The Phillips Curve as a Long-Run Phenomenon in a Macroeconomic Model with Complex Dynamics

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Abstract

In this paper we derive the Phillips curve as the image of a chaotic attractor of the state variables of a non-linear dynamical system describing the evolution of an economy. This has two important consequences: the Phillips curve in our model is a true long-run phenomenon and it cannot be used for policy purposes. The model is based on an overlapping-generations non-tâtonnement approach involving temporary equilibria with stochastic rationing in each period and price adjustment between successive periods. In this way we are able to obtain complex sequences of consistent allocations allowing for recurrent unemployment and inflation.

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1. Introduction

After the presentation of the Phillips curve as an empirical regularity (Phillips [1958]) economists and policy makers alike have tried to exploit it for policy purposes. Even before the oil shocks in the seventies and early eighties this has had mixed success only. With the advent of "stagflation" the Phillips curve seemed to be definitely dead but over the decades thereafter a more differentiated view has emerged. A majority of economists nowadays seems to agree that an inverse relationship between inflation and unemployment holds as long as changes in output and prices are demand-driven. Stagflation occurs only when supply-side shocks occur. As the latter happens much less frequently than shifts in aggregate demand, a revival of the interest in the Phillips curve has taken place. New explanations have been advanced both to investigate theoretically and estimate empirically the Phillips curve and to assess its potential for economic policy[1].

In this paper we aim to contribute to this discussion by highlighting a further aspect that so far seems not to have been taken into consideration. That is, we show that the Phillips curve can be derived as a dynamic phenomenon in the sense that it may be obtained as the image of a chaotic attractor of the state variables of a non-linear dynamical system describing the evolution of our economy. This has two important consequences: on the one hand the Phillips curve in our model is a true long-run phenomenon and, on the other, it cannot be used for policy purposes. The first aspect is compatible with Phillips' original finding whereas the second may contribute to explain why the attempts to take advantage of the apparent inflation-unemployment trade-off for economic policy have not been very successful.

The dynamic economy we consider is composed of overlapping-generations consumers, producers and a government who interact in a labor and a consumption goods market. Trades take place in each period even when prices are not at their Walrasian level. Therefore agents' effective transactions can differ from the desired ones which means that they are rationed. Prices may not be at their Walrasian levels because their adjustment to market imbalances is not instantaneous but proceeds with finite speed only; thus their functioning as an allocation device is imperfect, though not nil. As a consequence, quantity adjustments complement prices in their task of making trades feasible. In particular, we build on the fact that, in many markets, quantities adjust faster than prices, as emphasized for instance by Greenwald and Stiglitz [1989]. To account formally for this asymme-

[1] Recent examples of work that tries to account for and estimate the Phillips curve are T.F. Cooley and V. Quadrini [1999] and J. Gali and M. Gertler [1999], whose contributions have been collected, among others, in a special issue of the Journal of Monetary Economics (Vol. 44, N.2, 1999) on "The Return of the Phillips Curve".

try, and be able to work out its consequences most clearly, we assume that, while quantities adjust within any period so as to produce a feasible allocation, prices are fixed during each period but are adjusted when the economy moves from one period to the next.

This approach makes it possible to investigate disequilibrium phenomena like recurrent and permanent underemployment and inflation and, moreover, appears to reflect well the effective functioning of many markets. Labor markets, at least in Europe, are the foremost example, but also product markets often show a high degree of price inertia and stickiness (for empirical accounts see Carlton [1986] and Bean [1994])[2].

The adjustment of prices and wages follows the standard paradigm relating it to the sign of excess demand. Moreover, the size of price adjustment is based on the intensity of agents' rationing a measure for which is obtained through a mechanism of stochastic rationing[3]. This allows us to encompass a large variety of degrees of price and wage inertia/flexibility which turn out to be decisive for the type of dynamics that emerges.

More precisely, by deviating in the initial conditions from the values of the state variables corresponding to a stationary Walrasian equilibrium and simulating the model, we observe the possibility of the emergence of complex dynamics. This complexity of the dynamics is driven by the non-linearities present in

[2] On theoretical grounds, price stickiness has been explained by (menu) costs of adjustments of prices (Akerlof and Yellen [1985]), strategic interactions among oligopolists and coordination failure among monopolistic competitors (Ball and Romer [1991]), by certain characteristics of the production function and of the demand function, by imperfect and asymmetric information and by risk aversion of order one (Weinrich [1997]). Regarding wage rigidity, it has been derived for instance from insider-outsider arguments (Shaked and Sutton [1984]), fairness (Hahn and Solow [1995, ch. 5]), efficiency wages (Salop [1979], Solow [1979], Shapiro and Stiglitz [1984], Weiss [1991]) and uncertainty combined with imperfectly competitive markets (Holmes and Hutton [1996]). See also the recent contribution by Bewley [2000] who found by means of interviews of a large number of businessmen that for firms it is preferable to lay off workers in recessions rather than to lower wages.

[3] Although the idea just outlined is neither new nor complicated, it gives rise to two problems: how to find a consistent allocation when prices are not at their market clearing levels and how to define a sensible mechanism for the adjustment of prices. The first of these problems has been solved by introducing the concept of temporary equilibrium with quantity rationing, developed in the seventies for the case of deterministic rationing mainly by Drèze [1975] and Bénassy [1975].

As far as the second problem is concerned, a natural idea is to relate the adjustment of prices to the size of the dissatisfaction of agents with their (foregone) trades. As it has been argued by Green [1980], Svensson [1980], Douglas Gale [1979, 1981] and Weinrich [1984, 1988], a reliable measure of such dissatisfaction requires stochastic rationing, since - as opposed to deterministic rationing - it is compatible with manipulability of the rationing mechanism and therefore provides an incentive for rationed agents to express demands that exceed their expected trades. For a definition of manipulability see for example Böhm [1989] or Weinrich [1988].

the model and by the possibility of switching between different (dis)equilibrium regimes along the trajectory. An important parameter here is the speed of downward wage adjustment. When that speed is small, i.e. when wages are relatively sluggish downwards, the system converges to the stationary state. For higher speeds, instead, irregular behaviour and chaos result.

By looking at pairs of the rates of wage inflation and unemployment, the dynamic behavior of our economy displays, for a non pathological set of parameter values, a Phillips curve as the image of an attractor of the system. However, there is no perceived systematic relationship between the values obtained in one iteration and the ones obtained in the next one; the system typically jumps from one point of the curve to another moving from one period to the next. This implies that there is no way of pushing the economy toward a desired inflation-unemployment pair by affecting the values of parameters, thus eliminating, or at least reducing substantially, the relevance of the Phillips curve for economic policy.

The remainder of the paper is organized as follows. In section two we present the model and derive the behavior of consumers, producers and government. Section three studies temporary equilibria with rationing and proves the existence and uniqueness of equilibrium allocations. In section four we set up the dynamical system and in section five we present numerical simulations. Section six draws together the results of the previous sections to derive Phillips curves and discuss their validity and relevance. Section seven presents concluding remarks and an appendix contains the proofs of some technical results.

2. The Model

We consider an economy in which there are n OLG-consumers, n' firms and a government. Consumers offer labor inelastically when young and consume a composite consumption good in both periods. That good is produced by firms using an atemporal production function whose only input is labor. The government levies a proportional tax on firms' profits to finance its expenditure for goods. Nevertheless, budget deficits and surpluses may arise and are made possible through money creation or destruction.

2.1. Timing of the Model

The time structure of the model is depicted in Figure 2.1. In period $t-1$ producers obtain an aggregate profit of Π_{t-1} which is distributed at the beginning of period t in part as tax to the government ($tax\Pi_{t-1}$) and in part to young consumers ($(1 - tax)\,\Pi_{t-1}$), where $0 \leq tax \leq 1$. Also at the beginning of period t old consumers hold a total quantity of money M_t, consisting of savings generated in period $t-1$. Thus households use money as a means of transfer of purchasing power between periods.

Let X_t denote the aggregate quantity of the good purchased by young consumers in period t, p_t its price, w_t the nominal wage and L_t the aggregate quantity of labor. Then

$$M_{t+1} = (1 - tax)\,\Pi_{t-1} + w_t L_t - p_t X_t.$$

Denoting with G the quantity of goods purchased by the government and taking into account that old households want to consume all their money holdings in period t, the aggregate consumption of young and old households and the government is $Y_t = X_t + \frac{M_t}{p_t} + G$. Using that $\Pi_t = p_t Y_t - w_t L_t$, considering $\Pi_t - \Pi_{t-1} = \Delta M_t^P$ as the variation in the money stock held by producers before they distribute profits and denoting with $\Delta M_t^C = M_{t+1} - M_t$ the one referring to consumers, the following accounting identity obtains:

$$\Delta M_t^C + \Delta M_t^P = p_t G - tax\Pi_{t-1} = \text{ budget deficit}$$

2.2. The Consumption Sector

In his first period of life each consumer born at t is endowed with labor ℓ^s and an amount of money $(1 - tax)\,\Pi_{t-1}/n$ while his preferences are described by a utility function $u\,(x_t, x_{t+1})$. In taking any decision the young consumer has to meet the constraints

$$0 \leq x_t \leq \omega_t^i, 0 \leq x_{t+1} \leq \left(\omega_t^i - x_t\right) \frac{p_t}{p_{t+1}}, i = 0, 1 \tag{2.1}$$

where

$$\omega_t^1 = \frac{1 - tax}{p_t} \frac{\Pi_{t-1}}{n} + \frac{w_t}{p_t} \ell^s$$

denotes his real wealth when he is employed and

$$\omega_t^0 = \frac{1 - tax}{p_t} \frac{\Pi_{t-1}}{n}$$

when he is unemployed. Implicit in this is the assumption that rationing on the labor market is of type all-or-nothing and, moreover, that the labor market is visited before the goods market.

