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**FROM SIMPLE GROWTH TO NUMERICAL
SIMULATIONS: A PRIMER IN DYNAMIC
PROGRAMMING**

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Dipartimento di Economia e Finanza
Università Cattolica del Sacro Cuore
Largo Gemelli 1 - 20123 Milano – Italy
tel: +39.02.7234.2976 - fax: +39.02.7234.2781
e-mail: dip.economiaefinanza@unicatt.it

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FROM SIMPLE GROWTH TO NUMERICAL SIMULATIONS: A PRIMER IN DYNAMIC PROGRAMMING

GIANLUCA FEMMINIS

ABSTRACT. These notes provide an intuitive introduction to dynamic programming. The first two Sections, which can be skipped, present the standard deterministic Ramsey model using the Lagrangian approach. Section 3 reformulates the Ramsey problem by means of a Bellman equation, while Section 4 shows how to “guess” the maximum value function solving the problem (when this is possible). Section 5 is devoted to applications of the envelope theorem. Section 6 provides a “paper and pencil” introduction to the numerical techniques used in dynamic programming, and can be skipped by the uninterested reader. Sections 7 to 9 are devoted to stochastic modelling, and to the use of stochastic Bellman equations. Section 10 extends the discussion of numerical techniques. Two Appendixes provide details about the Matlab routines used to deal with the examples, and the solutions of the exercises proposed in the main text.

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Correspondence to: Gianluca Femminis, Università Cattolica, Largo Gemelli 1, 20123 Milano, Italy; *e-mail address:* gianluca.femminis@unicatt.it.

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1. UTILITY MAXIMIZATION IN A FINITE-HORIZON DETERMINISTIC SETTING

One of the ingredients that can be found in almost any growth model is the analysis of the agents' consumption behavior. In fact consumption, through savings, determines capital accumulation, which, in turn, is one of the key "engines of growth". In this Section, we consider the problem of the optimal determination of consumption in the easiest possible framework, namely the one in which the lifetime of a single consumer is of a finite and known length. We solve this intertemporal problem using the Lagrangian approach: once the problem is well understood, it will be easy to consider its infinite horizon counterpart and then to solve it by means of dynamic programming. This shall be done in Sections 2 and 3, respectively.

1.1. The problem. In our settings, a single consumer aims at maximizing her utility over her finite lifetime. Time is "discrete", i.e. it is divided into periods of fixed length (say, a year or a quarter), and our consumer is allowed to decide her consumption level only once per period. The consumption goods she enjoys are produced by means of a "neoclassical" production function.¹

We suppose that our consumer optimizes from time 0 onwards, and that her preferences are summarized by the following intertemporal utility function:

$$(1.1) \quad W_0 = \sum_{t=0}^T \beta^t U(c_t),$$

where $\beta \in (0, 1)$ is the subjective discount parameter, c_t is consumption at time t , and $T + 1$ is the length (in periods) of our consumer's lifetime. As for the single period utility function, $U(c_t)$, we accept the standard "neoclassical" assumptions, requiring that, in every period, the marginal utility is positive but decreasing, i.e. that $U'(c_t) > 0$, and $U''(c_t) < 0$. Moreover, we assume that: $\lim_{c_t \rightarrow 0} U'(c_t) = \infty$.

¹An analysis concerning a *single* consumer may seem very limited. In particular, as it will become clear in a while, a single agent—being alone—optimizes under the constraint of the production function. This appears to be in sharp contrast with what happens in the real world. In fact, in a market economy, agents do not directly manage a production function. Rather, they work, obtaining a wage income, and receive interests (or dividends) thanks to their past savings. In other words, optimizing agents take into account prices, wages, interest rates... However, it can be shown that *if* markets are competitive and agents are all alike, the resources allocation in our exercises is equivalent to the allocation of resources that is achieved by a competitive decentralized economy (which means by an economy where decisions are taken by a large number of economic agents, and based upon prices, wages...). Hence, while our *if* is a rather big one, our analysis is less limited than what it might seem at first sight.

Output (y_t) is obtained by means of a production function, the argument of which is capital (k_t):²

$$(1.2) \quad y_t = f(k_t).$$

As any well-behaved “neoclassical” production function, Eq. (1.2) satisfies some conditions. When capital depreciates at a constant rate $\delta \in [0, 1]$, these are:

- a:** $f'(k_t) > 0$,
- b:** $f''(k_t) < 0$,

(in words, the marginal productivity of capital is positive, but decreasing),

- c:** $f(0) = 0$,

(this means that capital is essential for production).

- d:** $f'(0) > \delta + 1/\beta - 1$,
- e:** $\lim_{k_t \rightarrow \infty} f'(k_t) = 0$,

(as it will become clear in what follows, hypothesis (d) implies that capital – at least at its lowest level – is productive enough to provide the incentive for building a capital stock, while assumptions (e) rules out the possibility that capital accumulation goes on forever.)³

At this point, it is commonly assumed that output, in our one-good economy, can be either consumed or invested, i.e. it is assumed that $y_t = c_t + i_t$ (which implies that we are abstracting from the presence of government expenditure). The stock of capital owned by our agent in period 1 is: $k_1 = i_0 + (1 - \delta)k_0$. Accordingly, Eq. (1.2) and the output identity, $y_t = c_t + i_t$, imply that k_1 can be written as:

$$k_1 = f(k_0) + (1 - \delta)k_0 - c_0.$$

Hence, in general, we have that

$$(1.3) \quad k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t,$$

²If you feel disturbed by the fact that capital is the unique production input, consider that we can easily encompass a fixed supply of labour in our framework. We might have specified our production function as $y_t = g(k_t, \bar{l})$, where \bar{l} is the labour fixed supply; in this case we could have normalized \bar{l} to unity and then we could have written $f(k_t) \equiv g(k_t, 1)$. An alternative, and more sophisticated, way of justifying Eq. (1.2) is to assume that output is obtained by means of a production function which is homogeneous of degree one in capital and labour, so that there are constant returns to scale. Then, one interprets k_t as the capital/labour ratio.

³What is really necessary is to accept that $\lim_{k_t \rightarrow \infty} f'(k_t) < \delta$, an hypothesis that can hardly be considered restrictive. The assumption in the main text allows for a slightly easier exposition.

for $t = 0, 1, \dots, T$. In addition to the above set of dynamic constraints, we require that

$$(1.4) \quad k_{T+1} \geq 0.$$

In words, this obliges our consumer to finish her life with a non-negative stock of wealth. This condition must obviously be fulfilled by a consumer which lives “in insulation” (a negative level of capital stock does not make any sense in this case); if our agent were settled in an economy where financial markets are operational, so that borrowing and lending are allowed, what a condition like (1.4) rules out is the possibility that our consumer dies in debt.

Summing up, we wish to solve the problem:

$$\max_{\{c_t\}_{t=0}^T} W_0 = \max_{\{c_t\}_{t=0}^T} \sum_{t=0}^T \beta^t U(c_t),$$

under the T constraints of the (1.3)-type, under the constraint (1.4), and given the initial amount of resources, k_0 . Notice that the solution of the problem requires the determination of $T+1$ consumption levels (c_0, c_1, \dots, c_T), and of $T+1$ values for the capital stock (k_1, k_2, \dots, k_{T+1}).

We can approach the consumer’s intertemporal problem by forming a Lagrangian:

$$(1.5) \quad \begin{aligned} L_0 = & U(c_0) + \beta U(c_1) + \beta^2 U(c_2) + \dots + \beta^T U(c_T) \\ & - \lambda_0 [k_1 - f(k_0) - (1 - \delta)k_0 + c_0] \\ & - \beta \lambda_1 [k_2 - f(k_1) - (1 - \delta)k_1 + c_1] \\ & - \dots \\ & - \beta^{T-1} \lambda_{T-1} [k_T - f(k_{T-1}) - (1 - \delta)k_{T-1} + c_{T-1}] \\ & - \beta^T \lambda_T [k_{T+1} - f(k_T) - (1 - \delta)k_T + c_T] \\ & + \beta^T \mu k_{T+1}, \end{aligned}$$

in which λ_t , for $t = 0, 1, \dots, T$, and μ are the Lagrange multiplier.⁴ Notice that λ_t , that pertains to the period t constraint, is multiplied by β^t .

To solve the problem, we must differentiate (1.5) with respect to c_t , k_{t+1} , λ_t (for $t = 0, 1, \dots, T$), and with respect to μ .⁵

⁴A neat interpretation of the role of Lagrange multiplier can be found in Dixit [1990].

⁵The conditions we imposed on the single period utility function and on the production function guarantee that we obtain a global maximum. See, e.g., Beavis and Dobbs [1990], or de la Fuente [2000].

By taking derivatives of (1.5) with respect to the $T+1$ consumption levels, we obtain the first order conditions

$$\begin{aligned}
 (1.6) \quad & U'(c_0) = \lambda_0 \\
 & U'(c_1) = \lambda_1 \\
 & \dots = \dots \\
 & U'(c_t) = \lambda_t \\
 & \dots = \dots \\
 & U'(c_T) = \lambda_T.
 \end{aligned}$$

Notice that each Lagrange multiplier λ_t expresses the consumer's marginal utility of period t consumption; had we not multiplied λ_t by β^t , the multipliers would have expressed the marginal utilities from the perspective of period 0.

When our agent optimizes with respect to the $T+1$ capital levels (from k_1 to k_{T+1}), she obtains:

$$\begin{aligned}
 (1.7) \quad & \lambda_0 = \beta\lambda_1[f'(k_1) + (1 - \delta)] \\
 & \beta\lambda_1 = \beta^2\lambda_2[f'(k_2) + (1 - \delta)] \\
 & \dots \\
 & \beta^t\lambda_t = \beta^{t+1}\lambda_{t+1}[f'(k_{t+1}) + (1 - \delta)] \\
 & \dots \\
 & \beta^{T-1}\lambda_{T-1} = \beta^T\lambda_T[f'(k_T) + (1 - \delta)] \\
 & \beta^T\mu = \beta^T\lambda_T.
 \end{aligned}$$

Of course, derivation of (1.5) with respect to the Lagrange multipliers λ_t , $t = 0, 1, 2, \dots, T$ yields the set of constraints (1.3). Moreover, derivation of (1.5) with respect to μ gives the constraint (1.4); in addition, being (1.4) an inequality constraint, one must consider the "complementary slackness" condition:

$$(1.8) \quad \beta^T\mu k_{T+1} = 0 \text{ and } \mu \geq 0,$$

which shall be commented upon in a while.

1.2. The Euler equation. Consider now any first order condition belonging to group (1.7): one immediately sees that those equations can be manipulated using the appropriate first order conditions of group (1.6). The typical result of this practice is:

$$(1.9) \quad U'(c_t) = \beta U'(c_{t+1})[f'(k_{t+1}) + (1 - \delta)]$$

This condition is known as the Euler equation, which is of remarkable importance not only to understand many growth models but also in consumption theory.

The Euler equation (1.9) tells us that an optimal consumption path must be such that – in any period – the marginal utility for consumption is equal to the following period marginal utility, discounted by β and capitalized by means of the net marginal productivity of capital. To gain some intuition about the economic meaning of Eq. (1.9), consider that it can be interpreted as prescribing the equality between the marginal rate of substitution between period t and period $t + 1$ consumptions (i.e. $U'(c_t)/\beta U'(c_{t+1})$), and the marginal rate of transformation, $f'(k_{t+1}) + (1 - \delta)$.⁶

To improve your understanding of this point, pick a consumption level for period t , say \check{c}_t .⁷ Then, choose a consumption level for the subsequent period $t + 1$, say \check{c}_{t+1} , which does not satisfy the Euler equation: we require only that it is feasible, i.e. that it can be produced given \check{k}_{t+1} , which is the capital stock implied by the initial capital level k_t , and by \check{c}_t (i.e., $\check{k}_{t+1} = f(k_t) + (1 - \delta)k_t - \check{c}_t$). Notice that, for a given k_t , \check{c}_t and \check{c}_{t+1} involve a specific level for k_{t+2} , say \check{k}_{t+2} . In deciding whether the choice to consume \check{c}_t in period t and \check{c}_{t+1} in period $t + 1$ is appropriate, our consumer must consider what would happen to her overall utility if – for given \check{k}_{t+2} – she decided to increase the time t consumption by a small amount ξ . In this case, her time t utility would increase by (approximately) $U'(\check{c}_t)\xi$.⁸ Moreover, because her savings would decrease by ξ , her next period resources would decrease by $[f'(k_{t+1}) + (1 - \delta)]\xi$, that is, by ξ multiplied by the productivity

⁶An alternative interpretation is based on the fact that our representative consumer – forsaking one unit of consumption today – obtains $f'(k_{t+1}) + (1 - \delta)$ unit of consumption tomorrow. Accordingly, $f'(k_{t+1}) + (1 - \delta)$ can also be interpreted as the price of current consumption if the price of *future* consumption is conceived as the numeraire, and hence fixed to unity. According to this interpretation, Eq. (1.9) can be seen as prescribing the equalization of the marginal rate of substitution $U'(c_t)/\beta U'(c_{t+1})$ with the price ratio $[f'(k_{t+1}) + (1 - \delta)]/1$.

⁷In these notes, we denote by an inverted hat an arbitrary level for a variable, with a star an optimal level, and by a hat a steady state level for that variable. As we shall see, a steady state is a situation such that all the variables do not change over time.

⁸If you do not “see” this, consider that the difference (in terms of period t utility) of the two policies is $U(\check{c}_t + \xi) - U(\check{c}_t)$. Applying Taylor’s theorem to $U(\check{c}_t + \xi)$ one obtains: $U(\check{c}_t + \xi) \simeq U(\check{c}_t) + U'(\check{c}_t)\xi$.

of the “marginal” savings. The reduction in period $t + 1$ utility is given by $U'(\check{c}_{t+1})[f'(k_{t+1}) + (1 - \delta)]\xi$. From the perspective of time t , this variation in utility must be discounted; hence, its period t value is $\beta U'(\check{c}_{t+1})[f'(k_{t+1}) + (1 - \delta)]\xi$.

