_t, \pi^1) = qs(r_{t+1}, w_{t+1}, w_t, \pi^1) + (1-q)s(r_{t+1}, w_{t+1}, w_t, \pi^2) < \overline{s}(r_{t+1}, w_{t+1}, w_t, q\pi^1 + (1-q)s^2) = \overline{s}(r_{t+1}, w_{t+1}, w_t, \overline{\pi})$. Since $\overline{k}$ puts an upper bound on k^* and thus a lower bound on r^*, we can find parameter configurations such that (24) holds, i.e. such that there is undersavings in the steady state (f.i. θ close to 1 and ρ and $\overline{\pi}$ close to 0) ■

[32] The idea of this condition was in essence suggested to me by Tiancheng Sun.

6 Conclusion

In the *King's Dilemma* there is, as noted earlier, arguably an objective negative externality from the actions of the individuals. If all three princes are allowed to compete for the princess' hand, two of them will suffer. If only one is allowed to compete, he will win. These facts are agreed upon by everyone. We do not rely on such an objective externality in our two main examples or in the OLG growth model we presented (see however Remark 5). It is thus worthwhile explaining how the reasoning regarding objective welfare in the context of our social security models flows from our reasoning regarding the King's Dilemma.

Suppose for the sake of concreteness that the King decides to randomly pick, with equal probability, the princes that should go. It is immaterial what belief about the princess' preferences is behind this decision, as long as that belief respects the objective facts that the princess will marry exactly one of the princes that show up. Based on such a belief the consequence of the King's rule is known to everyone and all princes are subjectively worse off, while according to the belief of any prince society is better off. This is the property of the King's Dilemma that we carry over to the models of social security.

In the two main examples, since individuals hold diverse expectations about the same variables (growth of the economy or return of the two trees), some agents must necessarily hold wrong beliefs, and as a consequence, society is not in an optimum. In the OLG growth model each individual acts on his own belief about hit own future productivity and these individual beliefs are not mutually inconsistent. It is only because we assume that each individual believes that the beliefs that other agents hold are incorrect (and, less importantly, that the correct belief about these agents' future productivity is the average one, $\overline{\Pi}$) that we are allowed to conclude that the social security policy is objectively improving.

All four cases are constructed in such a way that there is agreement among all individuals about the consequences for society of the policy of the decision maker, and in particular that that policy improves on the welfare of society, while at the same time each individual finds him- or herself not better of and sometimes worse off as a consequence of the policy.

As was stated in the introduction, our contribution has several aspects to it. Firstly it is normative in proposing a way of going beyond the Pareto criterion when individual beliefs are incompatible. Secondly it offers a possible explanation for the mandatory saving plans seen in many advanced economies, an explanation that however does not rely on an assumption of irrationality on the parts of agents. Finally and related to the first aspect, it provides a theoretical case for government intervention, even in the absence of the classical sources of market imperfections.

Related to the second aspect of our contribution, there are two assumptions that merit more discussion, namely about the willingness of people to give up on individual liberty for the sake of the common good and about the distribution of beliefs. Regarding the former, we already noted the poll from The Wall Street journal regarding social security showing such a tendency. There are many other policies, like the mandatory usage of safety belts, limitations on the usage of recreational drugs and mandatory insurance purchases for which it is probably the case that those in favor of the particular policy, would strictly prefer to be personally exempted from it. The reason we offer for such a position is not the only conceivable one. We mentioned the samaritan's dilemma, but possible negative externalities (loss of a breadwinner, a tax payer f.i.) could certainly also play a role. Regarding the distribution of beliefs, for the case of idiosyncratic shocks, if individuals would like to prevent people in general from taking certain actions but would like to be able to take these actions themselves, this may very well because they find the confidence of others, but (obviously) not their own, misguided. Such a situation is, in the context of rational beliefs, described by an assumption that according to each individual agent, the stationary measure applies to the idiosyncratic shocks of others, while his or her own idiosyncratic shock is forecast using a non-stationary rational belief. This was the assumption made in the OLG growth model.