Regarding the goods market the young household may be rationed according to the stochastic rule

$$x_t = \begin{cases} x_t^d & \text{with prob.} \quad \rho\gamma_t^d \\ c_t x_t^d & \text{with prob. } 1 - \rho\gamma_t^d \end{cases}$$

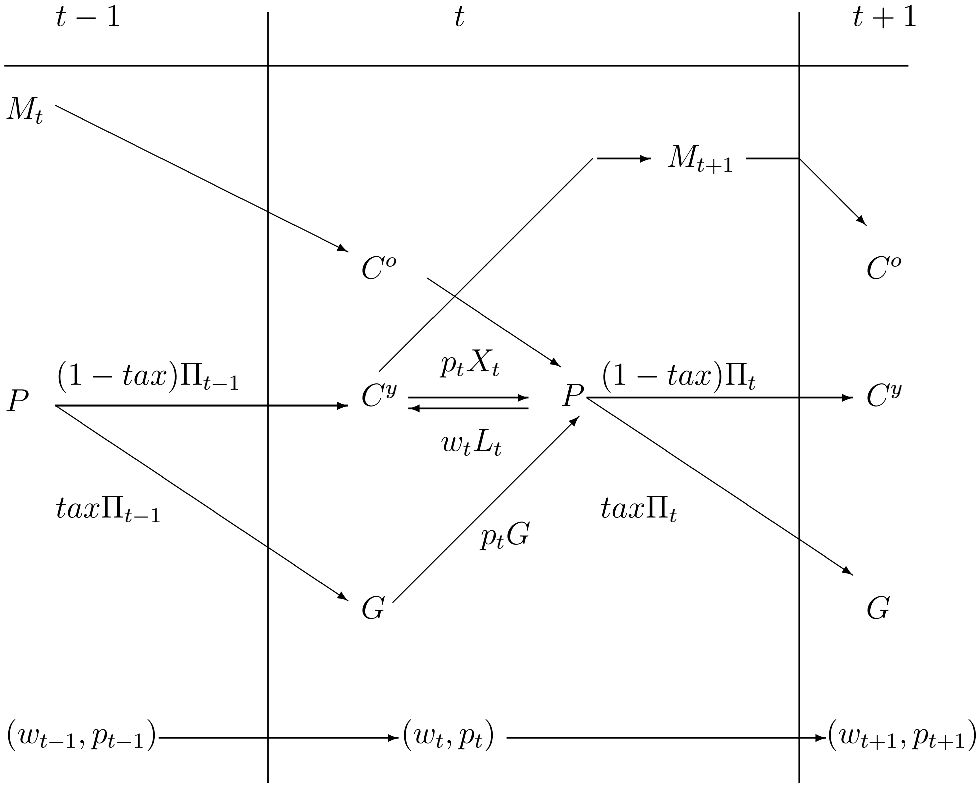


Figure 2.1: The time structure of the model

where x_t^d is the quantity demanded, $\rho \in [0,1]$ a fixed structural parameter of the rationing mechanism, $\gamma_t^d \in [0,1]$ a rationing coefficient which the household perceives as given but which will be determined in equilibrium and

$$c_t = \frac{\gamma_t^d - \rho\gamma_t^d}{1 - \rho\gamma_t^d}.$$

These settings are chosen such that the expected value of x_t is $\gamma_t^d x_t^d$, that is, expected rationing is proportional and hence manipulable.[4]

Denoting with $\theta_t^e = p_{t+1}^e / p_t$ the expected relative price for period t, the effective demand $x_t^{di}, i = 0, 1$, is obtained by solving the agent's expected utility maximization problem

$$\underset{x_t}{max}\ \ \rho\gamma_t^d u\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + \left(1 - \rho\gamma_t^d\right) u\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)$$

subject to the constraints (2.1). The resulting first-order condition is

$$\rho\gamma_t^d u_1\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + \left(1 - \rho\gamma_t^d\right) u_1\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right) c_t - \frac{\rho\gamma_t^d}{\theta_t^e} u_2\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) +$$

$$+ \left(1 - \rho\gamma_t^d\right) u_2\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)\left(-\frac{c_t}{\theta_t^e}\right) = 0$$

which yields

$$\frac{\rho u_1\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + (1-\rho) u_1\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)}{\rho u_2\left(x_t, \frac{\omega_t^i - x_t}{\theta_t^e}\right) + (1-\rho) u_2\left(c_t x_t, \frac{\omega_t^i - c_t x_t}{\theta_t^e}\right)} = \frac{1}{\theta_t^e}. \tag{2.2}$$

For a generic utility function it is hard to solve this equation for x_t but in the case that $u(x_t, x_{t+1}) = x_t^h x_{t+1}^{1-h}$ and $\rho = 1$ (i.e. 0/1-rationing), which we shall henceforth assume, we can prove that $x_t^{di} = h\omega_t^i, i = 0, 1$ (Lemma 1 in Appendix 1). In particular the young consumer's effective demand is independent of both γ_t^d and p_{t+1}^e.

The aggregate supply of labor is $L^s = n\ell^s$. Denoting with L_t^d the aggregate demand of labor and with $\lambda_t^s = \min\left\{\frac{L_t^d}{L^s}, 1\right\}$ the fraction of young consumers that will be employed, the aggregate demand of goods of young consumers is

$$X_t^d = \lambda_t^s n x_t^{d1} + (1 - \lambda_t^s) n x_t^{d0} \equiv X^d\left(\lambda_t^s; \frac{w_t}{p_t}, \frac{(1 - tax)\Pi_{t-1}}{p_t}\right).$$

[4] As has been shown by Green [1980] and Weinrich [1982], in case of rationing where the quantity signals are given by means of the aggregate values of demand and supply, the only mechanisms compatible with equilibrium are those for which the expected realization is proportional to the transaction offer.

The total aggregate demand of the consumption sector is then obtained by adding old consumers' aggregate demand M_t/p_t and government demand G:

$$Y_t^d = X^d\left(\lambda_t^s; \alpha_t, (1 - tax)\, \pi_t\right) + m_t + G_t$$

where $\alpha_t = w_t/p_t, \pi_t = \Pi_{t-1}/p_t$ and $m_t = M_t/p_t$.

2.3. The Production Sector

Each of the n' identical firms uses an atemporal production function $y_t = f\left(\ell_t\right)$. As with consumers, firms too may be rationed, by means of a rationing mechanism analogue to that assumed for the consumption sector.

Denoting the single firm's effective demand of labor by ℓ_t^d, the quantity of labor effectively transacted is

$$\ell_t = \begin{cases} \ell_t^d, & \text{with prob. } \lambda_t^d \\ 0, & \text{with prob. } 1 - \lambda_t^d \end{cases}$$

where $\lambda_t^d \in [0, 1]$. On the goods market the rationing rule is

$$y_t = \begin{cases} y_t^s, & \text{with prob. } \sigma\gamma_t^s \\ d_t y_t^s, & \text{with prob. } 1 - \sigma\gamma_t^s \end{cases},$$

where $\sigma \in (0, 1)\,, \gamma_t^s \in [0, 1]$ and $d_t = \left(\gamma_t^s - \sigma\gamma_t^s\right) / \left(1 - \sigma\gamma_t^s\right)$. σ is a fixed parameter of the mechanism whereas λ_t^d and γ_t^s are perceived rationing coefficients taken as given by the firm the effective value of which will be determined in equilibrium. The definition of d_t ensures that $\mathrm{E}y_t = \gamma_t^s y_t^s$. It is obvious that $\mathrm{E}\ell_t = \lambda_t^d \ell_t^d$.

The firm's effective demand $\ell_t^d = \ell^d\left(\gamma_t^s; \alpha_t\right)$ is obtained from the expected profit maximization problem

$$\max_{\ell_t^d} \gamma_t^s f\left(\ell_t^d\right) - \alpha_t \ell_t^d$$

subject to

$$0 \leq \ell_t^d \leq \frac{d_t}{\alpha_t} f\left(\ell_t^d\right)$$

while its effective supply is $y_t^s = f\left(\ell_t^d\right)$. The upper bound on labor demand reflects the fact that the firm must be prepared to finance labor service purchases even if rationed on the goods market (since the labor market is visited first it will know whether it is rationed on the goods market only after it has hired labor). In general the solution depends on this constraint but if we assume that $f\left(\ell\right) = a\ell^b$, $a > 0$, $0 \leq b \leq (1 - \sigma)$, then it is not binding (Appendix 1, Lemma 2).

The aggregate labor demand is $L_t^d = n'\ell_t^d\left(\gamma_t^s; \alpha_t\right) \equiv L^d\left(\gamma_t^s; \alpha_t\right)$ and, because only a fraction λ_t^d of firms can hire workers, the aggregate supply of goods is $Y_t^s = \lambda_t^d n' f\left(\ell^d\left(\gamma_t^s; \alpha_t\right)\right) \equiv Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t\right)$.

3. Temporary Equilibrium Allocations

For any given period t we can now describe a feasible allocation as a temporary equilibrium with rationing as follows.

Definition 3.1. *: Given a real wage α_t, a real profit level π_t, real money balances m_t, a level of public expenditure G and a tax rate tax, a list of rationing coefficients $\left(\gamma_t^d, \gamma_t^s, \lambda_t^d, \lambda_t^s, \delta_t, \varepsilon_t\right) \in [0,1]^6$ and an aggregate allocation $\left(\overline{L}_t, \overline{Y}_t\right)$ constitute* a temporary equilibrium with rationing *if the following conditions are fulfilled:*

(1) $\overline{L}_t = \lambda_t^s L^s = \lambda_t^d L^d\left(\gamma_t^s; \alpha_t\right)$;
(2) $\overline{Y}_t = \gamma_t^s Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t\right) = \gamma_t^d X^d\left(\lambda_t^s; \alpha_t, (1 - tax)\,\pi_t\right) + \delta_t m_t + \varepsilon_t G$;
(3) $\left(1 - \lambda_t^s\right)\left(1 - \lambda_t^d\right) = 0; \left(1 - \gamma_t^s\right)\left(1 - \gamma_t^d\right) = 0$;
(4) $\gamma_t^d\left(1 - \delta_t\right) = 0;\ \delta_t\left(1 - \varepsilon_t\right) = 0.$

Conditions (1) and (2) require that expected aggregate transactions balance. This means that all agents have correct perceptions of the rationing coefficients. Equations (3) formalize the short-side rule according to which at most one side on each market is rationed. The meaning of the coefficients δ_t and ε_t is that also old households and/or the government can be rationed. However, according to condition (4) this may occur only after young households have been rationed (to zero).