If $U'(\check{c}_t)\xi > \beta U'(\check{c}_{t+1})[f'(k_{t+1}) + (1 - \delta)]\xi$, it is convenient to increase period t consumption: the utility gain in that period is larger than the utility loss suffered at time $t + 1$ once the latter is discounted back to period t .

Likewise, if $U'(\check{c}_t)\xi < \beta U'(\check{c}_{t+1})[f'(k_{t+1}) + (1 - \delta)]\xi$, it is convenient to decrease period t consumption: the utility loss in that period is smaller than the discounted utility gain. From this reasoning, we can convince ourselves that Eq. (1.9) must hold true when the consumption sequence is optimally chosen. Notice moreover that the Euler equation is useful to relate the evolution of consumption over time to the existing capital stock. Assume that $c_{t+1} = c_t$, so that $\Delta c_t = 0$,⁹ and notice that $c_{t+1} = c_t$ implies $U'(c_t) = U'(c_{t+1})$. From Eq. (1.9), a constant consumption can be optimal if and only if $k_{t+1} = \hat{k}$, where \hat{k} is such that:

$$(1.10) \quad 1 = \beta[f'(\hat{k}) + (1 - \delta)].$$

In the steady state, i.e. when $k_t = \hat{k}$, and consumption is constant over time, the impatience parameter β exactly offsets the positive effects on savings exerted by the fact that they are rewarded by the marginal productivity of capital.¹⁰

Whenever $k_{t+1} < \hat{k}$, the marginal productivity of capital is higher than at \hat{k} (i.e. $f'(k_{t+1}) > f'(\hat{k})$);¹¹ hence $\beta[f'(k_{t+1}) + (1 - \delta)] > 1$. Therefore, the Euler equation is satisfied only for consumption levels such that $U'(c_t) > U'(c_{t+1})$. Hence, it must be true that $c_t < c_{t+1}$. In words, since the marginal productivity of capital is high, saving is very rewarding. Therefore, it is sensible to save a lot, by reducing consumption at the “early” date t . Because the “early” consumption is low, consumption increases over time.

By a similar argument, when $k_{t+1} > \hat{k}$, capital is “abundant” and its marginal productivity gets low. Therefore $\beta[f'(k_{t+1}) + (1 - \delta)] < 1$. Eq. (1.9) is satisfied if $U'(c_t) < U'(c_{t+1})$, which implies that $c_t > c_{t+1}$. Because the marginal productivity of capital is low, saving is ill-compensated. Therefore, it is sensible to choose a high consumption level at t , and decrease it over time.

⁹Following an often used convention, we denote, for any variable y_t , $\Delta y_t \equiv y_{t+1} - y_t$.

¹⁰Notice that the uniqueness of \hat{k} is granted by assumptions (a), (b), and (d).

¹¹This is granted by the assumptions of a positive but *decreasing* marginal productivity of capital, (a) and (b).

Finally, notice that Eq. (1.9)—relating the consumption level in period t to the one in period $t + 1$ —applies for $t = 0, 1, \dots, T - 1$ (consumption at time $T + 1$ does not make sense by our finite-live assumption). Hence, the Euler equation provides us with T relations that we exploit when we wish to solve analytically our maximization problem.

1.3. The solution. As already remarked, what we wish to determine are the $T + 1$ consumption values (i.e. c_0, c_1, \dots, c_T) and the $T + 1$ capital stocks (i.e. k_1, k_2, \dots, k_{T+1}). As already remarked, the Euler equation (1.9) provides us with T relations; the constraints like (1.3) are $T + 1$. Accordingly, to close the model we need a further equation.

This is obtained from the complementary slackness condition (1.8). This tells us that, if the consumer uses up her entire capital stock in the final period, so that $k_{T+1} = 0$, then μ is positive; alternatively, if $\mu = 0$, the final capital stock is positive. The last condition in (1.7) guarantees us that μ must actually be positive, since it is equal to λ_T , and λ_T is the marginal utility of consumption at time T , which can not be nought. Hence, the final period capital must be zero, which is very sensible in economic terms: because our agent will not consume anything in period $T + 1$, it is pointless for her to keep some capital: she can always improve her overall utility by eating up this stock of resources. In sum, we can be sure that $k_{T+1} = 0$, a condition “closing” the model in accordance with the complementary slackness condition (1.8).

The system composed by T equations like (1.9), of $T + 1$ equations like (1.3), and by $k_{T+1} = 0$ can – in principle – be solved for the $T + 1$ consumption levels and for the $T + 1$ capital levels. In practice, the solution of our problem is hardly achieved. In fact, in a few special cases only we can obtain a solution of all the c_t s and of all the k_t s in terms of the known elements (k_0 and the parameters). This motivates the use of numerical techniques, and, at least in part, the willingness to resort to the dynamic programming approach.

For future reference, bear in mind that the complementary slackness condition (1.8) can be reformulated, by means of the last conditions in (1.7) and (1.6), as:

$$(1.11) \quad \beta^T U'(c_T) k_{T+1} = 0.$$

2. THE INFINITE-HORIZON CONSUMPTION-GROWTH PROBLEM

Let us now imagine that our agent is living forever. At first sight, this might seem crazy. However, we may conceive our agent as a person who cares about her offsprings. In this case, she should consider that her sons

and daughters will be concerned about their offsprings' welfare and so on. Hence, she should optimize over the entire future. If you wish, you can think about our agent not as a person, but as a dynasty. An alternative interpretation of the model we are about to present is that the optimizing agent actually is a social planner who aims at maximizing a social welfare function whose arguments are the discounted utilities of all the agents who are alive now and in any possible future date. This alternative interpretation requires the determination of the optimal saving rate for the society, and in fact the British economist Frank P. Ramsey originally framed the question in these terms, and solved it on the ground of intertemporal optimization. The model presented here is essentially his.

Assuming that our agent optimizes from time 0 onwards, her preferences are now given by the following intertemporal utility function:

$$(2.1) \quad W_0 = \sum_{t=0}^{\infty} \beta^t U(c_t),$$

an expression which is analogous to (1.1), but for the fact that now the agent's horizon extends up to infinity.

Our problem is now to

$$\max_{\{c_t\}_{t=0}^{\infty}} W_0 = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t);$$

but notice that, while the constraints are given by equations such as (1.3), the fact that the consumer's planning horizon is infinite implies that we cannot impose a "terminal constraint" like (1.4).

The consumer's problem is tackled by means of the following "present value" Lagrangian:

$$(2.2) \quad \begin{aligned} L_0 = & U(c_0) + \beta U(c_1) + \dots + \beta^t U(c_t) + \dots \\ & - \lambda_0 [k_1 - f(k_0) - (1 - \delta)k_0 + c_0] \\ & - \dots \\ & - \beta^t \lambda_t [k_{t+1} - f(k_t) - (1 - \delta)k_t + c_t] \\ & - \dots \end{aligned}$$

Problem (2.2) differs from (1.5) because it involves an infinite number of discounted utility terms, and an infinite number of dynamic constraints; moreover – as already remarked – the constraint concerning the final level for the stock variable is missing.

We optimize (2.2) with respect to c_t , k_{t+1} , and λ_t for $t = 0, 1, \dots$ obtaining:

$$(2.3) \quad U'(c_t) = \lambda_t, \quad \forall t,$$

$$(2.4) \quad \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} [f'(k_{t+1}) + (1 - \delta)], \quad \forall t,$$

and, of course:

$$(2.5) \quad k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t, \quad \forall t.$$

These conditions are necessary, but they are not sufficient: the “final condition” is missing. In our infinite horizon model, the role of this final condition is played by the so-called transversality condition (henceforth *tvc*), which—in this set up—is:

$$(2.6) \quad \lim_{T \rightarrow \infty} \beta^T U'(c_T) k_{T+1} = 0.$$

Comparing the above expression with (1.11), we immediately notice that (2.6) is the limit, for $T \rightarrow \infty$, of (1.11). This suggests us that the *tvc* plays the role of the missing terminal condition. The *tvc* has an intuitive economic interpretation: it rules out policies implying a “too fast” capital accumulation in the long run. To understand this point, assume that our consumer follows a policy implying a growing capital stock. Assume, in particular, that – to allow for a fast capital accumulation – consumption is reduced over time. This policy would imply a high marginal utility of consumption, and an high and increasing value for the product $U'(c_T)k_{T+1}$. This, in itself, does not involve the violation of condition (2.6). In fact, condition (2.6) allows $U'(c_T)k_{T+1}$ to increase over time. Nevertheless, this growth must be slow enough to be compensated by the convergence to 0 of the term β^T . This is why we say that the transversality condition rules out policies implying a “too fast” long-run capital accumulation.

2.1. The qualitative dynamics. In our model, as it happens in many infinite horizon frameworks, it is useful to draw a “phase diagram”. This helps visualizing how consumption and capital evolve over time, which can be very useful whenever (which means, almost always) an analytic solution cannot be found.

To obtain the phase diagram, we first consider the stability loci for each of the two variables (i.e. we compute the set of points such that $\Delta c_t = 0$

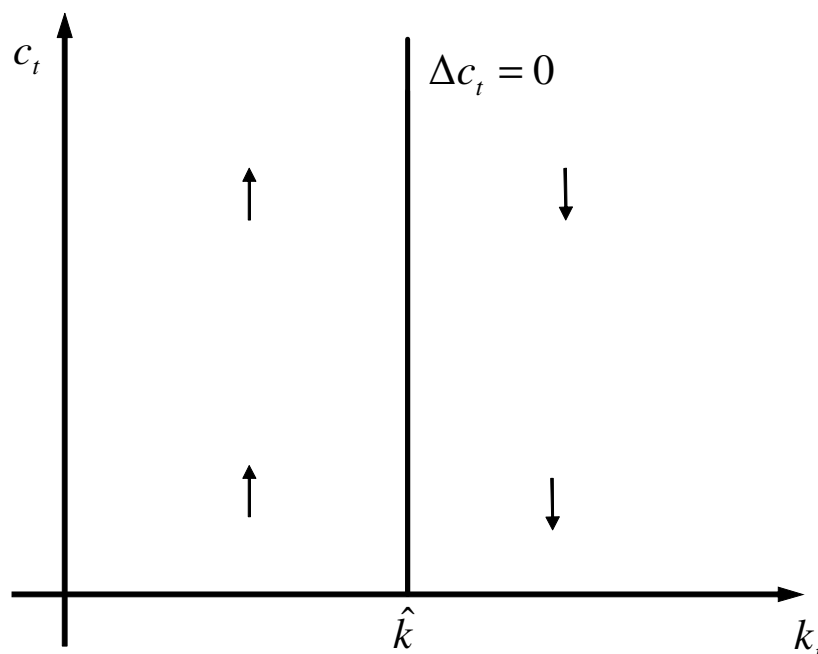


FIGURE 1.

and $\Delta k_t = 0$). This will help us to understand how consumption and capital change over time whenever they do not lie on their stability loci. Finally, we will jointly consider our knowledge for the dynamics of the two variables.

As for consumption, combining equation (2.4) with (2.3), we immediately see that optimality requires that the Euler equation (1.9) is satisfied. We already know that the Euler equation is useful to describe the evolution of consumption over time. In particular, we know that when $k_t = \hat{k}$, then $c_{t+1} = c_t$ and hence $\Delta c_t = 0$ (consider again how we obtained Eq. (1.10)). Hence, in Figure 1, we plot the locus implying stationarity for consumption as a the vertical line drawn at \hat{k} .

Whenever $k_t < \hat{k}$, it is optimal for our consumer to increase her consumption over time (hence, $c_t < c_{t+1}$). When instead $k_t > \hat{k}$, consumption must be shrinking over time ($c_{t+1} < c_t$). This behavior is summarized by the arrows in Figure 1.

From Eq. (2.5), we see that $\Delta k_t = f(k_t) - \delta k_t - c_t$, hence, capital is stationary when

$$(2.7) \quad c_t = f(k_t) - \delta k_t.$$

This relation can be portrayed as a function starting at the origin (by assumption (c)), with a maximum at \underline{k} (defined as the capital level such that

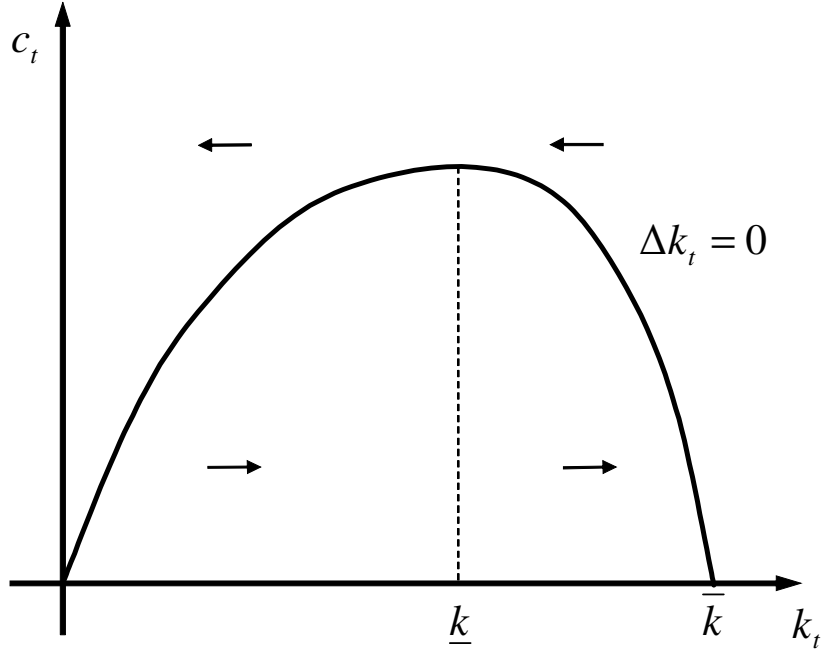


FIGURE 2.

$f'(k) = \delta$), and intersecting again the $c_t = 0$ axis at \bar{k} (which is the capital level such that $c_t = 0$, i.e. it is obtained solving the equation $f(\bar{k}) - \delta\bar{k} = 0$).¹² The behavior of the $\Delta k_t = 0$ locus is portrayed in Figure 2.