In our model everyone agrees on the average performance of exogenous variables of the economy. This is unlikely to be so in reality, however one would expect that, based on the historical experience, there is some consensus about how the economy is likely to perform in the future. For instance, historical evidence is often the reference point when economists state that stocks in general are "overvalued" or the housing market is "overheated". We have argued that such a consensus should be the basis for a decision about how much individuals are supposed to save up for retirement and how these savings should be invested.

Mandating individuals to save up a certain amount for retirement means not only overruling their subjective beliefs but also ignoring their private information about preferences, wealth and future income. An inventor who would like to place all his income in developing a new product with a high return in the future may be harmed if he is forced to place some of his income in lower earning financial instruments (and if he his unable or unwilling to obtain seed money from other sources). There is then a trade-off between ensuring that society as a whole acts in a consistent way and using privately available information in an efficient way. If one accepts the premise of our study it becomes an important question how to deal with this trade-off. We hope to address this question in more detail in future research.

Defining such a trade-off may also shed some new light on the classical debate about the merits of central planning versus decentralized markets. We agree with Hayek (1945) that each individual knows something of relevance for the optimal planning problem and that this information may (for many different reasons) not be (easily) communicated to a hypothetical central planning entity. However, the individual decisions of agents may involve incoherent models because of, as Hayek (1945) states (on p. 519), "all the separate individuals frequently contradictory knowledge" (a problem he ignores in his further arguments).In our model this "contradictory knowledge" is in the form of mutually contradictory statistical models by individual agents. Each of these models contains (is) knowledge since it is consistent with, i.e. embodies, past statistical observations, but none the less departs from all other models in its interpretation of this available information.

We have mostly considered the case where the government is fully informed about agents' beliefs, an assumption that was quite innocuous in our set-up. In less simple models than the ones studied here, the distribution of (rational) beliefs may become important for determining the effects of different economic institutions. In Nielsen (2003) and Nielsen (2009a) we have shown how one may model the government's uncertainty about the distribution of rational beliefs in the economy and in particular how to use a notion of "genericity" in the context of such beliefs.

Strategic behavior played no role in our model, however, in a model where agents have more choice variables, it could easily become an issue. From the point of view of public economics it would then become a problem how to design, in an ex-post sense, optimal pension plans when taking into consideration that agents' private savings and investment behavior may partially offset the effects of such plans.

Thus our study leaves many open questions to be addressed. Hopefully, some readers have been convinced that the concept of Pareto optimality is problematic not only in the context of stochastic exchange economies (which are after all theoretical constructs) but also for economies with production, where uncertainty becomes of key importance. There are many aspects of economic life where individual choice is restricted by laws and regulations. We have studied one such, namely savings for retirements, but the approach to economic policy, we proposed and used here, may be fruitful in other areas as well, to mention a few: health care (health insurance), traffic laws, and regulation of financial markets.

7 Appendices

7.1 Rational beliefs

By $\mathcal{N}$ we denote the set of natural numbers. The generic set of state variables is denoted S, a (Borel measurable) subset of $\Re^{\mathbf{N}}, N \in \mathcal{N}$. We let $\mathcal{B}(S^\infty)$ be the set of (Borel) measurable sets in the infinite product of S (with the product topology) and for $A \in \mathcal{B}(S^\infty)$, $1_A : S^\infty \to \{0,1\}$ is the indicator function. $T : S^\infty \to S^\infty$ is the shift transformation i.e. $T(s_1, s_2, \ldots) = (s_2, s_3, \ldots)$ and letting ν be a probability measure on $\mathcal{B}(S^\infty)$, $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ becomes a dynamical system. We often denote a sequence, $\{s_t\}_{t=1}^\infty$ in S^∞ by $\langle s \rangle$. For a given measure ν, $E_\nu(\cdot)$ denotes the expectation operator under ν. A set $A \in \mathcal{B}(S^\infty)$ is said to be invariant (T-invariant) if $T^{-1}A = A$. We denote by $\mathcal{I}$ the set of invariant sets in $\mathcal{B}(S^\infty)$. The dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is said to be ergodic if any invariant set has either measure 1 or measure 0. Finally, let $\mathcal{C}(S^\infty)$ be the cylinders in $\mathcal{B}(S^\infty)$ and $\mathcal{C}^d(S^\infty)$ the d-dimensional cylinders. The following definitions are from Kurz(1994):