As shown in the table below it is possible to distinguish different types of equilibrium according to which market sides are rationed: excess supply on both markets is called *Keynesian Unemployment* $[K]$, excess demand on both markets *Repressed Inflation* $[I]$, excess supply on the labor market and excess demand on the goods market *Classical Unemployment* $[C]$ and excess demand on the labor market with excess supply on the goods market *Underconsumption* $[U]$.

	K	I	C	U
λ_t^s	< 1	$= 1$	< 1	$= 1$
λ_t^d	$= 1$	< 1	$= 1$	< 1
γ_t^s	< 1	$= 1$	$= 1$	< 1
γ_t^d	$= 1$	< 1	< 1	$= 1$

Of course there are further intermediate cases which, however, can be considered as limiting cases of the above ones. In particular, when all the rationing coefficients are equal to one, we are in a Walrasian Equilibrium. Notice that in all equilibrium types different from the Walrasian equilibrium there is at least one rationing coefficient smaller than one, and therefore there are agents which are effectively rationed. Nevertheless these rationed agents express effective demands that exceed their expected transactions. To do this is rational for them because from an individual point of view rationing is manipulable and stochastic. This feature of equilibrium could not have been obtained with deterministic rationing since then rationing would have had to be nonmanipulable, and hence agents would not have had an incentive to exceed their rationing constraints.

Moreover, we can use the rationing coefficients $\lambda_t^s, \lambda_t^d, \gamma_t^s$ and γ_t^d associated to any equilibrium allocation as a measure for the strength of rationing on the various market sides. To see this, consider a K-equilibrium. Since there is zero-one rationing on the labor market, $1-\lambda_t^s = (L^s - \overline{L}_t)/L^s$ is the ratio of the number of unemployed workers and the total number of young households. Regarding the goods market, in a K-equilibrium $\overline{Y}_t = \gamma_t^s Y^s(1,\gamma_t^s)$, and therefore

$$\frac{d(1-\gamma_t^s)}{d\overline{Y}_t} = -\frac{1}{Y_t^s + \gamma_t^s \frac{\partial Y_t^s}{\partial \gamma_t^s}} < 0$$

since $\frac{\partial Y_t^s}{\partial \gamma_t^s}(1,\gamma_t^s) = n' f'\left(\ell^d(\gamma_t^s)\right)\frac{d\ell_t^d}{d\gamma_t^s} > 0$. So a decrease in $\overline{Y}_t$ (for example due to a reduction of government spending), and thus an aggravation of the shortage of aggregate demand for firms' goods, is unambiguously related to a decrease in γ_t^s which can therefore be interpreted as a measure of the strength of rationing on the goods market. A similar reasoning justifies the use as rationing measures of the terms λ_t^d and γ_t^d in the other equilibrium regimes.

To address the question of existence and uniqueness of temporary equilibrium we have to introduce some further concepts. In the remainder of this section we hold the variables α_t, m_t and π_t and the parameters G and tax fixed; therefore we omit them whenever possible as arguments in the subsequent functions.

Define the set

$$\overline{H} = \left\{\left(\lambda^s L^s, \gamma^d X^d(\lambda^s)\right) \mid \left(\lambda^s, \gamma^d\right) \in [0,1]^2\right\}$$

and its subsets $\overline{H}^K = \overline{H}\mid_{\gamma^d=1,\lambda^s<1}$, $\overline{H}^I = \overline{H}\mid_{\gamma^d<1,\lambda^s=1}$, $\overline{H}^C = \overline{H}\mid_{\gamma^d<1,\lambda^s<1}$ and $\overline{H}^U = \overline{H}\mid_{\gamma^d=1,\lambda^s=1}$. From these we derive the consumption sector's *trade curves*

$$\overline{H}_0^K = \overline{H}^K + \{(0, m_t+G)\} = \left\{\left(\lambda^s L^s, X^d(\lambda^s) + m_t + G\right) \mid \lambda^s \in [0,1)\right\},$$

$$\overline{H}_0^I = \left\{\left(L^s, \gamma^d X^d(1) + m_t + G\right) \mid \gamma^d \in (0,1)\right\} \cup \{(L^s, \delta m_t + G) \mid \delta \in (0,1]\} \cup \{(L^s, \varepsilon G) \mid \varepsilon \in [0,1]\}$$

and

$$\overline{H}_0^C = \left\{\left(\lambda^s L^s, \gamma^d X^d(\lambda^s) + m_t + G\right) \mid \left(\lambda^s, \gamma^d\right) \in [0,1)\times(0,1)\right\}$$
$$\cup\{(\lambda^s L^s, \delta m_t + G) \mid (\lambda^s, \delta) \in [0,1)\times(0,1]\} \cup \{(\lambda^s L^s, \varepsilon G) \mid (\lambda^s, \varepsilon) \in [0,1)\times[0,1]\}.$$

Similarly, starting from

$$\overline{F} = \left\{ \left(\lambda^d L^d \left(\gamma^s \right), \gamma^s Y^s \left(\lambda^d, \gamma^s \right) \right) \mid \left(\lambda^d, \gamma^s \right) \in [0,1]^2 \right\}$$

we define the production sector's *trade curves* as $\overline{F}^K = \overline{F} \mid_{\lambda^d=1,\gamma^s<1}$, $\overline{F}^I = \overline{F} \mid_{\lambda^d<1,\gamma^s=1}$, $\overline{F}^C = \overline{F} \mid_{\lambda^d=1,\gamma^s=1}$and $\overline{F}^U = \overline{F} \mid_{\lambda^d<1,\gamma^s<1}$. In the appendix (Lemma 3) we show that these curves are given by

$$\overline{F}^K = \overline{F}^I = \overline{F}^U = \left\{ \left(L, \frac{\alpha_t}{b} L \right) \mid 0 \leq L < L^d \left(1; \alpha_t \right) \right\} \tag{3.1}$$

and

$$\overline{F}^C = \left\{ \left(L^d \left(1; \alpha_t \right), \frac{\alpha_t}{b} L^d \left(1; \alpha_t \right) \right) \right\}.$$

Using the consumption sector's and the production sector's trade curves and indicating with S^c the closure of the set S, we now note that a pair $\left(\overline{L}, \overline{Y} \right) \in R_+^2$ is a temporary equilibrium allocation if and only if it is an element of the set

$$Z = \left(\left(\overline{H}_0^K \right)^c \cap \left(\overline{F}^K \right)^c \right) \cup \left(\left(\overline{H}_0^I \right)^c \cap \left(\overline{F}^I \right)^c \right) \cup \left(\left(\overline{H}_0^C \right)^c \cap \left(\overline{F}^C \right)^c \right)$$

Here equilibria of type U do not appear as they can be seen as limiting cases of K- as well as of I-type equilibria. To show existence of an equilibrium is equivalent to showing that Z is not empty. To this end consider first the locus

$$\left(\overline{H}_0^K \right)^c = \left\{ \left(\lambda_t^s L^s, X^d \left(\lambda_t^s \right) + m_t + G \right) \mid \lambda_t^s \in [0,1] \right\}$$

and recall that

$$X^d \left(\lambda_t^s \right) = nh \left(\lambda_t^s \omega_t^1 + \left(1 - \lambda_t^s \right) \omega_t^0 \right) = h \left(1 - tax \right) \pi_t + h \alpha_t \lambda_t^s L^s.$$

Defining the function

$$\Gamma_t \left(L \right) = h \left(1 - tax \right) \pi_t + h \alpha_t L + m_t + G, \;\; L \geq 0,$$

we see that $\left(\overline{H}_0^K \right)^c$ is the part of the graph of Γ_t for which $L \leq L^s$.

Next consider again the production sector's trade curves. From (3.1) we conclude that the locus $\left(\overline{F}^K \right)^c$ is the part of the graph of the function

$$\Delta_t \left(L \right) = \frac{\alpha_t}{b} L, \;\; L \geq 0,$$

for which $L \leq L^d(1)$. Notice that the graphs of the functions Γ_t and Δ_t always intersect. Indeed, $\Gamma_t \left(L \right) = \Delta_t \left(L \right)$ if and only if

$$L = \frac{b}{\alpha_t \left(1 - hb \right)} \left[h \left(1 - tax \right) \pi_t + m_t + G \right] = \widetilde{L} \left(\alpha_t, \pi_t, m_t, G, tax \right)$$

which is well defined and positive since $hb < 1$. Therefore the equilibrium level of employment is

$$\overline{L}_t = \min\left\{\widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right), L^d\left(1, \alpha_t\right), L^s\right\} = \mathcal{L}\left(\alpha_t, \pi_t, m_t, G, tax\right)$$

and the equilibrium level on the goods market is

$$\overline{Y}_t = \Delta_t\left(\overline{L}_t\right) = \mathcal{Y}\left(\alpha_t, \pi_t, m_t, G, tax\right).$$

This shows that the equilibrium allocation

$$\left(\overline{L}_t, \overline{Y}_t\right) = \left(\mathcal{L}\left(\alpha_t, \pi_t, m_t, G, tax\right), \mathcal{Y}\left(\alpha_t, \pi_t, m_t, G, tax\right)\right)$$

exists and is uniquely defined. More precisely, if $\min\{\cdot\} = \widetilde{L}(.)$, then $\left(\overline{L}_t, \overline{Y}_t\right) \in \left(\overline{H}_0^K\right)^c \cap \left(\overline{F}^K\right)^c$ and the resulting equilibrium is of type K or a limiting case of it. If $\min\{\cdot\} = L^d(1, \alpha_t)$, then $\left(\overline{L}_t, \overline{Y}_t\right) \in \left(\overline{H}_0^C\right)^c \cap \left(\overline{F}^C\right)^c$ and type C or a limiting case of it occurs. Finally, if $\min\{\cdot\} = L^s$, an equilibrium of type I or a limiting case results because then $\left(\overline{L}_t, \overline{Y}_t\right) \in \left(\overline{H}_0^I\right)^c \cap \left(\overline{F}^I\right)^c$. An equilibrium of type Keynesian unemployment is shown in Figure 3.1