To see what happens when the economic system is not on the stability locus (2.7), pick a capital level $\check{k} \in [0, \bar{k}]$; the corresponding consumption level guaranteeing stationarity for capital obviously is

$$\check{c} = f(\check{k}) - \delta\check{k}.$$

If the consumer chooses a consumption level $c_t > \check{c}$, her capital stock must decrease over time: the consumption is so high that our consumer lives using up part of her capital. More precisely, consumption is higher than the level, \check{c} , guaranteeing that the difference between gross production, $f(\check{k})$, and consumption is exactly equal to capital depreciation $\delta\check{k}$. Therefore the capital stock must decrease. The converse happens when our consumer chooses a consumption level c_t that is lower than \check{c} : her capital increases over time because a consumption lower than \check{c} implies that savings ($f(\check{k}) - \check{c}$) exceeds capital depreciation, and hence there is some net investment. This behavior is summarized by the arrows in Figure 2.

Merging Figures 1 and 2, we obtain Figure 3, which summarizes the dynamics of the model.

¹²Existence and uniqueness of \bar{k} are granted by assumptions (b) and (e).

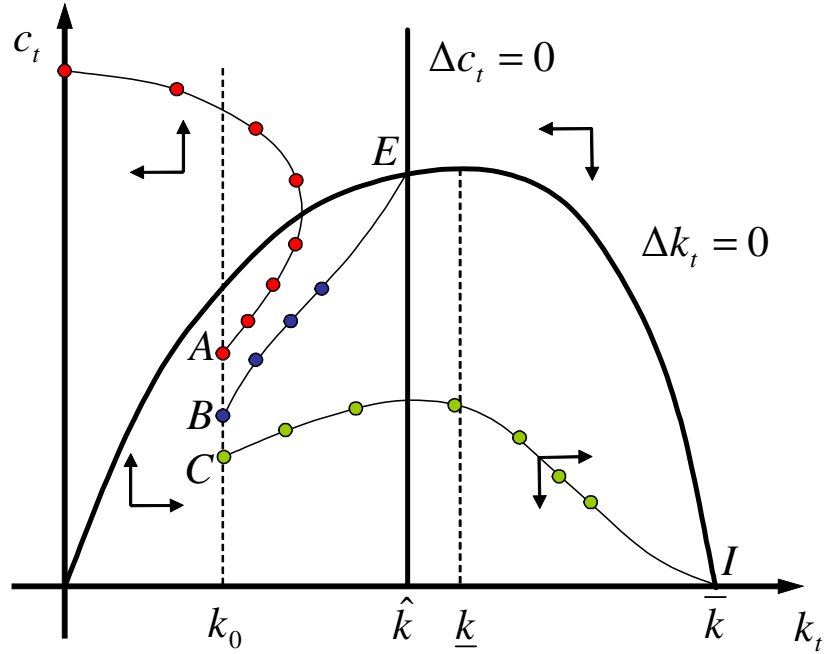


FIGURE 3.

Notice that there are three steady states (E , the origin, and I). It is easy to see that E is a steady state: it is the crossing point for the two loci $\Delta k_t = 0$ and $\Delta c_t = 0$. From Eqs. (1.3) and (1.9) it is clear that at E both consumption and capital do not change over time when

$$(2.8) \quad \begin{cases} 1 = \beta[f'(\hat{k}) + (1 - \delta)] \\ \hat{c} = f(\hat{k}) - \delta\hat{k} \end{cases}.$$

Hence, the consumption and capital level characterizing this steady state can be obtained solving the system above.

The origin is a resting point because of Assumption (c): if capital is 0, there is no production and hence no possibility of further capital accumulation. This resting point is usually considered uninteresting.

It is less obvious that also I is a steady state. To gain some intuition about the reason why I is a steady state, consider that the marginal utility of consumption increases very rapidly as consumption approaches 0 (this is because we assumed that $\lim_{c_t \rightarrow 0} U'(c_t) = \infty$). Hence, the increase in the marginal utility of consumption prescribed by Eq. (1.9) for $k_t > \hat{k}$ implies smaller and smaller reductions in consumption as c_t approaches 0. Hence, consumption does not become negative (which would of course have no economic meaning) and I is a resting point.

A less heuristic argument is presented in the next three paragraphs, that can be skipped by the uninterested reader.

To see that I is a steady state, rewrite (1.9) as:

$$U'(c_{t+1}) = U'(c_t) \left[\frac{1}{\beta[f'(k_{t+1}) + (1 - \delta)]} \right].$$

Pick a capital level, \check{k} , belonging to the interval (\hat{k}, ∞) : it is easy to see that $f'(\check{k}) + (1 - \delta) \in (1 - \delta, 1/\beta)$: this comes from (1.10) and from the assumptions: $f''(k_t) < 0$, and $\lim_{k_t \rightarrow \infty} f'(k_t) = 0$. Therefore, when $\check{k} \in (\hat{k}, \infty)$, the term in the big square brackets in the equation above must be larger than one, since the largest value for the denominator is “slightly smaller” than one. Therefore, the Euler equation not only tells us that the marginal utility of consumption must increase (and hence that consumption must decrease), but also that the rate of change of marginal utility is limited. In fact, we have

$$\frac{U'(c_{t+1}) - U'(c_t)}{U'(c_t)} = \frac{1}{\beta[f'(\check{k}) + (1 - \delta)]} - 1,$$

and the value of the right hand side of the expression above belongs to the interval $\left(0, \frac{1}{\beta(1-\delta)} - 1\right)$. This has a relevant implication: because the rate of change of the marginal utility is bounded, when we consider a sequence of consumption levels that – starting from a non negative value – fulfills the Euler equation, we see that this sequence cannot go to zero in finite time. This is because the marginal utility of consumption cannot “reach infinity” in finite time (bear in mind that $\lim_{c_t \rightarrow 0} U'(c_t) = \infty$). Because consumption takes an “infinite time” to reach its limiting value (that is, 0), “in the meantime” capital must reach \bar{k} (since consumption decreases, the system must reach at some time the area below the $\Delta k_t = 0$ locus where capital grows, approaching \bar{k}). Hence I is a stationary state for our system.

Let us now consider that our consumer is constrained by the fact that her initial stock of capital is given (at the level k_0). Hence, in choosing her optimal consumption path, she must take into account this constraint. In Figure 3, we have depicted some of the possible paths that our agent may decide to follow. These paths are intended to fulfill the Euler equation (1.9) and the capital accumulation constraint (1.3).¹³

The fact that there are many (actually, infinitely many) trajectories that are compatible with one initial condition cannot be surprising: while the

¹³While it is not possible to check that our paths in Figure 3 conform to what is prescribed by our difference equation, notice that they have been drawn respecting the “arrows”, which, in turn, summarize the “direction of movement” obtained from the difference equations (1.9) and (1.3).

capital stock k_0 is given, our consumer is free to pick her initial consumption level, which then determines the path for consumption and capital (via equations (1.9) and (1.3)).

2.2. The optimal path. So far, we have seen that there are multiple paths compatible with the same initial condition. What we need to do now, is to select the optimal one(s).

First, notice that the trajectory starting at A in Figure 3 cannot be optimal: because consumption is ever-increasing, capital must go to zero in finite time. Since by assumption capital is essential in production, at that time consumption must collapse, becoming nought. This big jump in consumption violates the Euler equation (which, to be fulfilled, would require a further increase in consumption). Hence, the path starting at A – ad all the paths akin to this one – cannot be optimal.

Second, consider the trajectory starting at B . In this case, our consumer chooses *exactly* the consumption level that leads the system to the stationary point E . This path not only fulfills the difference equations (1.9) and (1.3) but also the transversality condition (2.6). In fact, as time goes to infinity (i.e. as $t \rightarrow \infty$), capital and consumption approach their steady state levels, \hat{k} and \hat{c} , which are given by System (2.8). The fact that the long run levels for consumption and capital are positive and constant, tells us that, in the steady state, the marginal utility of consumption is finite. Hence, $\lim_{t \rightarrow \infty} \beta^t U'(c_t) k_{t+1} = \lim_{t \rightarrow \infty} \beta^t U'(\hat{c}) \hat{k} = 0$ simply because $\beta < 1$, and this second trajectory is optimal.

Third, consider the trajectory starting at C and leading to I . In this case, consumption and capital first increase together, but then consumption (as capital becomes larger than \hat{k}) starts to shrink while the capital accumulation process is still on. In the long run, our optimizing agent finds herself around I , devoting all the productive effort to maintain an excessive stock of capital, which is actually never used to produce consumption goods. Hence, this trajectory cannot be optimal, and the transversality condition is violated, because the marginal utility of consumption tends to be infinite.

In the next two paragraphs, that can be skipped, we give a less heuristic idea of the reasons why a path like the one starting at C violates the tvc.

To check whether a path of this type fulfills the tvc, imagine to be “very close” to I . Here, capital is (almost) \bar{k} . Recall that for $k_t \in (\underline{k}, \bar{k})$, $f'(k_t) + (1 - \delta) < 1$ (at \bar{k} , $f'(\bar{k}) = \delta$, and $f''(k_t) < 0$), and consider that consumption should evolve according to

$$U'(c_{t+1}) = U'(c_t) \frac{1}{\beta[f'(k_t) + (1 - \delta)]}.$$

Hence, for a capital stock close to \bar{k} , we have that

$$\beta U'(c_{t+1}) = U'(c_t) \frac{1}{f'(k_t) + (1 - \delta)} > U'(c_t).$$

Because “around” \bar{k} , $\beta U'(c_{t+1}) > U'(c_t)$, $\lim_{t \rightarrow \infty} \beta^t U'(c_t) > 0$ (In fact, suppose that at a given time T our system is already “very close” to I . In this case, in the following periods, i.e. for $t > T$, $\beta^{t-T} U'(c_t) > U'(c_T)$. Hence $\lim_{t \rightarrow \infty} \beta^t U'(c_t) = \lim_{t \rightarrow \infty} \beta^T \beta^{t-T} U'(c_t) > \beta^T U'(c_T)$) Hence, the tvc (2.6) is not fulfilled and any trajectory like the one starting at C is not optimal.

Summing up, the *unique* optimal path is the one leading to the steady state E ; this path prescribes a monotonic increasing relation between consumption and capital. Our consumer (or our economy) “jumps” on this path by adjusting the initial consumption to the level compatible with the existing capital stock, and with the behavior prescribed by our optimal trajectory.

3. THE DYNAMIC PROGRAMMING FORMULATION

In this Section, we solve the infinite-horizon growth model exploiting the dynamic programming approach, taking advantage of our previous understanding of the solution to provide an intuitive introduction for this new technique.

Our representative agent aims at maximizing, as before, the intertemporal utility function (2.1), so that her problem is to

$$\max_{\{c_t\}_{t=0}^{\infty}} W_0 = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t).$$

under the constraints given by $k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t$, for $t = 0, 1, \dots$

Obviously, our optimizing agent’s preferences can be written as:

$$W_0 = U(c_0) + \sum_{t=1}^{\infty} \beta^t U(c_t),$$

in which we have “separated” the utility obtained in the present, period 0, from the ones that will be enjoyed in the future. However, there is no change in the meaning for W_0 .

Notice that the expression above can be reformulated as:

$$(3.1) \quad W_0 = U(c_0) + \beta \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right].$$

Here, we have collected the factor β that is common to all the addenda expressing future utilities. The reason for this manipulation is that the term

in the big square brackets represents the consumer's preferences from the perspective of time 1.

Equation (3.1) allows us to write our problem as:

$$(3.2) \quad \max_{\{c_t\}_{t=0}^{\infty}} W_0 = \max_{\{c_t\}_{t=0}^{\infty}} \left\{ U(c_0) + \beta \left[\sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \right] \right\}.$$

under the constraints $k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t$, for $t = 0, 1, \dots$. In other words, we wish to determine the optimal consumption c_t^* , and the implied capital level k_{t+1}^* , for $t = 0, 1, \dots$

Now consider, in Figure 4, the optimal path starting from B and approaching E . This trajectory represents a function, say $\varphi(\cdot)$, relating the optimal consumption to the same period capital stock. We mean that c_1^* can be expressed as $c_1^* = \varphi(k_1)$; that c_2^* can be viewed as $c_2^* = \varphi(k_2)$, and so on. The function $c_t^* = \varphi(k_t)$ is unknown, and it can be very complex; actually, what usually happens is that the function $\varphi(\cdot)$ cannot be expressed in a closed-form analytical way. Nevertheless, the point that we underscore here is that Figure 4 powerfully supports the idea that we have just stated, i.e. that we can consider the optimal consumption as a function of contemporaneous capital.

It is worth underscoring that our function $c_t^* = \varphi(k_t)$ is “stationary”: it is always the same function, irrespective of the time period we are considering. Hence, the time dimension of the problem disappears. The intuition to understand why $\varphi(\cdot)$ is independent of time is to consider our consumer's perception of the future: because she lives forever, at time 0 her horizon is infinite, and so it is at time 1. Hence, she must not ground her decision on time, but just on capital, which therefore is the unique state variable in our model.

Notice that we have not proved that $\varphi(\cdot)$ is continuous and differentiable. However, the evolution of capital and consumption on the optimal path must fulfill equations (1.9) and (1.3), which are made up of continuous and differentiable functions. Hence, we have “good reasons to believe” that $\varphi(\cdot)$ actually is continuous and differentiable, and we skip the formal proof for these statements.