Definition 5 Stability
The dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ as well as the measure ν are said to be stable if for all cylinders $C \in \mathcal{C}(S^\infty)$:

$$\frac{1}{J}\sum_{j=0}^{J-1} 1_C(T^j\langle s \rangle) \tag{25}$$

converges as $J \to \infty$ for ν-a.a. $\langle s \rangle \in S^\infty$ ■

Definition 6 WAMS
The dynamical system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ as well as the measure ν are said to be Weakly Asymptotic Mean Stationary (WAMS) if for all cylinders $C \in \mathcal{C}(S^\infty)$, $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C)$ exists ■
One can show that $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is stable if and only if it is WAMS, (Proposition 2 of Kurz, 1994). If $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is WAMS there is (Proposition 3, Kurz 1994) an *associated stationary measure* $\overline{\nu}$ s.t. $\forall C \in \mathcal{C}(S^\infty)$: $\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} \nu(T^{-j}C) = \overline{\nu}(C)$. Furthermore, if ν is ergodic, then the convergence in (25) is to $\overline{\nu}(C)$, which is consequently called the empirical distribution. We always use a bar over a stable measure to denote the associated stationary measure. Note, that when the system is stationary, $\overline{\nu} = \nu$.

Rational Beliefs

We assume that it is known that the system $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$ is stable and ergodic, however that while $\overline{\nu}$ is known, ν is not. Any ergodic and stable probability measure μ on $(S^\infty, \mathcal{B}(S^\infty))$, s.t. $\overline{\mu} = \overline{\nu}$ is then called a Rational Belief (Kurz, 1994). The interpretation is that the empirical properties of the system $(S^\infty, \mathcal{B}(S^\infty), \mu, T)$ would exactly be like those of $(S^\infty, \mathcal{B}(S^\infty), \nu, T)$, and since the latter empirical properties are known, any stochastic model (hypothesis) should reproduce these. On the other hand, μ cannot be rejected based on the knowledge available to anyone. Notice that both ν and $\overline{\nu}$ are rational beliefs.

Let $f : S^\infty \to \Re$ be measurable w.r.t. to $\mathcal{C}^d(S^\infty)$ for some d. Then we have, for any rational belief μ, that

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=0}^{J-1} f(T^j\langle s \rangle) = E_{\overline{\nu}}(f) \text{ for } \mu - a.a.\langle s \rangle \tag{26}$$

The interpretation of this property is that the empirical expectation of any (finite dimensional) function will be the same for any rational belief, hence agents with different rational beliefs expect the same averages to be observed and these expectations are fulfilled.

7.2 Stability of steady state

To establish the global stability of the positive stationary equilibrium, k^* i.e. that $\phi(k) > k$ for $k < k^*$ and $\phi(k) < k$ for $k > k^*$, the first step is the following

Lemma 1 $\lim_{k\downarrow 0} \frac{\phi(k)}{k} = \infty$

Proof: Suppose to the contrary that there is a sequence $k_m \downarrow 0$ s.t. $\frac{\phi(k_m)}{k_m} \to \beta \geq 0$. Then $\frac{(1-\alpha)\phi(k_m)^\alpha}{(1-\alpha)k_m^\alpha} \to \beta^\alpha$ while $\alpha k_m^{\alpha-1} \to \infty$ and, as a consequence and using the FOC, $\overline{s}(\alpha\phi(k_m)^{\alpha-1}, (1-\alpha)\phi(k_m)^\alpha, (1-\alpha)k_m^\alpha, 0, \pi) \to 1$. It follows that