4. Dynamics

So far our analysis has been essentially static. For any given vector $(\alpha_t, \pi_t, m_t, G, tax)$ we have described a feasible allocation in terms of a temporary equilibrium with rationing. To extend now our analysis to a dynamic one we must link successive periods one to another. This link will of course be given by the adjustment of prices but also by the changes in the stock of money and in profits. Regarding the latter, this is automatic since by definition of these variables

$$\Pi_t = p_t\mathcal{Y}\left(\alpha_t, \pi_t, m_t, G, tax\right) - w_t\mathcal{L}\left(\alpha_t, \pi_t, m_t, G, tax\right),$$

$$\begin{aligned} M_{t+1} &= (1 - tax)\,\Pi_{t-1} + w_t\overline{L}_t - p_t\overline{X}_t \\ &= (1 - tax)\,\Pi_{t-1} + w_t\overline{L}_t - p_t\overline{Y}_t + \delta_t M_t + \varepsilon_t p_t G \\ &= (1 - tax)\,\Pi_{t-1} - \Pi_t + \delta_t M_t + \varepsilon_t p_t G \end{aligned}$$

Regarding the adjustment of prices we follow the standard hypothesis that, whenever an excess of demand (supply) is observed, the price rises (falls). In terms of the rationing coefficients observed in period t, this amounts to

$$p_{t+1} < p_t \Leftrightarrow \gamma_t^s < 1;\; p_{t+1} > p_t \Leftrightarrow \gamma_t^d < 1,$$

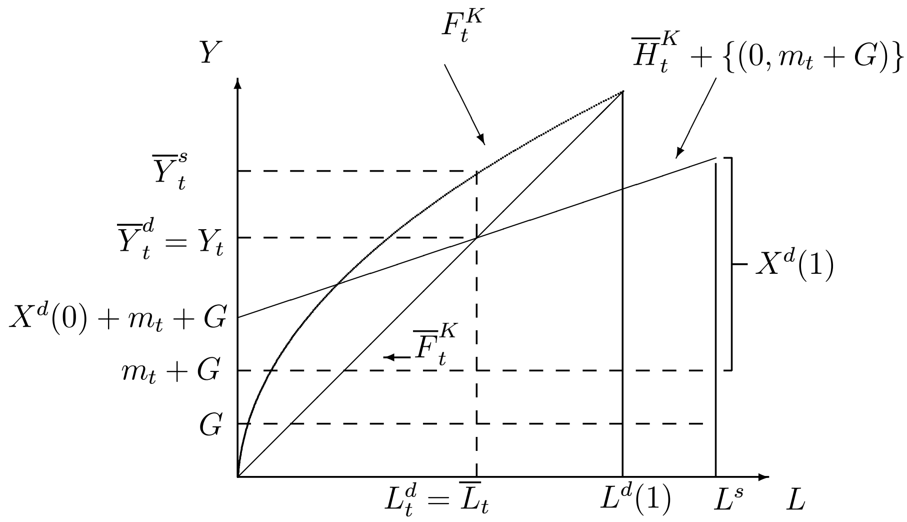


Figure 3.1: Keynesian Unemployment Equilibrium

$$w_{t+1} < w_t \Leftrightarrow \lambda_t^s < 1;\ w_{t+1} > w_t \Leftrightarrow \lambda_t^d < 1.$$

More precisely, in our simulation model we have specified these adjustments in two alternative ways. The first is a nonlinear mechanism, namely

$$p_{t+1} = (\gamma_t^s)^{\mu_1} p_t,\ \text{if } \gamma_t^s < 1;\ p_{t+1} = \left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2} p_t,\ \text{if } \gamma_t^d < 1,$$

$$w_{t+1} = (\lambda_t^s)^{\nu_1} w_t,\ \text{if } \lambda_t^s < 1;\ w_{t+1} = \left(\lambda_t^d\right)^{-\nu_2} w_t,\ \text{if } \lambda_t^d < 1.$$

where μ_1, μ_2, ν_1 and ν_2 are nonnegative parameters for the speeds of adjustment. This formalizes that the size of price adjustment depends on the strength of rationing and allows us to encompass a wide variety of circumstances. For example, wage flexibility upwards greater than downwards is obtained whenever $\nu_2 > \nu_1$ and wage rigidity downwards corresponds to $\nu_1 = 0$.

From the adjustment of nominal prices we obtain the one for the real wage as

$$\alpha_{t+1} = \frac{(\lambda_t^s)^{\nu_1}}{(\gamma_t^s)^{\mu_1}} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U,$$

$$\alpha_{t+1} = \frac{\left(\lambda_t^d\right)^{-\nu_2}}{\left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2}} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I,$$

$$\alpha_{t+1} = \frac{\left(\lambda_t^s\right)^{\nu_1}}{\left(\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)^{-\mu_2}} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in C.$$

Then for the growth factor of the price level $\theta_t = p_{t+1}/p_t$ there results

$$\theta_t = \left(\gamma_t^s\right)^{\mu_1} \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U,$$

$$\theta_t = \left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2} \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I \cup C.$$

The second adjustment mechanism is linear and in fact a linearization of the nonlinear rule. More precisely,

$$p_{t+1} = \left[1 - \mu_1 \left(1 - \gamma_t^s\right)\right] p_t, \text{ if } \gamma_t^s < 1, \tag{4.1}$$

$$p_{t+1} = \left[1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)\right] p_t, \text{ if } \gamma_t^d < 1, \tag{4.2}$$

$$w_{t+1} = \left[1 - \nu_1 \left(1 - \lambda_t^s\right)\right] w_t, \text{ if } \lambda_t^s < 1, \ w_{t+1} = \left[1 + \nu_2 \left(1 - \lambda_t^d\right)\right] w_t, \text{ if } \lambda_t^d < 1. \tag{4.3}$$

Then the adjustment equations for the real wage are

$$\alpha_{t+1} = \frac{1 - \nu_1 \left(1 - \lambda_t^s\right)}{1 - \mu_1 \left(1 - \gamma_t^s\right)} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U, \tag{4.4}$$

$$\alpha_{t+1} = \frac{1 + \nu_2 \left(1 - \lambda_t^d\right)}{1 + \mu_2 \left(1 - \frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I, \tag{4.5}$$

$$\alpha_{t+1} = \frac{1 - \nu_1 \left(1 - \lambda_t^s\right)}{1 + \mu_2 \left(1 - \frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in C \tag{4.6}$$

whereas θ_t is given by

$$\theta_t = 1 - \mu_1 \left(1 - \gamma_t^s\right) \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U, \tag{4.7}$$

$$\theta_t = 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right) \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I \cup C. \tag{4.8}$$

In both the linear and the nonlinear case the dynamics of the model in real terms is given by the sequence $\{(\alpha_t, m_t, \pi_t)\}_{t=1}^{\infty}$, where α_{t+1} is as above and

$$\pi_{t+1} = \frac{\left[\mathcal{Y}(.) - \alpha_t \mathcal{L}(.)\right]}{\theta_t}$$

and

$$m_{t+1} = \frac{1}{\theta_t} \left[\delta_t m_t + \varepsilon_t G + (1 - tax)\, \pi_t\right] - \pi_{t+1}.$$

5. Numerical Analysis

The economic model introduced in the previous sections represents a non-linear three-dimensional dynamical system with state variables α_t, m_t and π_t that cannot be studied with analytical tools only. Moreover, since there are three nondegenerate equilibrium regimes, the overall dynamic system can be viewed as being composed of three subsystems each of which may become effective through endogenous regime switching. (The complete equations of these systems are given in Appendix 2.)

In order to get some insights in these dynamics we report numerical simulations using computer programs based on the program packages GAUSS and MACRODYN[5]. The basic parameter set specifies values for the technological coefficients (a and b), the exponent of the utility function (h), the labor supply (L^s) and the total number of producers in the economy (n'), for the price adjustment speeds downward and upward (respectively μ_1 and μ_2) and the corresponding wage adjustment speeds (ν_1 and ν_2). We also have to specify initial values for the real wage, real money stock and real profit level (α_0, m_0 and π_0) and values for the government policy parameters (G and tax).

Starting from the following parameter values corresponding to a stationary Walrasian equilibrium

$$\begin{array}{ccccc} a = 1 & b = 0.85 & h = 0.5 & L^s = 100 & n' = 100 \\ \alpha_0 = 0.85 & m_0 = 46.25 & \pi_0 = 15 & G = 7.5 & tax = 0.5 \end{array}$$

we address the question of the impact of changes in the downward speed of adjustment of the wage rate. To this end we consider a reduction in the initial money stock to $m_0 = 40$. Employing the linear price and wage adjustment defined in the previous section, we allow for an adjustment of the price in both directions ($\mu_1 = \mu_2 = 1$) but impose downward wage rigidity ($\nu_1 = 0, \nu_2 = 1$). As can be seen from Figure 5.1, the restrictive money shock gives rise to a transitional phase of unemployment before the system returns to a Walrasian equilibrium with full employment. The unemployment phase can be shortened by allowing for downward wage flexibility (as was to be expected from textbook theory). This is shown in Figure 5.2 where ν_1 has been changed to 0.5. However, Figure 5.3, where $\nu_1 = 1$, suggests that further increasing this downward flexibility results in irregular behavior with frequently high unemployment rates.

To see which downward wage flexibility is "too much" we can consider the bifurcation diagram in Figure 5.4. It reveals that the system is stable with convergence to full employment for ν_1 smaller than about 0.7 whereas from approx-

[5]See Böhm,V., Lohmann, M. and U. Middelberg [1999], MACRODYN– a dynamical system's tool kit, version x99, University of Bielefeld, and Böhm and Schenk-Hoppe' [1998].