Now consider that if we can write $c_1^* = \varphi(k_1)$, then we are also able to express the capital stock at time 2 as a function of k_1 : in fact, from (1.3),

$$k_2^* = f(k_1) + (1 - \delta)k_1 - c_1^*,$$

hence, on the optimal path:

$$k_2^* = f(k_1) + (1 - \delta)k_1 - \varphi(k_1) = \zeta(k_1).$$

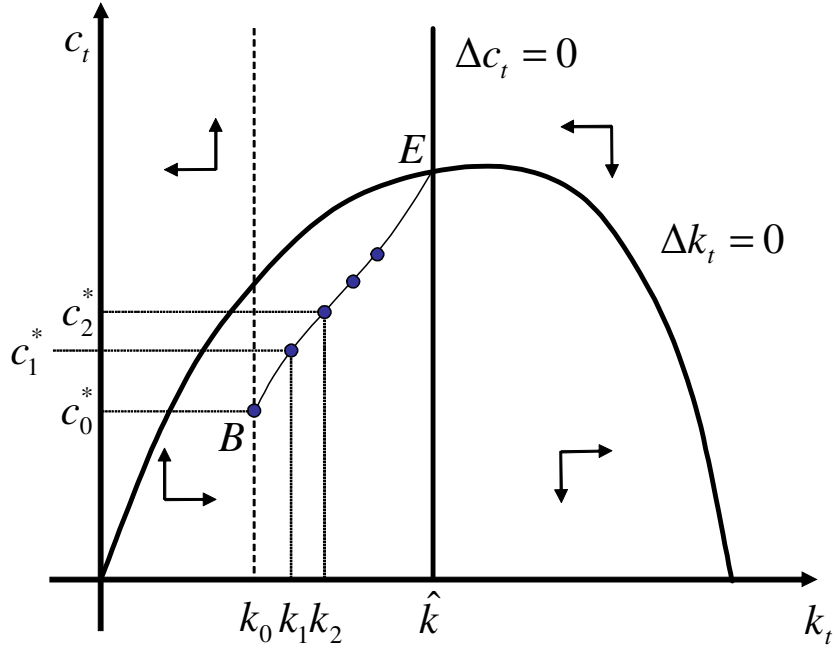


FIGURE 4.

The fact that $k_2^* = \zeta(k_1)$, allows us to consider the time 2 consumption as a function of k_1 : $c_2^* = \varphi(k_2^*) = \varphi(\zeta(k_1))$.

The important point here is to realize that *we can iterate this reasoning to express the whole sequence of optimal consumptions as a function of the time 1 capital stock*. (In fact, in general, $k_{t+1}^* = \zeta(k_t^*)$, and $c_{t+1}^* = \varphi(k_{t+1}^*) = \varphi(\zeta(k_t^*))$, and we can iterate the substitutions until we reach k_1).

Hence, when consumption is optimally chosen from period 1 onward, the group of addenda in the big square brackets in (3.2) can be expressed as a function of k_1 alone:

$$\sum_{t=1}^{\infty} \beta^{t-1} U(c_t^*) = V(k_1).$$

$V(k_1)$ is a “maximum value function”: it represents the maximum overall utility that can be obtained in period 1, when all the consumption levels are optimally chosen given the available capital stock, and given the need to fulfil the constraints of the (1.3)-type.

Because the value of all the future choices can be summarized in the function $V(k_1)$, we can think about the consumer’s intertemporal maximization problem in a way that is different from the initial one. We can imagine that, at time 0, she picks her optimal period 0 consumption, taking account of

the fact that, given the available capital, an increase in current consumption reduces the future capital stock (via Eq. (1.3)), and therefore negatively affects the future overall utility $V(k_1)$.

Accordingly, the consumer's intertemporal problem can be written as:

$$\begin{aligned} & \max_{c_0} \{U(c_0) + \beta V(k_1)\}, \\ & \text{s.t. } k_1 = f(k_0) + (1 - \delta)k_0 - c_0, \\ & \quad k_0 \text{ given.} \end{aligned}$$

Now, imagine that our consumer solves her period 0 constrained optimization problem, which means that she determines c_0^* as a function of k_0 . Our representative consumer, obtaining c_0^* , determines also her time 0 maximum value function, $V(k_0)$, which means that

$$\begin{aligned} & V(k_0) = \max_{c_0} \{U(c_0) + \beta V(k_1)\}, \\ (3.3) \quad & \text{s.t. } k_1 = f(k_0) + (1 - \delta)k_0 - c_0, \\ & \quad k_0 \text{ given.} \end{aligned}$$

The above problem is said to be expressed as a “in recursive form” or as a “Bellman equation”.

While it is easy to understand that Problem (3.3) gets its name from Richard Bellman, the Author of the well know [1957] book, it is not so simple to explain in plain English the meaning of the word “recursive”.

Let us try. A procedure is recursive when one of the steps that makes up the procedure requires a new running of the procedure itself. Hence, a recursive procedure involves some degree of “circularity”. As a simple example, consider the following recursive definition for the factorial of an integer number:

$$\begin{aligned} & \ll \text{if } n > 1, \text{ then the factorial for } n \text{ is } n! = n(n-1)!; \\ & \quad \text{when } n = 1, \text{ then } 1! = 1 \gg. \end{aligned}$$

Clearly, the above procedure defines $n!$ by means of $(n-1)!$, that is defined exploiting $(n-2)!$ and so on. This procedure goes on until 1 is reached, at this point the “termination clause”, $1! = 1$, enters into stage.

The logic of problem (3.3) is quite similar: the maximum value $V(k_0)$ is obtained choosing the current consumption to maximize the sum of the current utility and of the next-period discounted maximum value, which, in turn is obtained by choosing future consumption in order to maximize

the sum of the future period utility and of the two periods ahead maximum value ...

The difference between (3.3) and the factorial number example lies in the fact that our Bellman equation does not have a termination clause. This is due to the fact that the planning horizon for our representative agent is infinite. If she were to optimize only between period 0 and a given period T – for example, because she is doomed to die at T – it would have been natural to introduce a terminal condition: in this case, the maximum value at time $T+1$ of any stock of capital k_{T+1} would have been equal to nought.¹⁴

Notice that the example we have just sketched implies that the maximum value function depends not only on capital, but also on time. In fact, when the agent's time horizon is finite, the maximum value function typically depends on the remaining optimization horizon: how long you are going to live typically matters a lot for you, and it also affects how you evaluate your stock of wealth. Accordingly, at time t the maximum value function is characterized by some terms involving $T - t$.

In contrast – as already underscored – in Problem (3.3) the maximum value function depends only on capital. This sometimes strikes sensitive students: in fact, considering $V(k_t)$ as a *function of capital alone*, we imply that $V(k_t)$ does not change over time despite the fact that – moving backward from time 1 to time 0 – we discount the previous maximum value function, and then we add to it the term $U(c_0^*)$. In other words, in problem (3.3), we have the very same function $V(k_t)$ both on the left and on the right hand side.

The intuition to understand why $V(k_t)$ is independent of time is to consider again that *our consumer's perception of the future is the same at time 0 and at time 1*. This occurs simply because she lives forever. Hence, she bases her consumption decision only on capital, which therefore is the unique state variable in our model. Because the function $V(k_t)$ summarizes the optimal consumption decisions from period t onward, this function depends only on capital.

The independence on time of the maximum value function in infinite horizon frameworks is one of the reasons why these frameworks are so popular: their maximum value function – having a unique state variable – is less *complex* to compute. Notice however that the absence of a termination clause makes the problem of finding a solution conceptually more *difficult*. In fact, when we have a terminal time, we know the maximum value function for that period: this is just the final period utility function. Hence, we can

¹⁴Accordingly, the period T maximum value would have simply been $V(k_T) = U(c_T)$ with $c_T = f(k_T) + (1 - \delta)k_T$.

always solve the problem for the terminal time, and then work “backward” toward the present. This simple approach is precluded in infinite horizon models. Solving a finite horizon problem is similar to decide how to send to the surface of the Moon a scientific pod, while the solution of an infinite horizon model is analogous to finding an optimal lunar orbit for the pod. In the first case, you can analyze all the possible locations to pick the most convenient one, say the Tranquility sea. Then you realize that to send your pod to the Tranquility sea, you need a lunar module; then—working backward—you determine that a spaceship is needed to get close to the moon and finally you understand that it takes a Saturn-*V* missile to move the spaceship out of the Earth atmosphere. Notice that, in this case, you have a clear hint on how to start to work out your sequence of optimal decisions, and your sequence is composed of a finite number of steps. On the contrary, if you need to have the scientific pod orbiting around the Moon, your time horizon is (potentially) infinite, and you need to devise an infinite sequence of decisions. Moreover, because your pod is continuously orbiting, you do not have a clearly specified terminal condition from which to move backward.

The Bellman Equation (3.3) is useful also to introduce some the dynamic programming jargon. The single period payoff (utility) function is often called the *return function*, the dynamic constraint is referred to as the *transition function*, while a function relating in an optimal way the control variable(s) to the state variable(s) is called the *policy function*. In our example, the policy function is $c_t^* = \varphi(k_t)$, the transition function is $k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t$, while the return function is, of course, $U(c_t)$.

Before studying how an infinite horizon problem can be solved, we need to understand under which conditions the solution for problem (3.3) exists and is unique.

Stokey, Lucas, and Prescott [1989] assure us that

Theorem 1. *If: i) $\beta \in (0, 1)$, ii) the return function is continuous, bounded and strictly concave, and iii) the transition function is concave, then the maximum value function $V(k_t)$ not only exists and is unique, but it is also strictly concave, and the policy function is continuous.*

Assumption i) is usually referred to as the “discounting” hypothesis, Assumption ii) requires that the utility function $U(c_t)$ is continuous, bounded and strictly concave, while Assumption iii) constrains the production function to be concave, so that the transition function $k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t$ is concave for any given c_t .

The above result is neat, but it suffers from a relevant drawback: it does not allow us to work with unbounded utility functions, and hence we cannot

assume, for example, that $U(c_t) = \ln(c_t)$. In general, it does not seem easy to justify an assumption that precludes utility to grow without bounds.¹⁵

Stokey, Lucas, and Prescott discuss the case of unbounded returns in some details (see their Theorem 4.14); here we follow Thompson [2004] in stating a more restrictive theorem that will however suffice for many applications.

Theorem 2. *(A theorem for unbounded returns). Consider the dynamic problem (3.3). Assume that $\beta \in (0, 1)$, and that the term $\sum_{t=0}^{\infty} \beta^t U(c_t)$ exists and is finite for any feasible path $\{k_t\}_{t=0}^{\infty}$, given k_0 . Then there is a unique solution to the dynamic optimization problem.*

Theorem 2 essentially restricts the admissible one-period payoffs to sequences that cannot grow too rapidly relative to the discount factor.¹⁶ To see how this theorem can be applied, consider for instance Figure 3. Clearly, when $k_0 < \bar{k}$, capital cannot become larger than \bar{k} . Hence, given $k_0 \in [0, \bar{k}]$, $U(c_t)$ is finite, and so is $\sum_{t=0}^{\infty} \beta^t U(c_t)$, for any feasible path $\{k_t\}_{t=0}^{\infty}$.¹⁷

Before considering how to solve a Bellman problem, here you find a *caveat lector*: bear in mind that we have been able to easily reformulate our intertemporal maximization problem in the recursive form (3.3) because the payoff function and the intertemporal constraint are time-separable.¹⁸

Let us now eventually tackle the problem of finding the solution for the Bellman equation. This is a functional equation, because we must determine the form of the unknown function $V(\cdot)$. Notice that from the way we have formulated Problem (3.3) it is now “natural” to look for the function $V(\cdot)$, while our discussion in the previous Sections may have induced to think that what is really important is the policy function. It will soon become clear that once we have obtained the maximum value function $V(\cdot)$ it is easy to obtain the policy function.

¹⁵Unfortunately, the boundedness of the return function is an essential component of the proof for the results stated in Theorem 1. In fact, to prove existence and uniqueness for $V(k_t)$, one needs to use an appropriate fixed point theorem, because the function $V(k_t)$ is the fixed point of Problem (3.3), and the proofs of fixed point theorems require the boundedness of the functions involved.

¹⁶The idea behind the proof, that we skip, is that *if* the term $\sum_{t=0}^{\infty} \beta^t U(c_t)$ exists and is finite, then the maximum value function is finite, and we can apply an appropriate fixed-point theorem. Thompson (2004) is a very good introduction to the existence issues of the maximum value function. He also provides several result useful to characterize the maximum value function. The key reference in the literature is Stokey, Lucas, and Prescott [1989], which is however much more difficult.

¹⁷The conditions stated in Theorem 1 or 2 imply the fulfilment of Blackwell’s sufficient conditions for a contraction.

¹⁸Moreover, if our agent’s utility depended not only upon current but also upon future (expected) consumption levels, we would need to tackle a time inconsistency problem. For a simple introduction to this specific issue, see de la Fluenta [2000, ch. 12].

Problem (3.3) tells us that, to obtain $V(k_0)$, it is necessary to maximize, with respect to c_0 , the expression $U(c_0) + \beta V(k_1)$. Provided that $V(k_1)$ is continuous and differentiable – a point that we are ready to accept – the necessary condition for a maximum is obtained by differentiating $U(c_0) + \beta V(k_1)$ with respect to current consumption, which yields:

$$(3.4) \quad U'(c_0^*) + \beta V'(k_1) \frac{\partial k_1}{\partial c_0} = 0,$$

where, of course, $\partial k_1 / \partial c_0 = -1$, from the capital accumulation equation.

The first order condition above relates the control variable (current consumption) with the current and future values of the state variable (k_0 and k_1). We can exploit the first order condition (3.4), the Bellman equation, and the constraint (1.3) to form the system:

$$(3.5) \quad \begin{cases} V(k_0) = U(c_0^*) + \beta V(k_1), \\ k_1 = f(k_0) + (1 - \delta)k_0 - c_0^*, \\ U'(c_0^*) = \beta V'(k_1), \\ k_0 \text{ given.} \end{cases}$$

The above system determines the form of the maximum value function $V(\cdot)$, and pins down k_1 , and c_0^* , for a given k_0 . Notice that the “max” operator in the Bellman equation has disappeared, simply because we have already performed this operation, via Eq. (3.4). Now consider what we often do when we deal with a functional equation: when we need to solve a difference or a differential equation, we (try to) guess the solution. Here, we can proceed in the same way. The next Section provides examples for which this strategy is successful.

4. GUESS AND VERIFY

4.1. Logarithmic preferences and Cobb-Douglas production. In this Sub-section, we analyze a simplified version of the Brock and Mirman (1972) optimal growth model. We shall propose a tentative solution, and then we will verify that the guess provides the correct solution.