$$\frac{\phi(k_m)}{k_m} = \frac{\overline{s}(\alpha k_m^{\alpha-1}, (1-\alpha)\phi(k_m)^\alpha, (1-\alpha)k_m^\alpha, 0, \pi)}{(1+n+\delta)(1+g)}(1-\alpha)k_m^{\alpha-1} \to \infty$$

a contradiction ■

It follows from the the lemma that $\phi(k) > k$ for k close to 0. This gives $\phi(k) > k$ for $k < k^*$. The uniqueness of the stationary equilibrium and the upper bound (19) on ϕ then implies that $\phi(k) < k$ for $k > k^*$ (for else there would be two positive stationary states).

7.3 Undersavings

We now keep r_{t+1}, w_t and w_{t+1} fixed and, for notational simplicity, write the savings function $s(\cdot)$ as a function of π only. Setting $K = \frac{(1+r_{t+1})^{1-\theta}}{1+\rho}$, $A = \delta(1+g_H)(1+g)\frac{w_{t+1}}{w_t(1+r_{t+1})} > B = \delta(1+g_L)(1+g)\frac{w_{t+1}}{w_t(1+r_{t+1})}$ the FOC, (13) can (with $v = 0$) be rewritten as

$$H(s,\pi) \equiv -(1-s)^{-\theta} + K\{\pi[A+s]^{-\theta} + (1-\pi)[B+s]^{-\theta}\} = 0 \tag{27}$$

The condition for an interior solution is

$$K\{\pi A^{-\theta} + (1-\pi)B^{-\theta}\} > 1 \tag{28}$$

(which holds for $w_t = w_{t+1} = w^*$ and $r_{t+1} = r^*$)

and the SOC is

$$H'_s = -\theta\left[(1-s)^{-(1+\theta)} + K\{\pi[A+s]^{-(1+\theta)} + (1-\pi)[B+s]^{-(1+\theta)}\right] < 0$$

Suppose for $0 \leq q \leq 1$, $q\pi^1 + (1-q)\pi^2 = \overline{\pi}$. We would like to know whether $s(\overline{\pi}) > qs(\pi^1) + (1-q)s(\pi^2))$ or not, i.e. whether $s(\cdot)$ is strictly concave or not.

By definition, $H(s(\pi),\pi) \equiv 0$ and so $H'_s(s(\pi),\pi)s'(\pi) + H'_\pi(s(\pi),\pi) \equiv 0$ and

$$\begin{aligned}H''_{ss}(s(\pi),\pi)s'(\pi)^2 + H''_{s\pi}(s(\pi),\pi)s'(\pi) + H'_s(s(\pi),\pi)s''(\pi) + \\ H''_{\pi s}(s(\pi),\pi)s'(\pi) + H''_{\pi\pi}(s(\pi),\pi) \equiv 0\end{aligned}$$

which, noting that $H''_{\pi\pi} = 0$, gives us that

$$s''(\pi) = -\frac{H''_{ss}(s(\pi),\pi)s'(\pi)^2 + 2H''_{s\pi}(s(\pi),\pi)s'(\pi)}{H'_s(s(\pi),\pi)} \tag{29}$$

$$H''_{s\pi} = -\theta K\left\{[A+s]^{-(1+\theta)} - [B+s]^{-(1+\theta)}\right\} > 0$$

and

$$H''_{ss} = \theta(1+\theta)\left\{-(1-s)^{-(2+\theta)} + K[\pi[A+s]^{-(2+\theta)} + (1-\pi)[B+s]^{-(2+\theta)}]\right\}$$