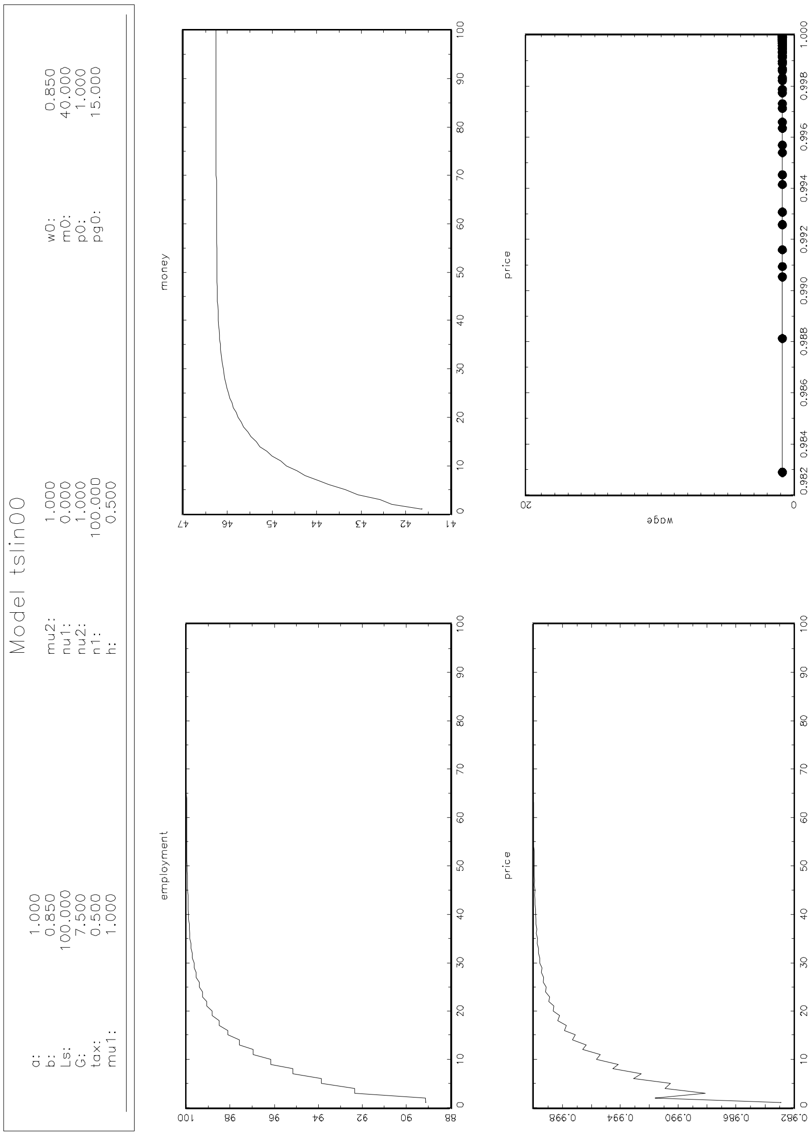


Figure 5.1: The time series in the figure show the emergence of transitional unemployment when the real stock of money is reduced from the Walrasian equilibrium level of 46,25 to 40 and the coefficients for the adjustment of prices and wage are $\mu_1 = \mu_2 = \nu_2 = 1$ and $\nu_1 = 0$.

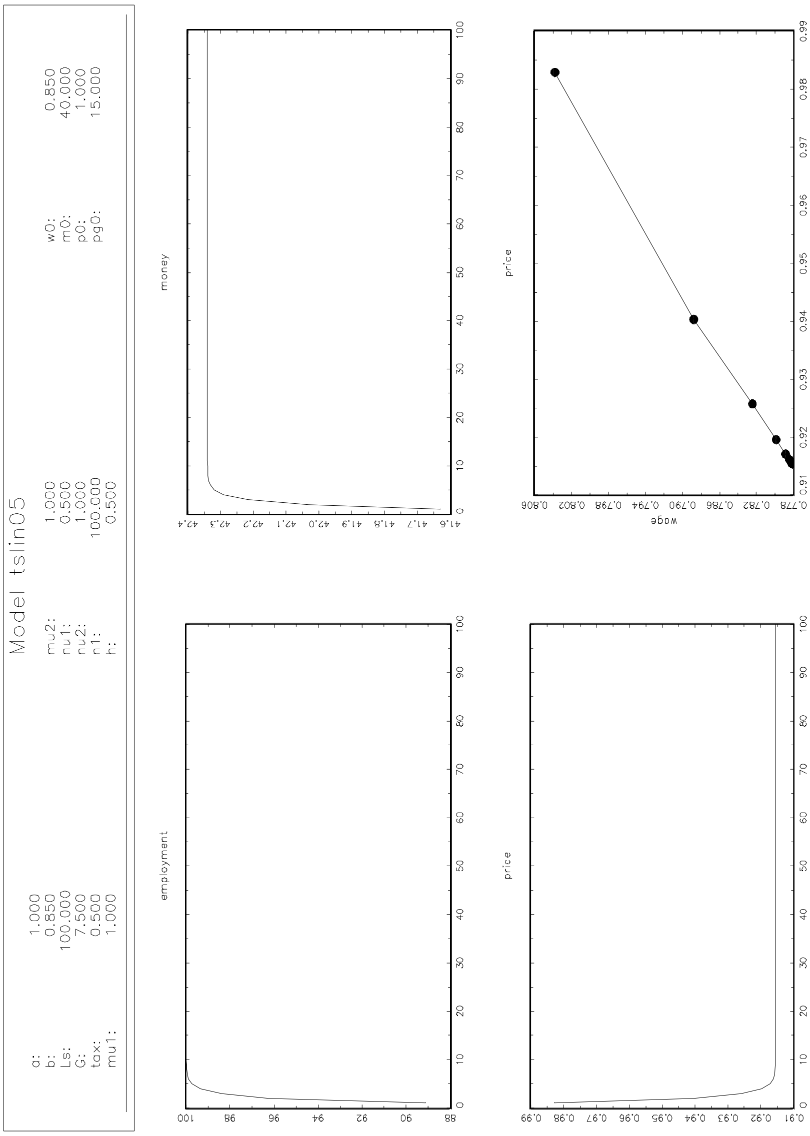


Figure 5.2: The unemployment phase is shortened when downward wage flexibility is allowed. The parameter set is the same as in the previous figure, except that $\nu_1 = 0.5$.

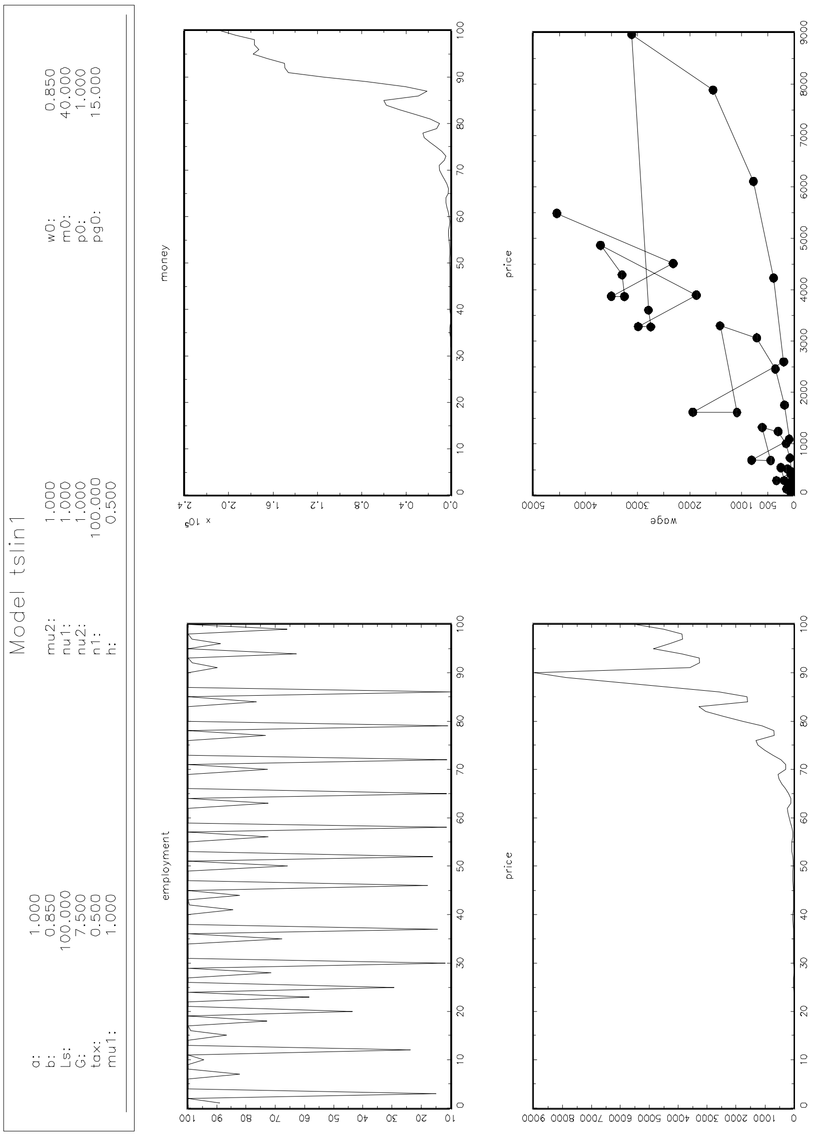


Figure 5.3: The behavior of the system becomes highly irregular when the downward flexibility increases. In the simulation represented here ν_1 is 1.