In this framework the consumer’s time horizon is infinite, and her preferences are logarithmic:

$$(4.1) \quad W_0 = \sum_{t=0}^{\infty} \beta^t \ln(c_t),$$

with $\beta \in (0, 1)$. Using the jargon, we say that we are analyzing the case of a logarithmic return function. The production function is a Cobb-Douglas

characterized by a depreciation parameter as high as unity (i.e. $\delta = 1$, hence capital entirely fades away in one period). Accordingly, we have that

$$(4.2) \quad k_{t+1} = Ak_t^\alpha - c_t,$$

where the “total factor productivity” parameter A is constant and $\alpha \in (0, 1)$.

Our problem is to solve:

$$\begin{aligned} V(k_0) &= \max_{c_0} \{ \ln(c_0) + \beta V(k_1) \}, \\ \text{s.t. } k_1 &= Ak_0^\alpha - c_0, \\ k_0 &\text{ given.} \end{aligned}$$

The first order condition, corresponding to equation (3.4), is: $1/c_0^* = \beta V'(k_1)$. This condition must be exploited in the problem above to substitute out consumption. This yields:

$$(4.3) \quad \begin{cases} V(k_0) = \ln([\beta V'(k_1)]^{-1}) + \beta V(k_1) \\ k_1 = Ak_0^\alpha - \frac{1}{\beta V'(k_1)} \\ k_0 \text{ given.} \end{cases}$$

This formulation makes it apparent once again that the Bellman equation is a functional equation: it involves the function $V(\cdot)$ and its derivative $V'(\cdot)$; the constraint incorporates the initial condition for the state variable. Notice that, because k_0 is given, once $V'(\cdot)$ is known the constraint determines k_1 , and therefore the evolution for the capital stock. Notice also that system (4.3) corresponds to (3.5), but for the fact that here we have directly substituted out c_0^* thanks to the explicit formulation $c_0^* = [\beta V'(k_1)]^{-1}$.

Our weapon to seize the solution of the functional equation problem, the tentative solution, in this case takes the form:

$$(4.4) \quad V(k_t) = e + f \ln(k_t),$$

where e and f are two constants to be determined, i.e. two undetermined coefficients.¹⁹ Spend a few seconds in considering the guess (4.4). It is a linear transformation of the utility function, where the control variable has been substituted by the state variable. When trying to devise a sensible tentative solution, it is usually sensible to proceed as we did, i.e. to start with a guess that is “similar” to the return function.

¹⁹Notice that we are bold: we think we know the correct functional form for the maximum value function. However, we are not arrogant: we do not claim that we can imagine the maximum value function complete with all its details.

Taking the guess seriously, and exploiting it in (4.3), we obtain, for a given k_0 :

$$(4.5) \quad \begin{cases} e + f \ln(k_0) = \ln\left(\frac{k_1}{\beta f}\right) + \beta[e + f \ln(k_1)] \\ k_1 = Ak_0^\alpha - \frac{k_1}{\beta f} \end{cases} .$$

One immediately notices that the second equation may be solved for k_1 :

$$(4.6) \quad k_1 = \frac{\beta f}{1 + \beta f} Ak_0^\alpha,$$

so that

$$c_1 = \frac{1}{1 + \beta f} Ak_0^\alpha.$$

Substituting the above result in the first equation in (4.5) gives:

$$e + f \ln(k_0) = \ln\left(\frac{1}{1 + \beta f} Ak_0^\alpha\right) + \beta \left[e + f \ln\left(\frac{\beta f}{1 + \beta f} Ak_0^\alpha\right) \right].$$

Exploiting the usual properties enjoyed by logarithmic functions, we obtain:

$$\begin{aligned} e + f \ln(k_0) &= -\ln(1 + \beta f) + \\ &+ \ln A + \alpha \ln k_0 + \beta e + \beta f \ln(\beta f) - \beta f \ln(1 + \beta f) + \beta f \ln A + \alpha \beta f \ln k_0. \end{aligned}$$

The above equation must be satisfied for *any* k_0 and for *any* admissible value of the parameters A, β , and α . Hence, it must be true that:

$$\begin{cases} f = \alpha + \alpha \beta f \\ e = -\ln(1 + \beta f) + \ln A + \beta e + \beta f \ln(\beta f) - \beta f \ln(1 + \beta f) + \beta f \ln A \end{cases} .$$

From the first equation in the system above we obtain:

$$f = \frac{\alpha}{1 - \alpha \beta}.$$

Notice that $f > 0$, because $\alpha, \beta \in (0, 1)$. This implies $V'(k) > 0$, a sensible result which tells us that a richer consumer enjoys a higher overall utility. Substituting f in the second equation gives:

$$(4.7) \quad e = \frac{1}{1 - \beta} \left[\frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta) + \ln(1 - \alpha \beta) + \frac{1}{1 - \alpha \beta} \ln A \right].$$

The equation above provides us with some intuition concerning the reason why we need the assumption $\beta < 1$: were this requirement not fulfilled (i.e. were $\beta = 1$), the maximum value function would “explode” to infinity.

Because f and e are independent from capital, our guess (4.4) is verified.

Substituting f into (4.6), we obtain: $k_1 = \alpha\beta Ak_0^\alpha$, which is the specific form of the function $k_{t+1} = \zeta(k_t)$ introduced in Section 3.

From the first order condition, we know that $c_0^* = 1/[\beta V'(k_1)]$, hence, using our guess, the computed value for f , and the fact that $k_1 = \alpha\beta Ak_0^\alpha$, we obtain: $c_0^* = (1 - \alpha\beta)Ak_0^\alpha$, which is the function $c_t = \varphi(k_t)$ which applies for this example.

The consumption function $c_0^* = (1 - \alpha\beta)Ak_0^\alpha$ is a neat but somewhat economically uninteresting result: it prescribes that the representative agent must consume, in each period, a constant share $(1 - \alpha\beta)$ of her current income, Ak_0^α . Notice however that this share decreases with both β and α , which makes economic sense: an increase in β implies that the future is more important, while an increase in α , involving a higher marginal productivity for capital, increases the incentive to save.

Notice that our consumption function relates the control variable to the state variable: hence, it is a policy function; finally notice also that, from $k_1 = \alpha\beta Ak_0^\alpha$, we can easily obtain the steady state level for capital, that is: $\hat{k} = (\alpha\beta A)^{\frac{1}{1-\alpha}}$.

Having already obtained the maximum value function, it is certainly funny to ask ourselves to prove that it exists and is unique. Nevertheless, this is exactly what we are going to do in the next five paragraphs, in which we show that Theorem 2 can be applied to the utility and production functions under consideration. This effort is motivated by the fact that we wish to develop a line of reasoning that may be helpful also when it is not possible to find a closed-form solution for the maximum value function. The uninterested reader may skip these paragraphs. In this case, however, Exercise 4 will prove rather difficult.

Recall that $U(c_t) = \ln c_t$, and that $k_{t+1} = Ak_t^\alpha - c_t$. Notice that the path for capital characterized by the fastest possible growth for capital itself is obtained by choosing zero consumption in each period. This path is given by $\check{k}_1 = Ak_0^\alpha$, $\check{k}_2 = A\check{k}_1^\alpha$, ... Hence, in general, we have that

$$\begin{aligned} \ln \check{k}_t &= \ln A + \alpha \ln \check{k}_{t-1} = \ln A + \alpha \ln A + \alpha^2 \ln \check{k}_{t-2} = \\ &= \dots = \frac{1 - \alpha^t}{1 - \alpha} \ln A + \alpha^t \ln k_0. \end{aligned}$$

Notice also that – in period t – the largest one-period utility is obtained by consuming the entire output that the representative agent can produce at that time, i.e.

$$U(\check{c}_t) = \ln(A\check{k}_t^\alpha) = \ln A + \alpha \ln \check{k}_t.$$

Therefore, if we follow the policy prescribing to save everything up to period $t - 1$ and then to consume the entire output, we obtain:

$$U(\check{c}_t) = \ln A + \alpha \ln \check{k}_t = \frac{1 - \alpha^{t+1}}{1 - \alpha} \ln A + \alpha^{t+1} \ln k_0.$$

Imagine, counterfactually, that the above policy could be followed in *every* period (may be thanks to the intervention of a powerful wizard). In this case, the lifetime utility for the representative agent would be:

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \ln \check{c}_t &= \sum_{t=0}^{\infty} \beta^t \left(\frac{1 - \alpha^{t+1}}{1 - \alpha} \ln A + \alpha^{t+1} \ln k_0 \right) = \\ &= \frac{1}{(1 - \beta)(1 - \beta\alpha)} \ln A + \frac{\alpha}{1 - \beta\alpha} \ln k_0. \end{aligned}$$

It is obvious that the above expression is finite. Clearly, any *feasible* path would yield a lower lifetime utility, therefore, any feasible sequence of payoffs must be bounded (in present value). This implies that the maximum value function must also be bounded. Because $\sum_{t=0}^{\infty} \beta^t U(c_t)$ exists and is finite for any feasible path, Theorem 2 applies and the maximum value function is unique.

Exercise 1. Assume that i) the single period utility function is: $\ln(c_t) + \gamma \ln(1 - l_t)$, where $l_t \in [0, 1]$ is the share of time devoted to labour, and $\gamma > 0$; ii) the dynamic constraint is $k_{t+1} = Ak_t^\alpha l_t^{1-\alpha} - c_t$, where $A > 0$ and $0 < \alpha < 1$. Find the value function, and the related policy functions.

(Hint: because we have two control variables, the first order conditions)

Exercise 2. (Habit persistence) Assume that i) the single period utility function is: $\ln(c_t) + \gamma \ln(c_{t-1})$, where $\gamma < 0$; ii) the dynamic constraint is $k_{t+1} = Ak_t^\alpha - c_t$, where $A > 0$ and $\alpha \in (0, 1)$. Find the value function, and the related policy function.

(Hint: consider past consumption as a state of the system, hence the value function has two arguments: k_t and c_{t-1})

Exercise 3. Assume that i) the single period utility function is: $c_t^{1-\gamma}/(1-\gamma)$, where $\gamma \in [0, 1) \cup (1, \infty)$; ii) the dynamic constraint is $k_{t+1} = Ak_t - c_t$, where $A > 0$. Find the value function, and the related policy function.

Exercise 4. Provide – for the return and the transition functions used in the previous Exercise 3– a condition on γ such that the boundedness conditions in Theorem 2 is satisfied.

4.2. Quadratic preferences with a linear constraint. We now consider a consumer, whose time horizon is again infinite, characterized by “quadratic” preferences:

$$(4.8) \quad W_0 = \sum_{t=0}^{\infty} \beta^t \left(\varepsilon + \gamma c_t - \frac{\eta}{2} c_t^2 \right), \text{ with } \beta \in (0, 1).$$

Because the parameters γ and η are assumed to be positive, the marginal utility for our consumer is positive for $c_t < \gamma/\eta$, while it is negative for $c_t > \gamma/\eta$. Hence, the single-period utility is at its maximum when $c_t = \gamma/\eta$, which is called the “bliss point” in consumption (the corresponding utility is $\varepsilon + \gamma^2/(2\eta)$).²⁰ The intertemporal constraint is linear in the state variable k_t :

$$(4.9) \quad k_{t+1} = (1+r)k_t - c_t.$$

Notice that we may interpret k_t as the consumer’s financial assets and r as the interest rate. Notice also that, because the utility function is strictly concave and bounded and the transition function is concave, we can be sure that the value function is unique and strictly concave (Theorem 1).

Our problem is to solve:

$$\begin{aligned} V(k_0) &= \max_{c_0} \left\{ \varepsilon + \gamma c_0 - \frac{\eta}{2} c_0^2 + \beta V(k_1) \right\}, \\ \text{s.t. } k_1 &= (1+r)k_0 - c_0, \\ k_0 &\text{ given.} \end{aligned}$$

As before, we find the first order condition, which is: $\gamma - \eta c_0^* = \beta V'(k_1)$. From the dynamic constraint we obtain $c_0 = (1+r)k_0 - k_1$, which is used to substitute consumption out of the Bellman equation and out of the first order condition. This yields

$$(4.10) \quad \begin{cases} V(k_0) = \varepsilon + \gamma[(1+r)k_0 - k_1] - \frac{\eta}{2}[(1+r)k_0 - k_1]^2 + \beta V(k_1) \\ \gamma - \eta[(1+r)k_0 - k_1] = \beta V'(k_1), \\ k_0 \text{ given.} \end{cases}$$

Notice that system (4.10) corresponds to (3.5), but for the fact that we have substituted out c_0^* exploiting the linear constraint (4.9).

The logic of the solution method is the same we have experienced in the previous example: accordingly, we now introduce the tentative solution,

²⁰We do not take a position about ε : it can be positive, negative or nought. For example, a negative ε implies that our consumer needs a minimal amount of consumption to start enjoying life (and hence having a positive utility).

which takes the same functional form of the return function:

$$(4.11) \quad V(k_t) = g + hk_t + \frac{m}{2}k_t^2,$$

where g , h , and m are the undetermined coefficients. The (small) difference with the example in Sub-section 4.1 is that in the present case we shall set up a three equations system, because we need to determine three coefficients.

As a preliminary, from the second equation in (4.10), we obtain exploiting our guess (4.11):

$$k_0 = \frac{1}{1+r} \left[\frac{\gamma - \beta h}{\eta} + \frac{\eta - \beta m}{\eta} k_1 \right].$$

The equation above grants us that – in the present example – the function $k_{t+1} = \zeta(k_t)$ is linear.