From (29) and the facts that $s'(\pi) < 0$ and $H'_s(s(\pi), \pi) < 0$, $s''(\pi) < 0$ is equivalent to

$$H''_{ss}(s,\pi)s'(\pi) + 2H''_{s\pi} =$$

$$K\left\{[A+s]^{-\theta} - [B+s]^{-\theta}\right\} \frac{(1+\theta)\left\{-(1-s)^{-(2+\theta)} + K[\pi[A+s]^{-(2+\theta)} + (1-\pi)[B+s]^{-(2+\theta)}]\right\}}{(1-s)^{-(1+\theta)} + K\{\pi[A+s]^{-(1+\theta)} + (1-\pi)[B+s]^{-(1+\theta)}\}}$$
$$-2\theta K\left\{[A+s]^{-(1+\theta)} - [B+s]^{-(1+\theta)}\right\} > 0$$

which can be rewritten as

$$\begin{aligned}&\left\{[A+s]^{-\theta} - [B+s]^{-\theta}\right\}(1+\theta)\left\{-(1-s)^{-(2+\theta)} + K[\pi[A+s]^{-(2+\theta)} + (1-\pi)[B+s]^{-(2+\theta)}]\right\} - \\ &2\theta\left\{[A+s]^{-(1+\theta)} - [B+s]^{-(1+\theta)}\right\}\left\{(1-s)^{-(1+\theta)} + K\{\pi[A+s]^{-(1+\theta)} + (1-\pi)[B+s]^{-(1+\theta)}\}\right\} > 0\end{aligned} \tag{30}$$

for all s and π s.t (13) holds.

If $-(1-s)^{-(2+\theta)} + K[\pi[A+s]^{-(2+\theta)} + (1-\pi)[B+s]^{-(2+\theta)}] < 0$, the desired inequality holds. One sufficient condition for this is $1 - s < B + s$ (which must hold if $B \geq 1$). To see this, note that, since in this case $\ln(1-s) < \ln(B+s) < \ln(A+s)$,

$$\frac{\partial}{\partial x}\left\{-(1-s)^{-x} + K[\pi[A+s]^{-x} - (1-\pi)[B+s]^{-x}]\right\} =$$
$$\ln(1-s)(1-s)^{-x} - K[\pi\ln(A+s)[A+s]^{-x} + (1-\pi)\ln(B+s)[B+s]^{-x}] < 0$$

whenever $\{-(1-s)^{-x} + K[\pi[A+s]^{-x} + (1-\pi)[B+s]^{-x}]\} = 0$. Note that if $K(1-\pi) \geq 1$ then $B + s > 1 - s$ in (27), i.e. we have $K(1-\pi) < 1$ whenever $1 - s \geq B + s$. For the case where $1 - s \geq B + s$ (which implies $s < 1/2$) the following condition is sufficient for the desired inequality to hold:

$$\text{Condition} \quad \left[\frac{3\theta+1}{1+\theta}\right]^{-\theta/2} < K(1-\pi)$$

We need only consider the case where $\left\{-(1-s)^{-(2+\theta)} + K[\pi[A+s]^{-(2+\theta)} + (1-\pi)[B+s]^{-(2+\theta)}]\right\} > 0$ which means that we can rewrite the desired condition as

$$\frac{\left\{[A+s]^{-(1+\theta)} - [B+s]^{-(1+\theta)}\right\}\left\{(1-s)^{-(1+\theta)} + K\{\pi[A+s]^{-(1+\theta)} + (1-\pi)[B+s]^{-(1+\theta)}\}\right\}}{\left\{-(1-s)^{-(2+\theta)} + K[\pi[A+s]^{-(2+\theta)} + (1-\pi)[B+s]^{-(2+\theta)}]\right\}\left\{[A+s]^{-\theta} - [B+s]^{-\theta}\right\}} > \frac{1+\theta}{2\theta} \tag{31}$$

Note that

$$K(\pi(A+s)^{-(2+\theta)} + (1-\pi)(B+s)^{-(2+\theta)}) < K(\pi(A+s)^{-\theta} + (1-\pi)(B+s)^{-\theta})(B+s)^{-2} = (1-s)^{-\theta}(B+s)^{-2}$$