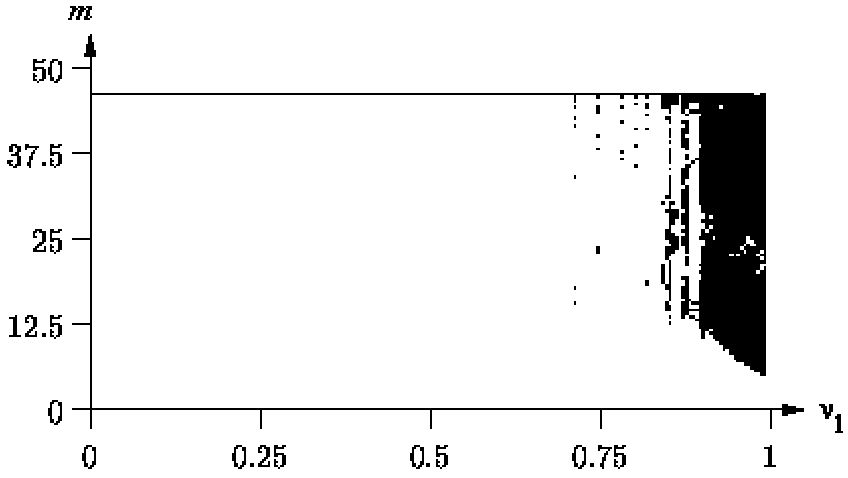


Figure 5.4: The bifurcation diagram shows that for values of ν_1 smaller than 0.7 the system converges, but for values larger than 0.9 it displays chaotic behavior.

imately 0.9 on its behavior is chaotic. The latter can also be seen by looking at chaotic attractors as the one shown in Figure 5.5. It depicts points (α_t, m_t) obtained from the same parameter set as the one used in Figure 5.3, i.e. with $\nu_1 = 1$. More precisely the diagram represents the projection into the (α, m) −plane of a part of the sequence $\{(\alpha_t, m_t, \pi_t)\}_{t=0}^{\infty}$. The total set of t_1 iterations is divided in two subsets, one $(0 \leq t < t_0)$ including the transient phase generating points not plotted and one subsequent $(t_0 \leq t \leq t_1)$ producing points that are shown.[6] In case of convergence, as with the parameter sets of Figures 5.1 and 5.2, all the points generated during the transient phase fall into the first subinterval and then the diagram shows one point only, namely the limit of the sequence. Instead in the case of chaos, during the simulation that produces an attractor, the latter's persistence is reflected on the computer screen also by the fact that the second subinterval is in turn partitioned into six sub-subintervals such that points belonging to iterations stemming from a specific sub-subinterval are shown in a corresponding specific colour. If these colours cover each other, it means that the geometric object produced does not move any more which supports the fact that it is an attractor.

[6]In our simulations we have worked with $t_0 = 10$ million and $t_1 = 100$ million.

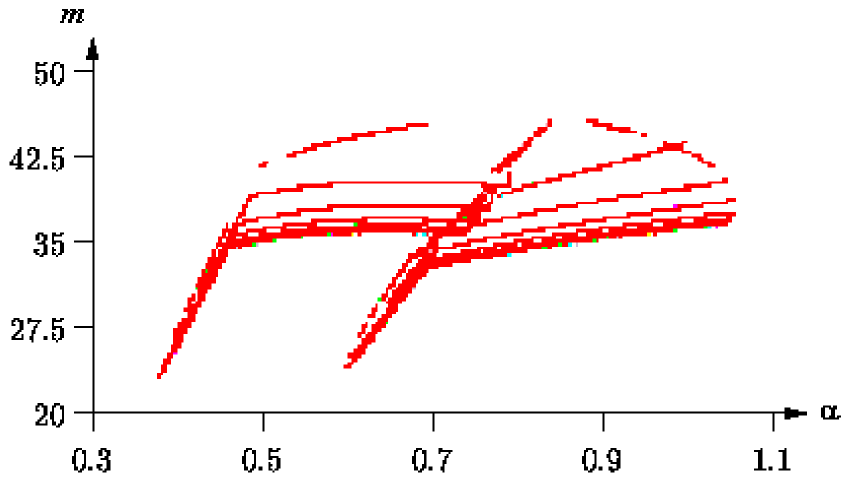


Figure 5.5: A chaotic attractor in the (α, m) - space.

6. The Phillips Curve

By definition the Phillips curve plots the rate of wage inflation against the unemployment rate. Any sequence $\{(\alpha_t, m_t, \pi_t)\}_{t=0}^{\infty}$ gives rise, for given ν_1, to a sequence $\{(u_t, v_t)\}_{t=0}^{\infty} = \{F(\alpha_t, m_t, \pi_t)\}_{t=0}^{\infty}$ where $u_t = \left(L^s - \overline{L}_t\right)/L^s$ is the unemployment rate and $v_t = (w_{t+1} - w_t)/w_t$ the rate of wage inflation. Before we look at numerically generated sequences $\{(u_t, v_t)\}_t$ we establish the theoretical relationship between u_t and v_t.

Given any $u_t \in [0, 1]$, the associated employment level is $\overline{L}_t = L^s(1 - u_t)$ and therefore $\lambda_t^s = \overline{L}_t/L^s = 1 - u_t$. If wage adjustment is linear then $v_t = -\nu_1(1 - \lambda_t^s) = -\nu_1 u_t$. Solving for u_t and taking account of the fact that it cannot be negative, we obtain

$$u_t = \begin{cases} -\frac{1}{\nu_1} v_t, & v_t \leq 0 \\ 0 & , v_t \geq 0 \end{cases} . \tag{6.1}$$

On the other hand, if wage adjustment is nonlinear, $v_t = (\lambda_t^s)^{\nu_1} - 1 = (1 - u_t)^{\nu_1} - 1$. Therefore in this case

$$u_t = \begin{cases} 1 - (1 + v_t)^{1/\nu_1}, & v_t \leq 0 \\ 0 & , v_t \geq 0 \end{cases} . \tag{6.2}$$

Both (6.1) and (6.2) are functional relationships between the rate of wage inflation and the unemployment rate the graphs of which are potential Phillips curves. In

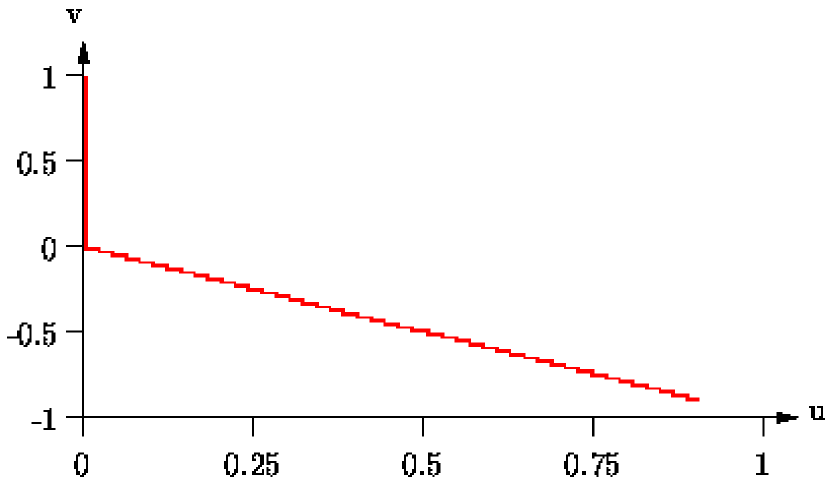


Figure 6.1: A long run Phillips-curve as an attractor, generated with the parameter values of Figures 5.3 and 5.5.

fact, for any given value of the speed of downward wage adjustment ν_1, any trajectory of $\{(u_t, v_t)\}_{t=0}^{\infty}$ must lie on the corresponding curve, independently of all other parameter values.

Figure 6.1 shows a Phillips curve as a plot from t_0 to t_1 of a sequence $\{(u_t, v_t)\}_{t=0}^{\infty}$ resulting from the sequence $\{(\alpha_t, m_t, \pi_t)\}_{t=0}^{\infty}$ generating Figure 5.5.

Note that this Phillips curve emerges by very construction as a true long-run phenomenon as it is the image $\{F(\alpha_t, m_t, \pi_t)\}_{t=t_0}^{t_1}$ of an attractor. Notice also the remarkable fact that the trajectory $\{(u_t, v_t)\}_{t=t_0}^{t_1}$ lies on a curve in spite of the fact that the corresponding trajectory $\{(\alpha_t, m_t)\}_{t=t_0}^{t_1}$ does not lie on a one-dimensional geometric object. The proof of this is given by (6.1) resp.(6.2).

From these considerations follows that it would be wrong to interprete our Phillips curve as a policy instrument in terms of a trade-off between unemployment and inflation. Any point on the curve is but one element on the trajectory $\{(u_t, v_t)\}_t$ and successive points of this trajectory may lie far away one from the other. Thus, even if the government could choose a specific point on the curve in one period, in the next period already the system may go to a very different point on the curve.

To understand still better how the curve comes about, we can look at the diagrams in the second row of charts of Figure 5.3. The left-hand chart plots the price for periods 1 to 100. Until about period 50 the price does not change very much but then it starts to alternatingly increase and decrease quite substantially.

The right-hand chart depicts the trajectory of price and wage couples. It starts out in the lower left angle and then has a tendency to move upwards and to the right. If, in a given period, the economy finds itself in a state of Keynesian unemployment, both the price and the wage are increased, whereas the opposite is true in a state of repressed inflation. Furthermore, in a state of classical unemployment the price increases but the wage diminishes. Therefore the chart displaying price and wage couples shows that the economy visits all three types of equilibria along its trajectory. From the chart displaying employment, on the other hand, it is obvious that unemployment rates may vary substantially and therefore the points on the Phillips curve may jump consirably from one period to the next.

How robust is the emergence of a Phillips curve? It is clear that, when the economy converges, the Phillips curve reduces to one point. This is due to the fact that in the attractor plotting the transient phase is excluded. Therefore, when the picture produced is a full curve, this can only be an attractor. To see how sensitive to the parameter values is this phenomenon, we can use a technical device called cyclogram and developed by Lohmann and Wenzelburger [1996]. It permits to establish a relationship between the values of the relevant parameters and the structure of the resulting dynamics (although it is not able to distinguish between regular quasi-periodic behavior and "true" chaotic motion).