Substituting the guess (4.11) for $V(k_0)$, and $V(k_1)$ in the first equation in (4.10), and exploiting the above expression for k_0 , we obtain a quadratic equation in k_1 . Because this equation must be satisfied for any value of k_1 , it must be true that:

$$(4.12) \quad \begin{cases} g + \frac{h}{1+r} \left(\frac{\gamma - \beta h}{\eta} \right) + \frac{m}{2(1+r)^2} \left(\frac{\gamma - \beta h}{\eta} \right)^2 = \varepsilon + \frac{\gamma}{\eta}(\gamma - \beta h) - \frac{(\gamma - \beta h)^2}{2\eta} + \beta g \\ \frac{h}{1+r} \left(\frac{\eta - \beta m}{\eta} \right) + \frac{m}{(1+r)^2} \left(\frac{\eta - \beta m}{\eta} \right) \left(\frac{\gamma - \beta h}{\eta} \right) = -\frac{\gamma \beta m}{\eta} + \frac{(\gamma - \beta h)\beta m}{\eta} + \beta h \\ \frac{m}{2(1+r)^2} \left(\frac{\eta - \beta m}{\eta} \right)^2 = -\frac{(\beta m)^2}{2\eta} + \frac{\beta m}{2} \end{cases} .$$

Inspection of the above system reveals that it is convenient to solve first the third equation: the unique unknown it involves is m . From this equation, we immediately obtain:

$$(4.13) \quad m = \frac{\eta}{\beta} [1 - \beta(1+r)^2].$$

Inserting (4.13) into the second equation in (4.12) gives an equation which can be solved for h , yielding:

$$(4.14) \quad h = \frac{\gamma}{\beta} \frac{[\beta(1+r)^2 - 1]}{r}.$$

Finally, we can use (4.14) and (4.13) into the first equation in (4.12), to obtain g :

$$(4.15) \quad g = \frac{\varepsilon}{1-\beta} + \frac{\gamma^2}{2\eta\beta(1-\beta)} \left[\frac{1 - \beta(1+r)}{r} \right]^2.$$

From (4.15) we see once again why the assumption $\beta < 1$ is crucial: it prevents the maximum value function from exploding to infinity. Because g , h and m are independent of the state variable, that is capital, our guess (4.11) is correct.²¹

4.2.1. *A particular case.* This discussion requires a preamble. In a competitive economy, the marginal productivity of capital (net of depreciation) equals the interest rate. In fact, when the factor markets are competitive, capital is paid its marginal productivity, and the interest rate is equalized to the latter by competition on the financial market. When we accept the assumption of competitive markets, we accordingly interpret the interest rate r_t as the net marginal productivity of capital, $f'(k_t) - \delta$.

Coming back to our linear quadratic example, we notice that, for the dynamic constraint to be linear (as it is assumed by (4.9)), the interest rate, and hence $f'(k_{t+1})$ must not change; this can happen with a potentially varying capital stock only in two cases. Either $f(k_t)$ is linear, or $f'(k_{t+1})$ stays constant because the capital stock actually does not change over time, which means that the economy is in its steady state. Linearity of the production function is a strong assumption precisely because it amounts to accept that the marginal productivity of capital is constant. Hence, we assume that capital is constant at its long run value, \hat{k} . Notice, that, at this level of the capital stock, it must be true that $\beta (f'(\hat{k}) + 1 - \delta) = 1$ (Equation (1.10)), and hence that $\beta(1 + r) = 1$. In the steady state, in fact, the impatience parameter β exactly offsets the positive effects on saving exerted by its reward.

For $\beta(1 + r) = 1$, which means “around” the steady state, the value function parameters (4.13-4.15) simplify to:

$$\begin{cases} m = -\frac{\eta}{\beta}r \\ h = \frac{\gamma}{\beta} \\ g = \frac{\epsilon}{1-\beta} \end{cases} .$$

In this particular—but sensible—case, it is easy to find the policy function. Exploiting the tentative solution (4.11), and the parameters computed above, the second equation in (4.10) becomes: $\gamma - \eta[(1+r)k_0 - k_1] = \gamma - \eta r k_1$, which confirms $k_1 = k_0$. Hence, from the dynamic constraint (4.9), we get: $c_0^* = r k_0$. It is also immediate to check that the transversality condition holds. In fact, in this model the tvc is:

$$\lim_{t \rightarrow \infty} \beta^t (\gamma - \eta c_0^*) k_0,$$

²¹Our example has required a good deal of calculations. When a linear-quadratic model involves two (or more) state variables, it has to be solved by means of numerical techniques involving matrix manipulations. Refer to Chow [1997] or to Sargent and Ljungqvist [2012].

which converges to 0 simply because $\lim_{t \rightarrow \infty} \beta^t = 0$.

Exercise 5. (a) Study the dynamics for c_t and k_t if $\beta(1+r) > 1$. Discuss whether the transversality condition is always satisfied. (b) Study the dynamics for c_t and k_t if $\beta(1+r) < 1$. Discuss whether the transversality condition is always satisfied.

4.2.2. *An application to the growth model.* Let us now consider again the growth model with logarithmic preferences, so that the consumer's overall utility is given by (4.1). Assume that the production function is Cobb-Douglas; but now take into account the fact that capital depreciates slowly: actually, apart maybe some fancy electronic devices, physical capital does not depreciate in one period. Hence, our intertemporal constraint is:

$$(4.16) \quad k_{t+1} = Ak_t^\alpha + (1 - \delta)k_t - c_t.$$

Because there is no way to obtain a closed form analytic solution, one approach that we can use relies on a linear-quadratic approximation of our model. This, of course, implies the need to choose a point around which to approximate. The standard choice for this point is the steady state, which is certainly sensible if our research project involves, for example, the introduction of some sort of shock, and the study of its effects, in a "mature" economic system.

Hence, we determine first the steady state. With the production function (4.16), the steady state equations (2.8) become:

$$(4.17) \quad \begin{cases} \beta \left(\alpha A \hat{k}^{\alpha-1} + 1 - \delta \right) = 1 \\ \hat{c} = A \hat{k}^\alpha - \delta \hat{k} \end{cases}.$$

The above system allows to determine the consumption and capital steady-state levels.²² We now apply Taylor's theorem to the logarithmic utility function, obtaining:

$$(4.18) \quad \ln(c_t) = \ln(\hat{c}) + \frac{1}{\hat{c}}(c_t - \hat{c}) - \frac{1}{2\hat{c}^2}(c_t - \hat{c})^2.$$

As for the capital accumulation constraint, we truncate the Taylor's approximation to the first term, which yields:

$$k_{t+1} = A\hat{k}^\alpha + (1 - \delta)\hat{k} + \alpha A\hat{k}^{\alpha-1}(k_t - \hat{k}) + (1 - \delta)(k_t - \hat{k}) - \hat{c} - (c_t - \hat{c}),$$

which immediately becomes:

²²Which are: $\hat{k} = \left(\frac{\alpha\beta A}{1-\beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$, and $\hat{c} = \left(\frac{1-\beta[1-\delta(1-\alpha)]}{\alpha\beta} \right) \left(\frac{\alpha\beta A}{1-\beta(1-\delta)} \right)^{\frac{1}{1-\alpha}}$.

$$k_{t+1} = A\hat{k}^\alpha + (1 - \delta)\hat{k} - \hat{c} + \left(\alpha A\hat{k}^{\alpha-1} + 1 - \delta \right) (k_t - \hat{k}) - (c_t - \hat{c}),$$

and hence, using (4.17):

$$(4.19) \quad k_{t+1} - \hat{k} = \frac{1}{\beta}(k_t - \hat{k}) - (c_t - \hat{c}).$$

Equations (4.18) and (4.19) lead to the very same structure that can be found in (4.8) and (4.9), and hence we can solve the approximated problem using the tentative solution postulated for the linear-quadratic case (the relevant variables being the deviations of capital and consumption from the steady state).

5. TWO USEFUL RESULTS

The “guess and verify” technique is useful only when a closed form solution exists. Unfortunately, only a few functional forms for the payoff function and for the dynamic constraint allow for a closed form maximum value function: the examples and the exercises in the previous Section almost work out the list. Hence, it is often useful to “qualify” the solution, identifying some of its characteristics or properties, without looking for the (actually non-existent) analytic formulation for the maximum value function. In this Section, we review two important results, that may be helpful in analyzing the solution for a Bellman problem.

5.1. The Envelope Theorem. We now apply the envelope theorem to the growth model formulated in Problem (3.3). In other words, we concentrate – for simplicity – on a specific application of the envelope theorem. However, the results that we obtain, besides being important, are of general relevance.

To save on notation, we now denote the dynamic constraint by $k_1 = g(k_0, c_0)$, accordingly, the dynamic programming formulation for our utility-maximization problem becomes:²³

$$V(k_0) = \max_{c_0} \{U(c_0) + \beta V(k_1)\},$$

s.t. $k_1 = g(k_0, c_0)$, k_0 given.

We already know that the first order condition with respect to the control variable is $U_c(c_0^*) + \beta V_k(k_1) \frac{\partial k_1}{\partial c_0} = 0$. Consider now the Bellman problem

²³Partial derivatives will soon come into play. Hence, we start denoting derivatives by a subscript. Accordingly, $g_k(k_0, c_0)$ is the partial derivative of $g(k_0, c_0)$ with respect to capital, and so on. This convention shall be maintained but when we deal with models or exercises in which we only need to differentiate functions with a single argument.

above, assuming to be on the optimal path. In this case, we have $V(k_0) = U(c_0^*) + \beta V(k_1)$ (the max operator disappears exactly because we already are on the path along which consumption is optimal). The total differential for the last equation is: $dV(k_0) = dU(c_0^*) + \beta dV(k_1)$, or: $V_k(k_0)dk_0 = U_c(c_0^*)dc_0 + \beta V_k(k_1)dk_1$. The differential for k_1 can be easily obtained from the dynamic constraint: $dk_1 = g_k(k_0, c_0)dk_0 + g_c(k_0, c_0)dc_0$.

Exploiting dk_1 , the total differential for the Bellman equation becomes:

$$V_k(k_0)dk_0 = U_c(c_0^*)dc_0 + \beta V_k(k_1) [g_k(k_0, c_0^*)dk_0 + g_c(k_0, c_0^*)dc_0].$$

This expression, using the first order condition for c_0 , reduces to:

$$V_k(k_0) = \beta V_k(k_1)g_k(k_0, c_0^*).$$

This is an application of the Envelope theorem: we have simplified the total differential precisely because we are on the optimal path, and hence the first order condition must apply.

Notice that we could have expressed the above result as follows:

$$(5.1) \quad V_k(k_0) = \beta V_k(k_1) \frac{\partial k_1}{\partial k_0},$$

where $\partial k_1 / \partial k_0$ is the partial derivative, $g_k(k_0, c_0^*)$.

Equation (5.1) can be useful in several contexts. In fact, it can be reformulated in a very convenient way. Because the first order condition states that: $U_c(c_0^*) = \beta V_k(k_1)$, it must also be true that $U_c(c_1^*) = \beta V_k(k_2)$. With these facts in mind, we forward once equation (5.1), and we obtain:

$$(5.2) \quad U_c(c_0^*) = \beta U_c(c_1^*)g_k(k_1, c_1^*),$$

which is the Euler equation (bear in mind that, in the growth example, $g_k(k_1, c_1) = f'(k_1) + 1 - \delta$, and refer to (1.9)). Not only it is often easier to solve numerically equation (5.2) than the Bellman's one – as we shall see in Sub-section 6.2 – but equation (5.2) can be interpreted without referring to the still unknown maximum value function.

That Equation (5.2) corresponds exactly to Equation (1.9) reassures us about the fact that the Bellman's approach and the Lagrange's one lead to the same result.

Exercise 6. *Exploiting the envelope theorem, find as many details as possible of the maximum value function for the non-stochastic version of the Brock and Mirman model discussed in Sub-section 4.1.*

5.2. The Benveniste and Scheinkman formula. In this Sub-section – that can be skipped during the first reading – we consider a more general

framework, where the return function, now denoted by $Q(\cdot)$, does not depend only on the control variables, but it also depends on the state. In this case the dynamic programming formulation is:

$$V(k_0) = \max_{c_0} \{Q(k_0, c_0) + \beta V(k_1)\},$$

$$s.t. \ k_1 = g(k_0, c_0), \ k_0 \text{ given.}$$

The first order condition with respect to the control variable yields:

$$Q_c(k_0, c_0^*) + \beta V_k(k_1) \frac{\partial k_1}{\partial c_0} = 0.$$

Assume, as in the previous Sub-section, to be on the optimal path, so that: $V(k_0) = Q(k_0, c_0^*) + \beta V(k_1)$. The total differential for the last equation is: $dV(k_0) = dQ(k_0, c_0^*) + \beta dV(k_1)$, or:

$$V_k(k_0)dk_0 = Q_c(k_0, c_0^*)dc_0 + Q_k(k_0, c_0^*)dk_0 + \beta V_k(k_1)dk_1.$$

The differential for the period 1 capital is obtained from the dynamic constraint, and is $dk_1 = g_k(k_0, c_0)dk_0 + g_c(k_0, c_0)dc_0$. Hence, the total differential for the Bellman's equation becomes:

$$\begin{aligned} V_k(k_0)dk_0 &= \\ &= Q_c(k_0, c_0^*)dc_0 + Q_k(k_0, c_0^*)dk_0 + \beta V_k(k_1) [g_k(k_0, c_0^*)dk_0 + g_c(k_0, c_0^*)dc_0]. \end{aligned}$$

Using the first order condition, the equation above reduces to:

$$(5.3) \quad V_k(k_0) = Q_k(k_0, c_0^*) + \beta V_k(k_1)g_k(k_0, c_0^*).$$

This is the Benveniste and Scheinkman formula, which has been obtained as an application of the Envelope theorem.

The usefulness of the Benveniste and Scheinkman formula (5.3) can be appreciated considering that, in many problems, there is not a unique way to define states and controls. For example, in the growth model, one could define as control variable not consumption, but gross savings (which means that the control is $s_t = f(k_t) - c_t$). When depreciation is complete (i.e. when $\delta = 1$), this modification implies that the next-period state is equal to the current control ($k_{t+1} = s_t$), and the Bellman problem (3.3) becomes:

$$(5.4) \quad \begin{aligned} V(k_0) &= \max_{s_0} \{Q(f(k_0) - s_0) + \beta V(k_1)\}, \\ &\text{s.t. } k_1 = s_0, \\ &k_0 \text{ given.} \end{aligned}$$

With this formulation, the first order condition is: $Q_c(f(k_0) - s_0^*) = \beta V_k(k_1)$.²⁴

Differentiating the Bellman equation “on the optimal path”, we obtain:²⁵

$$V_k(k_0)dk_0 = Q_c(f(k_0) - s_0^*)(df(k_0) - ds_0) + \beta V_k(k_1)dk_1.$$

Because $dk_1/ds_0 = 1$, using the first order condition, the differential can be reduced to:

$$(5.5) \quad V_k(k_0) = Q_c(f(k_0) - s_0^*)f_k(k_0),$$

which is how the Benveniste and Scheinkman formula applies to this problem.