- by the FOC, (27) - and that

$$\frac{[A+s]^{-(1+\theta)} - [B+s]^{-(1+\theta)}}{[A+s]^{-\theta} - [B+s]^{-\theta}} = \frac{[A+s]^{-\theta}[A+s]^{-1} - [B+s]^{-\theta}[B+1]^{-1}}{[A+s]^{-\theta} - [B+s]^{-\theta}} >$$
$$\frac{[A+s]^{-\theta}[B+s]^{-1} - [B+s]^{-\theta}[B+1]^{-1}}{[A+s]^{-\theta} - [B+s]^{-\theta}} = [B+s]^{-1} > [1-s]^{-1}$$

Thus the expression on the left hand side of (31) is larger than

$$[1-s]^{-1}\frac{(1-s)^{-(1+\theta)}}{-(1-s)^{-(2+\theta)} + (1-s)^{-\theta}(B+s)^{-2}} = \frac{(B+s)^2}{(1-s)^2 - (B+s)^2}$$

which, since (27) implies that $1-s < [K(1-\pi)]^{-1/\theta}(B+s)$, in turn is larger than

$$\frac{(B+s)^2}{\left\{[K(1-\pi)]^{-2/\theta}(B+s)\right\}^2-(B+s)^2}=\frac{1}{[K(1-\pi)]^{-2/\theta}-1}$$

By the condition assumed (and the fact that $K(1-\pi)<1$),

$$\frac{1}{[K(1-\pi)]^{-2/\theta}-1}>\frac{1}{\frac{3\theta+1}{1+\theta}-1}=\frac{1}{\frac{2\theta}{1+\theta}}=\frac{1+\theta}{2\theta}$$

which proves that (31) holds.

7.4 The irrelevance of dynamic inefficiency

In the new stationary state with social security, at date t the young optimists' change in consumption is $-\Delta s_1 A_t w^* < 0$ while that of the pessimists is $-\Delta s_2 A_t w^* > 0$. At the following date the change in consumption for the old optimists is approximately $s^* A_t w^* \Delta r + \Delta s_1 A_t w^*(1+r^*) + vA_{t+1}$ and that of the old pessimists $s^* A_t w^* \Delta r + \Delta s_2 A_t w^*(1+r^*) + v_2 A_{t+1}$. In period $t+1$ the change in consumption for the young is $-\Delta s_1 A_{t+1} w^*$ and $-\Delta s_2 A_{t+1} w^*$ for the optimists and pessimists respectively. Summing over young and the part related to increased savings by the old at date $t+1$, we get:

$$[q\Delta s_1+(1-q)\Delta s_2](1+r^*)A_t w^* L_t^y-[q\Delta s_1+(1-q)\Delta s_2]A_{t+1}w^* L_{t+1}^y=[q\Delta s_1+(1-q)\Delta s_2]A_t w^*[1+r^*-(1+g)(1+n)]L_t^y$$

$$[=q\Delta s_1+(1-q)\Delta s_2]A_t w^*[1+\alpha k^{*\alpha-1}-(1+g)(1+n)]L_t^y$$

which is negative or positive depending on whether we have dynamic inefficiency or not.

For the other part of the change in the consumption of the old at date $t+1$ we have

$$q[s_1^* A_t w^* \Delta r+v_1 A_t]+(1-q)[s_2^* A_t w^* \Delta r+v_2 A_{t+1}]=\overline{s}^* A_t w^* \Delta r+(1+n+\delta)A_{t+1}(\hat{w}-w^*)=$$
$$A_t[\overline{s}^* w^* \Delta r+(1+n+\delta)(1+g)(\hat{w}-w^*)]=A_t[(1+n+\delta)(1+g)k^*\Delta r+(1+n+\delta)(1+g)(\hat{w}-w^*)]=$$
$$A_t(1+n+\delta)(1+g)[k^*\Delta r+(\hat{w}-w^*)]\approx A_t(1+n+\delta)(1+g)[k^*\alpha k^{*\alpha-2}\Delta k+(\alpha(1-\alpha)k^{*\alpha-1}\Delta k]=0$$

Even in the case of dynamic inefficiency, there is then a welfare gain. However a higher welfare gain could be achieved by an unfunded social security program.

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