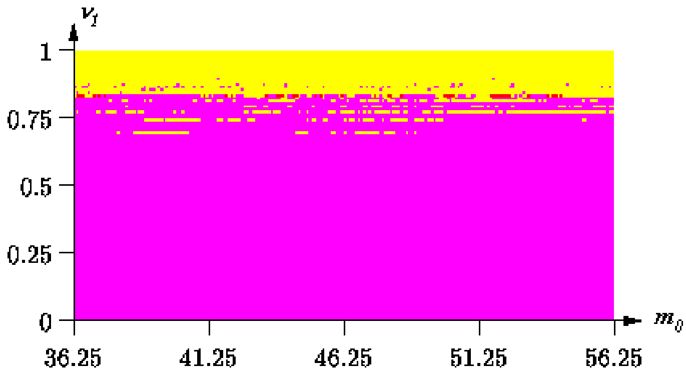
More precisely, the first chart in Figure 6.2 displays a cyclogram where the color attached to each point (m_0, ν_1) in the rectangle $[36.25, 56.25] \times [0, 1]$ reflects either convergence (red) or irregular behaviour (yellow). From this it is evident that all velocities of wage reduction beyond 0.9 give rise to irregular behaviour and thus the Phillips curve shown in Figure 5.3 can be obtained for all values of ν_1 between 0.9 and 1. Analogous cyclograms can be obtained by varying on the horizontal axis the initial real wage α_0 or the initial profit π_0 around their Walrasian values.[7]

The Phillips curve shown in Figure 6.1 has its downward sloping part for negative values of wage inflation only. This is due to our strict application of the rule that prices are increased only if there is excess demand. A more flexible - and probably more realistic - formulation allows for a "natural rate of inflation" $v^n \geq 0$ which is the rate at which prices and wages are increased when markets clear. Then equations (4.1) - (4.3) become

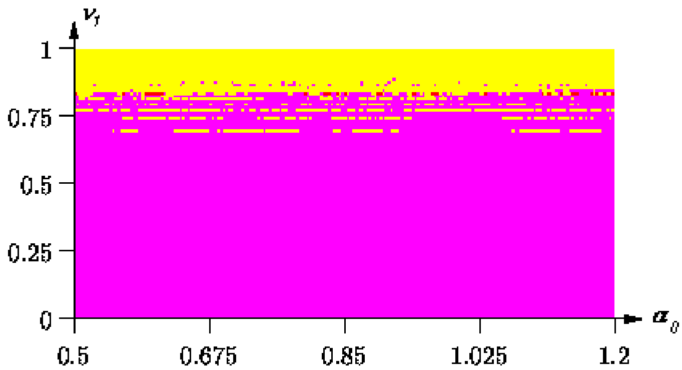
$$p_{t+1} = (1 - \mu_1 (1 - \gamma_t^s)) (1 + v^n) p_t, \text{ if } \gamma_t^s < 1,$$

$$p_{t+1} = \left(1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)\right) (1 + v^n) p_t, \text{ if } \gamma_t^d < 1,$$

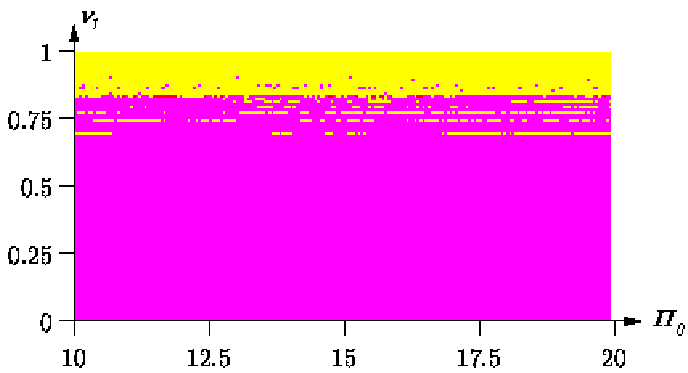
[7] The cyclograms in Figure 6.2 have been obtained for speeds of adjustment $\mu_1 = \mu_2 = \nu_2 = 1$ and $0 \leq \nu_1 \leq 1$. Qualitatively similar results can be obtained also for values $\mu_1 = \mu_2 = \nu_2 = s \in [0.4, 1]$ and $0 \leq \nu_1 \leq s$.



Cyclogram in (m_0, ν_1) space



Cyclogram in (α_0, ν_1) space



Cyclogram in (Π_0, ν_1) space

Figure 6.2: These cyclograms show the dynamic behavior of the economy when the initial values of the state variables m_0, α_0 and π_0 vary respectively in the range $(36.25; 56.25)$, $(0.5; 1.2)$ and $(10; 15)$. Notice that downward wage flexibility favors irregular behavior.

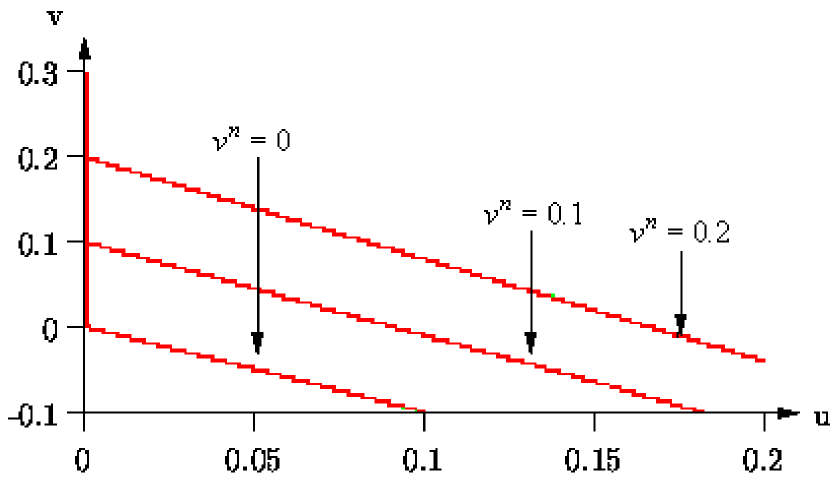


Figure 6.3: Phillips curves for $v^n = 0$, $v^n = 0.1$ and $v^n = 0.2$.

$$w_{t+1} = (1 - \nu_1 (1 - \lambda_t^s)) (1 + v^n) w_t, \text{ if } \lambda_t^s < 1,$$
$$w_{t+1} = \left(1 + \nu_2 \left(1 - \lambda_t^d\right)\right) (1 + v^n) w_t, \text{ if } \lambda_t^d < 1.$$

Equations (4.4) - (4.6) do not vary whereas (4.7) and (4.8) now are

$$\theta_t = (1 - \mu_1 (1 - \gamma_t^s)) (1 + v^n) \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U,$$

$$\theta_t = \left(1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)\right) (1 + v^n) \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I \cup C.$$

This means that the dynamics of real variables is unchanged relative to the case $v^n = 0$ whereas nominal variables do change. The Phillips curve (6.1) then becomes

$$u_t = \begin{cases} \frac{v^n - v_t}{\nu_1 (1 + v^n)}, & v_t \leq v^n \\ 0 & , v_t \geq v^n \end{cases} .$$

Two curves for $v^n = 0.1$ and $v^n = 0.2$, together with the previous case $v^n = 0$, are shown in Figure 6.3.

The fact that for different values of v^n we obtain different Phillips curves does not impair the validity of this paper's main messages. To recall them, they are that a Phillips curve can be generated as a long-run result of the working of a dynamical system describing the evolution of an economy and that it cannot be exploited for policy uses. In our model this is true for any fixed value of v^n. On the other hand, the model is not rich enough to be able to directly account for supply-side shocks. Empirically, it is these shocks that are typically giving rise to stagflation and to an increase in the expected inflation rate, thereby shifting the Phillips curve to the right. The latter is what happens in our model when v^n is augmented. Thus that parameter can be seen as a shortcut to account for the role of shifts in inflation expectations. Therefore the model can indirectly accomodate supply shocks, in spite of the fact that its formulation of the production sector is too simple to do this in a direct way.

7. Concluding Remarks

In this paper we have presented a dynamic macroeconomic model capable of generating long-run Phillips curves. The specific property of the model is that it allows for trade also when prices are not market clearing. This renders possible to have recurrent unemployment and inflation. Although prices are adjusted from period to period when there are market imbalances, convergence to the Walrasian equilibrium may fail and, to the contrary, complex dynamics may occur. More precisely, we have adopted a non-tâtonnement approach involving temporary equilibria with stochastic quantity rationing and price adjustment between successive equilibria. In this way we have obtained a process which allows us to describe consistent allocations involving disequilibrium features in every period but which at the same time displays a well-defined dynamics.

While the dynamics of the resulting three-dimensional system is too complex to be completely understood by means of analytical tools only, we have been able to shed some light on it using simulations in the form of time series, bifurcation diagrams, attractors and cyclograms. From these simulations it is apparent that the economy may produce sequences of pairs of unemployment rates and inflation rates which constitute a Phillips curve. The specific feature of these so generated curves is that they are a true long-run phenomenon since they are images of strange attractors. However, these curves cannot be exploited for policy purposes as any point on them represents but one realization of a whole trajectory and successive points of the trajectory may lie far away one from the other. Therefore those economists who have criticized as a misunderstanding of the workings of an economy the use of the Phillips curve as policy tool can be confirmed by our results. On the other hand, our results are compatible with empirical findings such as the original one by Phillips [1958].

The occurence of a Phillips curve as an attractor in our model requires that wages are sufficiently flexible. Indeed, when wages are rigid downwards, the economy typically converges to the market-clearing equilibrium. It is true that that convergence is speeded up when wages become flexible downwards but, when that flexibility is large enough, convergence is destroyed and irregular behaviour emerges. To summarize this in a accentuated way is to say that, contrary to the standard paradigm, wage rigidity favors the stability of the economy and its return after a shock to the Walrasian equilibrium whereas wage flexibility may be destabilizing.