Formula (5.5) is interesting because it shows that the Benveniste and Scheinkman result can be used to highlight a relation between the maximum value function, the return function and the dynamic constraint. The Benveniste and Scheinkman formula leads to such relation *when the transition function is such that the period $t+1$ state variable does not depend on the period t state (i.e. when the partial derivative of the dynamic constraint with respect to period t state variable is 0)*. For this to be true, it is necessary that the initial period state variable (k_0) is excluded from the dynamic constraint, as it happens in problem (5.4). The example we have just developed shows that this can be achieved by means of an appropriate variable redefinition (see Sargent [1987] pp. 21-26 for a discussion and an alternative example).

Exercise 7. *Apply the Benveniste-Scheinkman formula to the traditional (Ramsey) growth model, and show that the maximum value function is concave.*

Exercise 8. *By means of an appropriate variable redefinition, find the equivalent of formula (5.5) for the traditional growth model if $\delta < 1$.*

²⁴Alternatively, defining $Q(c_0) (= Q(f(k_0) - s_0)) \equiv Z(k_0, s_0)$, yields $Z_s(k_0, s_0^*) + \beta V_k(k_1) = 0$.

²⁵Notice that $Z_k(k_0, s_0)dk_0 = Q_c(f(k_0) - s_0)df(k_0)$ and $Z_s(k_0, s_0)ds_0 = -Q_c(f(k_0) - s_0)ds_0$.

6. A “PAPER AND PENCIL” INTRODUCTION TO NUMERICAL TECHNIQUES

As already underscored, the “guess and verify” approach is useful only in the few cases in which a closed form solution for the maximum value function exists. In this Section, we illustrate two alternative techniques that can be used to approximate numerically the maximum value function: the value function iteration method and the (more up-to-date) collocation technique. We do this by means of simple examples, which rely on the elementary version of the Brock and Mirman model solved in Sub-section 4.1.

6.1. Value function iteration based on the discretization of the state space. Let us consider again the version of Brock and Mirman model that we faced in Sub-section 4.1, assuming, however, that we are not able to find the explicit solution for this problem, so that we need to compute a numerical approximation for the solution. The reason why we solve a well-understood problem by means of a numerical technique is to allow for the comparison of the approximated solution, that we obtain, with the exact one, that we already know.

Because we are moving in the direction of using numerical techniques, the first thing we need to do is to pick the values we want to assign to our parameters. This is readily done. A sensible value for α is 0.3. In fact, α represents the capital income share of output, a value which is between 0.25 and 0.33 for most OECD countries. As for β we choose 0.97: we know from equation (1.10) that – in the steady state – β is equal to the reciprocal of the marginal productivity of capital, which represents also the interest factor. Hence, if a period represents a year, $\beta = 0.97$ implies a long-run annual (real) interest factor approximately equal to 1.03, a realistic value. As for the total factor productivity parameter A , we choose a value such that the long-run capital level, $\hat{k} = (\alpha\beta A)^{\frac{1}{1-\alpha}}$, is unity. Hence, $A \simeq 3.43643$. This is a normalization: we have decided to measure output using a reference unit such that the long-run value for capital is exactly equal to one.

Accordingly, our problem is to solve:

$$(6.1) \quad \begin{aligned} V(k_0) &= \max_{c_0} \{ \ln(c_0) + 0.97V(k_1) \}, \\ s.t. \quad k_1 &= 3.43643k_0^{0.3} - c_0, \\ &k_0 \text{ given.} \end{aligned}$$

We assume to be interested in solving the problem for values for capital that are around the steady state.²⁶

The state variable is continuous, nevertheless we now consider it as if it were discrete: it is this approximation that allows to use the numerical

²⁶Once you have read the current Sub-section, justify this choice.

technique we are describing. To fix the ideas, we consider only five possible levels for capital, which are $\{0.98, 0.99, 1, 1.01, 1.02\}$.

Notice that, if we knew the function $V(k_1)$, it would be relatively easy to solve problem (6.1): the solution would be the consumption level c_0^* that allows to obtain the highest $V(k_0)$, provided that $k_0, k_1 \in \{0.98, 0.99, 1, 1.01, 1.02\}$. For example, assume that $V(k_1) = 20k_1$ (this choice is completely arbitrary; actually we know that it is wrong: we just want to illustrate what we mean when we say that – knowing $V(k_1)$ – it is easy to solve problem (6.1)). When $k_0 = 0.98$, the problem becomes:

$$(6.2) \quad \begin{aligned} V(k_0 = 0.98) &= \max_{c_0} \{ \ln(c_0) + 0.97 \times 20k_1 \}, \\ \text{s.t. } k_1 &= 3.43643(0.98)^{0.3} - c_0, \\ k_0 &= 0.98. \end{aligned}$$

To tackle the problem, we must leave our standard tool – the derivative – on the shelf, because our problem is not continuous, since we have decided to compute $V(k_1)$ only in a few points. Hence, we need to perform all the required calculations. First, we express the constraint as: $c_0 = 3.43643(0.98)^{0.3} - k_1 = 3.41566 - k_1$, and we compute the consumption levels that allow k_1 to take one of the five feasible values. These are:

k_1	0.98	0.99	1	1.01	1.02
c_0	2.43566	2.42566	2.41566	2.40566	2.39566

Second, we consider the postulated maximum value function: for $V(k_1) = 20k_1$ the right hand side of problem (6.2) takes, in correspondence of the five couples $\{k_1, c_0\}$ computed above, the following values:

$$r.h.s.(6.2) \quad 19.90222 \quad 20.09210 \quad 20.28197 \quad 20.47182 \quad 20.66166$$

According to the above calculations, the highest value is obtained for the consumption choice $c_0 = 2.39566$, hence $V(k_0 = 0.98)$ takes the value 20.66166.

This calculation should be repeated for every remaining $k_0 \in \{0.99, 1, 1.01, 1.02\}$, this would give us the period 0 maximum value function for each state.

However, there is no particular usefulness in performing all these calculations: in fact the few calculations we have presented above are already enough to conclude that the postulated function $V(k_1) = 20k_1$, is incorrect. From our analysis in Section 3, we know that – in infinite horizon models – the maximum value function is time independent. Accordingly, the functional form we have assumed for period 1 must apply also to period 0.

Hence, we should expect to find that $V(k_0 = 0.98) = 19.6$, which is not the case.

While the usefulness of the above example lies in the fact that we have understood which kind of computations are required, we must now tackle the real issue: we need to know how to obtain the unknown maximum value function.

The strategy to get it prescribes to:

- (Step 1) specify a set of arbitrary values to $V(k_1)$. (Denote this initial set of values by $V_0(k_1)$);
- (Step 2) solve the problem (6.1) for each state k_0 , finding a new (and hence different) maximum value function (which, of course, substantiates in a new set of values, denoted by $V_1(k_0)$);
- (Step 3) obtain $V_2(k_0)$, using $V_1(k_1) = V_1(k_0)$ as a new initial guess for the value function;²⁷
- (Step 4) iterate the step above until $V_n(k_0)$ and $V_{n+1}(k_0)$ are “sufficiently close”.

The convergence of the above procedure to a set of values that represent the true maximum value function is guaranteed whenever the hypothesis in Theorem 1 or in Theorem 2 are satisfied. In these cases, successive iterations of the value function converge to the true value function, and this convergence takes place for any starting point, i.e. for any initial arbitrary value function.

We now provide some further details on how to apply the above procedure in our extremely simplified example.

We start by choosing a set of values for $V(k_1)$. A commonly chosen set of initial values is $V_0(k_1) = 0$ for any k_1 . This choice corresponds to Step 1 in our procedure.

As for Step 2, we start by noticing that the above choice implies that we need to face problems of the following type:

$$\begin{aligned} V(k_0) &= \max_{c_0} \{\ln(c_0)\}, \\ \text{s.t. } k_1 &= 3.43643k_0^{0.3} - c_0, \\ &k_0 \text{ given.} \end{aligned}$$

Notice that, as before, both k_0 and k_1 must take one of the possible values for capital (i.e. $\{0.98, 0.99, 1, 1.01, 1.02\}$): this restrict the set of feasible consumption levels. For example, if $k_0 = 0.98$, for $k_1 = \{0.98, 0.99, 1, 1.01,$

²⁷If you feel confused by this, just go on reading.

1.02}], we compute, as before: $c_0 = \{2.43566, 2.42566, 2.41566, 2.40566, 2.39566\}$; the corresponding utilities are $\{0.89022, 0.88610, 0.88197, 0.87782, 0.87366\}$. The highest utility level is reached for $c_0 = 2.43566$, which is the consumption level implying that the next period capital is $k_1 = 0.98$.

Hence, the solution for this specific problem is $V_1(k_0 = 0.98) = 0.89022$.

Repeating this reasoning we obtain: $V_1(k_0 = 0.99) = 0.89449$, $V_1(k_0 = 1) = 0.89871$, $V_1(k_0 = 1.01) = 0.90288$, $V_1(k_0 = 1.02) = 0.90701$. In every problem, the solution corresponds to the maximum feasible consumption level, which is the one corresponding to $k_1 = 0.98$. In other words, for any $k_0 \in \{0.98, 0.99, 1, 1.01, 1.02\}$, c_0 is such that $k_1 = 0.98$. This is hardly surprising: since we have arbitrarily chosen $V_0(k_1) = 0$, capital bears no future value, and hence it is sensible to choose to consume as much as possible.²⁸

The key point is that our maximization procedure has led to a set of values for $V_1(k_0)$. Accordingly, we have completed Step 2 in the procedure.

Hence – Step 3 – we use these values as a new guess for the maximum value function. This bit of the procedure is inspired by the fact that the true maximum value function is time independent: at different dates it must take the same values for any capital stock. Accordingly, we now assume $V_1(k_1) = \{0.89022, 0.89449, 0.89871, 0.90288, 0.90701\}$.

In this second iteration for the maximization problem, when $k_0 = 0.98$, consumption may again take the values $c_0 = \{2.43566, 2.42566, 2.41566, 2.40566, 2.39566\}$. When c_0 is 2.43566, (that is, when it takes the value guaranteeing $k_1 = 0.98$), the corresponding “overall utility” is $\ln(2.43566)$ plus β times $V_1(k_1 = 0.98) = 0.89022$ (which gives $V_2(k_0 = 0.98) = 1.75373$). In words, this is the “overall value” obtained by choosing the consumption level that allows to maintain the current capital level in the next period.

Considering the whole set of choices, $c_0 = \{2.43566, 2.42566, 2.41566, 2.40566, 2.39566\}$, we find that the values associated are $\{\ln(2.43566) + 0.97 \times 0.89022, \ln(2.42566) + 0.97 \times 0.89449, \ln(2.41566) + 0.97 \times 0.89871, \ln(2.40566) + 0.97 \times 0.90288, \ln(2.39566) + 0.97 \times 0.90701\}$, which are $\{1.75373, 1.75376, 1.75372, 1.75362, 1.75346\}$.

Notice that the second value in the row, 1.75376 is the highest one. Hence, the optimal choice is now to opt for a consumption level such that the next period capital is 0.99. In words, if the initial capital is 0.98 and if we take account of the future period (discounted) utility, it is optimal to save 0.01 units of capital.

When we repeat this reasoning for the other capital values in our grid, we obtain that: if $k_0 = 0.99$, the consumption optimal choice is such that k_1

²⁸Here the logic is the same we used in Sub-section 1.3 to argue that $k_{T+1} = 0$.

remains at 0.99; when $k_0 = 1$, the consumption optimal choice is such that k_1 is again 0.99; when $k_0 = 1.01$ or 1.02, the consumption optimal choice is such that $k_1 = 1$. In correspondence to the optimal choices described above, the value function is: $V_2(k_1) = \{1.75376, 1.75804, 1.76228, 1.76649, 1.77065\}$. The obtaining of this set of values concludes Step 3 in our procedure.

Of course, a normal person is already fed up with all these calculations. This is the point to let our computer do the calculations, hence performing Step 4. It is easy to write a program using Matlab (or Python) that carries out this task.²⁹ The results of this program are reported in Table 1.

TABLE 1

iter.#	$V(k_0=0.98)$	$V(k_0=0.99)$	$V(k_0=1.00)$	$V(k_0=1.01)$	$V(k_0=1.02)$
0	0	0	0	0	0
1	0.890218	0.894487	0.898707	0.902881	0.907009
2	1.753757	1.758043	1.762281	1.766486	1.770648
3	2.591406	2.595692	2.599944	2.604152	2.608314
4	3.403926	3.408223	3.412478	3.416686	3.420850
5	4.192081	4.196381	4.200636	4.204844	4.209008
6	4.956594	4.960894	4.965149	4.969357	4.973521
7	5.698172	5.702472	5.706727	5.710935	5.715099
8	6.417502	6.421802	6.426058	6.430265	6.434430
9	7.115253	7.119553	7.123808	7.128016	7.132180
10	7.792071	7.796371	7.800626	7.804834	7.808998
20	13.538224	13.542524	13.546779	13.550987	13.555151
50	23.204550	23.208850	23.213105	23.217313	23.221477
100	28.264686	28.268986	28.273241	28.277449	28.281613
200	29.608749	29.613049	29.617304	29.621512	29.625676
300	29.672662	29.676962	29.681218	29.685425	29.689590
375	29.675528	29.679828	29.684084	29.688291	29.692456
376	29.675538	29.679838	29.684093	29.688301	29.692465
true	29.675860	29.680156	29.684409	29.688619	29.692788

In the simulation, the convergence criterion we have chosen is as follows: we let the computer calculate—for each gridpoint—the difference between the final and the initial value of $V(k_0)$ for each iteration (i.e. $V_{n+1}(k_0) - V_n(k_0)$); then we considered the absolute values of these differences and we let the routine to pick the largest. When this is below $10^{(-5)}$, we let the computer to stop. On a i7-4600 CPU computer, this happens after about 30/100 of a second, at the 376th iteration.

²⁹In the Appendix, we spell out in details the routines used to solve our examples.