Appendix 1: Lemma 1 - 3

Lemma 1. *The solution of the young consumer's maximization problem is independent both of γ_t^d and p_{t+1}^e when his utility function is $u(x_t, x_{t+1}) = x_t^h x_{t+1}^{1-h}$ and $\rho = 1$. More specifically, in that case $x_t^{di} = h\omega_t^i, i = 0, 1$.*

Proof. For notational simplicity let $x = x_t^d, \omega = \omega_t^i, c = c_t, \theta = \theta_t^e$ and define

$$f(x) = u_1\left(x, \frac{\omega - x}{\theta}\right) = h\left(\frac{\omega - x}{\theta x}\right)^{1-h};$$

$$g(x) = u_2\left(x, \frac{\omega - x}{\theta}\right) = (1-h)\left(\frac{\theta x}{\omega - x}\right)^{h};$$

Then

$$f(cx) = u_1\left(cx, \frac{\omega - cx}{\theta}\right) = h\left(\frac{\omega - cx}{\theta cx}\right)^{1-h};$$

$$g(cx) = u_2\left(x, \frac{\omega - cx}{\theta}\right) = (1-h)\left(\frac{c\theta x}{\omega - cx}\right)^{h}.$$

Observe that $g(x)$ and $g(cx)$ can be written as functions of $f(x)$ and $f(cx)$ respectively:

$$g(x) = \frac{1-h}{h}\frac{\theta x}{\omega - x}f(x);$$

$$g(cx) = \frac{1-h}{h}\frac{c\theta x}{\omega - cx}f(cx).$$

By substituting these functions in equation (2.2) we get

$$\frac{\rho f(x) + (1-\rho) f(cx)}{\rho\frac{1-h}{h}\frac{\theta x}{\omega - x}f(x) + (1-\rho)\frac{1-h}{h}\frac{c\theta x}{\omega - cx}f(cx)} = \frac{1}{\theta}$$

and, remembering the definition of $f(x)$ and $f(cx)$, after some algebraic manipulations follows

$$\rho(\omega - cx)^h(h\omega - x) + (1-\rho)\left(\frac{1}{c}\right)^{1-h}(\omega - x)^h(h\omega - cx) = 0 \qquad \text{(a1)}$$

Recall that by assumption $\rho = 1$ which implies $c = 0$. Hence the term $(1-\rho)\left(\frac{1}{c}\right)^{1-h}$ is undetermined and therefore we need to consider the limit

$$\lim_{\rho\to 1}(1-\rho)\left(\frac{1}{c}\right)^{1-h} = \lim_{\rho\to 1}(1-\rho)\left[\frac{(1-\rho)\gamma}{1-\rho\gamma}\right]^{h-1} = 0.$$

By substituting this into equation (a1) the result follows; in fact,

$$h\omega - x = 0 \Rightarrow x_t^{di} = h\omega_t^i, i = 0,1 \blacksquare$$

Lemma 2. *When the production function is $f(\ell) = a\ell^b$, with $a > 0$ and $0 < b \leq 1-\sigma$, the solution to the firm's maximization problem is independent of the constraint $\ell_t^d \leq \frac{d_t}{\alpha_t} f\left(\ell_t^d\right)$.*

Proof. The first order condition for an interior solution of the firm's problem is

$$\gamma^s f'(\ell) = \alpha \Leftrightarrow \gamma^s \frac{bf(\ell)}{\ell} = \alpha \Leftrightarrow \ell = \gamma^s \frac{bf(\ell)}{\alpha}.$$

Moreover the inequalities $\frac{1}{b} \geq \frac{1}{1-\sigma} \geq \frac{1-\gamma^s\sigma}{1-\sigma}$ yield $1 \leq \frac{1-\sigma}{b(1-\gamma^s\sigma)}$. From this follows

$$\ell \leq \frac{\gamma^s(1-\sigma)}{1-\gamma^s\sigma}\frac{1}{\gamma^s}\frac{1}{b}\ell = d\frac{1}{\gamma^s}\frac{1}{b}\ell = d\frac{1}{\gamma^s}\frac{1}{b}\gamma^s\frac{bf(\ell)}{\alpha} = \frac{d}{\alpha}f(\ell),$$

which proves our claim. $\blacksquare$

Lemma 3. *When the production function is $f(\ell) = a\ell^b$, with $a > 0$ and $0 < b \leq 1-\sigma$, the producers' trade curves are given by*

$$\overline{F}^K = \overline{F}^I = \overline{F}^U = \left\{\left(L, \frac{\alpha_t}{b}L\right) \mid 0 \leq L < L^d(1;\alpha_t)\right\}$$

and $\overline{F}^C = \left\{\left(L^d(1;\alpha_t), \frac{\alpha_t}{b}L^d(1;\alpha_t)\right)\right\}$.

Proof. From $f(\ell) = a\ell^b$ follows $f'(\ell) = b\frac{f(\ell)}{\ell}$, which implies $f(\ell) = \frac{1}{b}f'(\ell)\ell$ and

$$Y = \gamma^s Y^s\left(\lambda^d, \gamma^s\right) = \gamma^s\lambda^d n' f\left(\ell^d(\gamma^s;\alpha_t)\right) = \gamma^s\lambda^d n'\frac{1}{b}f'\left(\ell^d(\gamma^s;\alpha_t)\right)\ell^d\left(\gamma^{s;\alpha_t}\right)$$

But $\gamma^s f'\left(\ell^d(\gamma^s;\alpha_t)\right) = \alpha_t$ from any producer's optimizing behavior, and thus

$$Y = \frac{\alpha_t}{b}\lambda^d L^d(\gamma^s;\alpha_t) = \frac{\alpha_t}{b}L \blacksquare$$

Appendix 2: The complete dynamic system

The dynamic system is given by three different subsystems, one for each of the equilibrium types K, I and C, and endogenous regime switching. For given (G, tax), any list (α_t, π_t, m_t) gives rise to a uniquely determined equilibrium allocation $\left(\overline{L}_t, \overline{Y}_t\right)$ being of one of the above types (or of an intermediate one). This type is determined according to the procedure described in section 3. More precisely,

$$\overline{L}_t = \min\left\{\widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right), L^d\left(1, \alpha_t\right), L^s\right\}$$

with

$$\widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right) = \frac{b}{\alpha_t\left(1 - hb\right)}\left[h\left(1 - tax\right)\pi_t + m_t + G\right],$$

$$L^d\left(1, \alpha_t\right) = n'\left(\frac{\alpha_t}{ab}\right)^{\frac{1}{b-1}}.$$

When $\overline{L}_t = \widetilde{L}(\cdot)$, the K-subsystem applies whereas when $\overline{L}_t = L^d(\cdot)$ or L^s, the systems associated to the types C or I, respectively, are the ones to be used. Regime switching may occur because $\left(\overline{L}_t, \overline{Y}_t\right)$ may be of type $T \in \{K, I, C\}$ and $\left(\overline{L}_{t+1}, \overline{Y}_{t+1}\right)$ of type $T' \neq T$. Regarding the subsystems, they are the following. (We present the case of linear price and wage adjustment only; the nonlinear case is analogous.)

Keynesian unemployment system

Employment level: $\overline{L}_t = \widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right)$; output level: $\overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t$; rationing coefficients: $\lambda_t^s = \frac{\overline{L}_t}{L^s}, \lambda_t^d = 1, \gamma_t^s = \left[\frac{\overline{Y}_t}{n'a}\left(\frac{\alpha_t}{ab}\right)^{\frac{b}{1-b}}\right]^{1-b}, \gamma_t^d = 1, \delta_t = \varepsilon_t = 1$; price inflation: $\theta_t = 1 - \mu_1\left(1 - \gamma_t^s\right)$; real wage adjustment: $\alpha_{t+1} = \frac{1-\nu_1(1-\lambda_t^s)}{1-\mu_1(1-\gamma_t^s)}\alpha_t$; real profit: $\pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right)$;

real money stock: $m_{t+1} = \frac{1}{\theta_t}\left[m_t + G + \left(1 - tax\right)\pi_t\right] - \pi_{t+1}$.

Repressed inflation system

$\overline{L}_t = L^s; \overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t; \lambda_t^s = 1, \lambda_t^d = \frac{L^s}{L^d(1,\alpha_t)}; \gamma_t^s = 1;$
if $\overline{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t\overline{L}_t}, \delta_t = \varepsilon_t = 1;$
if $G + m_t > \overline{Y}_t \geq G$, then $\gamma_t^d = 0, \delta_t = \frac{\overline{Y}_t - G}{m_t}, \varepsilon_t = 1;$
if $\overline{Y}_t < G$, then $\gamma_t^d = \delta_t = 0, \varepsilon_t = \frac{\overline{Y}_t}{G};$

$\theta_t = 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right); \alpha_{t+1} = \frac{1+\nu_2\left(1-\lambda_t^d\right)}{1+\mu_2\left(1-\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)}\alpha_t; \pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t \overline{L}_t\right);$
$m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1 - tax)\,\pi_t\right] - \pi_{t+1}.$

Classical Unemployment System

$\overline{L}_t = L^d\left(1, \alpha_t\right); \overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t; \lambda_t^s = \frac{\overline{L}_t}{L^s}, \lambda_t^d = 1, \gamma_t^s = 1;$
if $\overline{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t \overline{L}_t}, \delta_t = \varepsilon_t = 1;$
if $G + m_t > \overline{Y}_t \geq G$, then $\gamma_t^d = 0, \delta_t = \frac{\overline{Y}_t - G}{m_t}, \varepsilon_t = 1;$
if $\overline{Y}_t < G$, then $\gamma_t^d = \delta_t = 0, \varepsilon_t = \frac{\overline{Y}_t}{G};$
$\theta_t = 1 + \mu_2 \left(1 - \frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right); \alpha_{t+1} = \frac{1-\nu_1\left(1-\lambda_t^s\right)}{1+\mu_2\left(1-\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)}\alpha_t; \pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t \overline{L}_t\right);$
$m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1 - tax)\,\pi_t\right] - \pi_{t+1}.$

The underconsumption case is not represented with an own dynamical system because choosing $\gamma_t^s = \gamma_t^s \lambda_t^d$ and $\lambda_t^d = 1$ it can be treated as a special case of the Keynesian case.

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