Because the computed value function converges to the true maximum value function, when the set of values representing $V_{n+1}(k_0)$ is “almost identical” to the one used to represent $V_n(k_0)$, they are also “almost identical” to the true set of values, which is to the true $V(k_0)$.

Having chosen an exercise for which we know the exact solution, we have been able to provide, as a reference point, the true values for $V(k_1)$.

Exercise 9. *Check that the values for $V(k)$ in the last line of Table 1 are correct.*

From Table 1, we can see that the values for $V(k_1)$ smoothly converge toward their true value: we are in the position to “observe in action” the (vector) fixed point nature of our problem.

In our case, the convergence is slow. This is due to the fact that β is close to unity: what happens in the future matters a lot, hence the initial arbitrary values for $V(k_1)$ do not lose rapidly their weight.

The nice feature of this approach consists in the fact that it is immediate to change the interval for the state variable and the number of gridpoints, therefore adapting a numerical routine to a new situation. For example, we can use the program written for the problem above to study what happens for $k \in [0.7, 1.1]$ with 1600 gridpoints. Panel (a) in Figure 5 plots the maximum value function, while Panel (b) shows the differences between the true value function and the approximated one. Notice that the computing time increases dramatically: it takes about three minutes and one half to achieve convergence.³⁰

This is probably the right time to become aware of one of the sad facts of life, which is usually referred to as “the curse of dimensionality”. By these words, one usually refers to the fact that the speed of a procedure such as the one we have briefly outlined decreases more than proportionally with the number of the state variables. To understand this fact, bear in mind that, to find the value function by iteration, we have discretized the state space. In our case, the state space is a line, because we have just one state variable, that is physical capital. If we had two state variables, say physical and human capital, we would have a 2-dimensional state space (which means, a plane). In this case, it would have been necessary to divide the state space in small “squares”, the number of which is of course the square of the number of points we use to discretize each dimension. Accordingly, the number of “evaluation points” increases with the power of the number of state variables. Hence, the required computer time grows with the power of the number of

³⁰This is exploiting again an Intel i7-4600 CPU @ 2.70 Ghz.

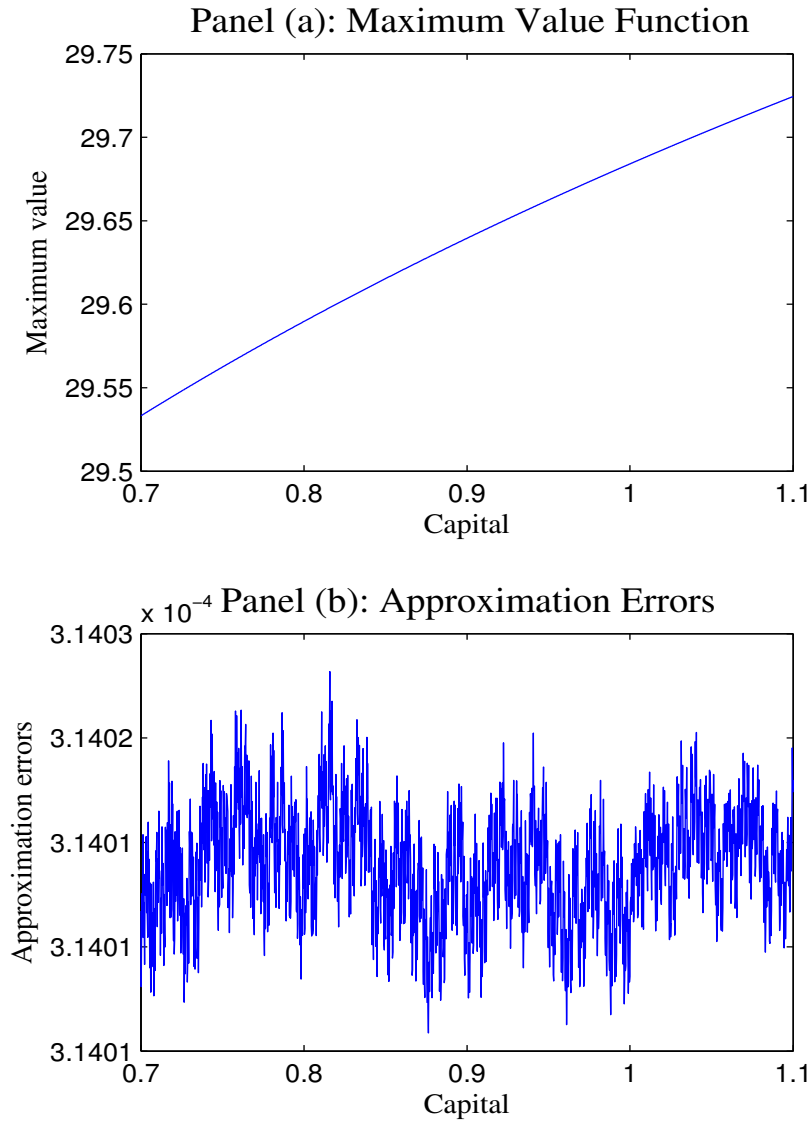


FIGURE 5.

state variables.³¹ For example, in a model where the representative agent decides her consumption level and her schooling effort as functions of both human and physical capital, we need to compute the value function for every combination of human and physical capital in the two-dimensional grid.

In short: if our problem is large, we need to use a more efficient technique, which is the topic of the next Sub-section. Nevertheless, the value function

³¹If one chooses a different number of grid points for the two state variables—therefore dividing the plane in small rectangles—the argument in the main text must be properly adapted.

iteration method can be of some help when one or more dimensions of the state space are naturally discrete. This happens when the researcher wishes to study problems where the agent may be employed or unemployed, may decide whether to get a degree or not, whether to retire or not, and so on. This method can be of some help also if the transition function is subject to a constraint. Suppose, to mention the most famous example, that investment is irreversible, so that $k_1 \geq k_0(1 - \delta)$. This problem can be easily dealt with in our framework: it is sufficient not to consider the consumption level that implies an infringement of the irreversibility constraint.³²

Before concluding this Section, consider that there are ways to reduce the computing time.

One can move in the direction of providing a better initial guess, given that our initial value function ($V_0(k) = 0$) proved to be quite poor. With this purpose in mind, consider a particular version of problem (6.1), a version in which the representative agent is not allowed to change her capital stock. In this case, the maximum value attainable by the representative agent would be, for any given k_0 ,

$$V(k_0) = \ln(c_0) + 0.97V(k_1),$$

$$\text{with } k_1 = k_0 = 3.43643k_0^{0.3} - c_0.$$

Notice that the “max” operator has been omitted, since no decision can be taken by the representative agent, whose consumption decision is actually constrained. Accordingly, the maximum value that can be obtained is

$$V(k_0) = \frac{1}{0.03} \ln(3.43643k_0^{0.3} - k_0),$$

or, in general,

$$V(k) = \frac{1}{1 - \beta} \ln(Ak^\alpha - k).$$

The expression above is a commonly used initial guess for the maximum value function; notice moreover that the logic can be applied for different

³²The presence of an irreversibility constraint is much more interesting in a stochastic model than in a deterministic one. Figure 3 suggests that whenever the initial capital stock is below the steady-state, capital grows over time until it reaches its steady state value. Accordingly, an irreversibility constraint can bind only if our representative agents receives, as a gift, a capital level that is so much higher than the steady state one, that she wants to decumulate it at a fast rate. Instead, in a stochastic model, a shock may be sufficient to make the irreversibility constraint binding.

Notice that, when the irreversibility constraint is binding, our consumer is forced to consume less than what would be optimal. Hence, she would like to avoid finding herself in this situation. Therefore, the awareness that then irreversibility constraint may be binding, even if with a low probability, can be sufficient to deeply influence the representative agent behavior. We shall briefly return on models with stochastically binding constraints in Section (10.2).

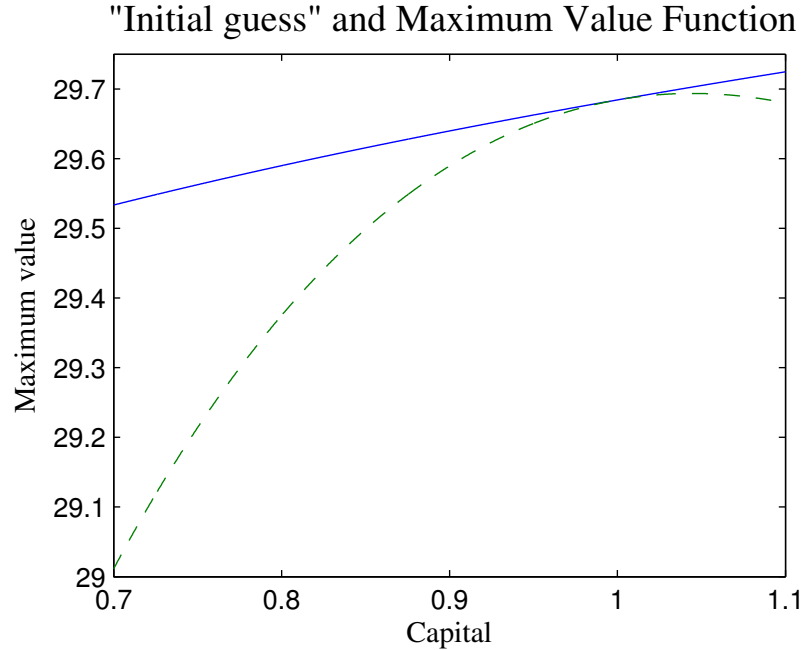


FIGURE 6.

utility functions, and for $\delta \in [0, 1]$. Figure 6 shows that the maximum value function (continuous line) lies above the initial guess (dashed line), since the latter has been obtained by imposing a constraint to the agent's behavior (and the maximum value function is attained by unconstrained and fully optimizing agents). Actually, the difference between the two functions can be interpreted as the utility cost imposed by the constraint. In other words, when the capital stock is below the steady state, what the figure highlights is the present discounted value of the gain in future utilities that can be obtained by saving; when instead the capital stock is above the steady state, Figure 6 shows the cost of being obliged to maintain an excessively high capital stock.

Notice that, not surprisingly, the two functions take the same value in the steady state, which is the point in which the representative agent is not willing to modify the capital stock, and hence the constraint is not binding. An element of interest is that the computing time collapses to about three seconds (from around 210).

When looking for ways to reduce the computing time, another (important) direction consists on iterating on the policy function instead of iterating on the value function. This approach is often referred to as the "Howard's improvement algorithm", which often converges faster than does the value function iteration. On this we refer e.g. to Ljungqvist and Sargent (2012).

6.2. Collocation techniques. As already remarked, the previous technique is rather inefficient, because it requires the evaluation of the value function in a large number of points. Here, we present a more efficient method to solve dynamic models, which is known as “collocation technique”.

Because the application of this technique to a Bellman equation is relatively complex, we move in small steps. We first underscore that the collocation technique is a way to “approximate” or to “interpolate” a function. Hence, it can be applied to any function, say to trigonometric or to exponential ones, and not necessarily to a maximum value function. Second, we show how this technique can be used to solve a simple – actually the simplest – differential equation, and finally we deal with a Bellman problem.

To understand the idea underlying this technique, we need to introduce the Weierstrass Theorem, which says:

Theorem 3. *Any continuous real function $f(x)$ can be approximated in a interval X by a polynomial $p_n(x) = \sum_{i=0}^n a_i x^i$.*

The words “can be approximated” mean that for any number $\nu > 0$, there is a polynomial, of suitable degree, such that $\max_{x \in X} |f(x) - p_n(x)| < \nu$. In practice, a reduction in ν , i.e. in the approximation error, usually calls for an increase in the degree of the polynomial.³³

This is interesting and rather intuitive, but it does not tell us how to proceed in practice. Hence, let us consider a specific example, choosing a relatively challenging function: imagine that we need to approximate $\sin(x)$ over the interval $X = [0, 2\pi]$. Let us start with the third degree polynomial, $p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Our problem is to determine the four a_i coefficients so that our polynomial behaves as the $\sin(x)$ function in the interval X . Because we need to determine four unknown coefficients, we choose four points in X . In fact, imposing equality between $\sin(x)$ and $p_3(x)$ at these points, we obtain four equations in the a_i 's, which can be solved for the unknown coefficients.

³³Bear in mind, however, that the relation between ν and the degree of the approximating polynomial is not always monotonic.

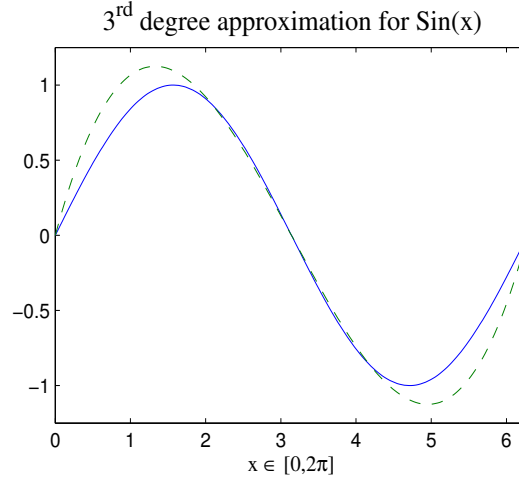


FIGURE 7.

For example, choose $x = \{0, 2/3\pi, 4/3\pi, 2\pi\}$.³⁴ At these points, the collocation method forces the approximating polynomial to be identical to the function which needs to be approximated. This yields:

$$(6.3) \quad \begin{cases} 0 = a_0 \\ 0.86602 = a_0 + a_1 2.09440 + a_2 4.38649 + a_3 9.18704 \\ -0.86602 = a_0 + a_1 4.18879 + a_2 17.54596 + a_3 73.49636 \\ 0 = a_0 + a_1 6.28318 + a_2 39.47842 + a_3 248.05021 \end{cases}$$

Solving for the a_i s this linear system, one obtains that the approximating third-degree polynomial is: $0 + 1.860680x - 0.888409x^2 + 0.094263x^3$.

For a “first impression judgement” of our results, we plot our polynomial and the $\sin(x)$ function over $[0, 2\pi]$. In Figure 7, the continuous line represents $\sin(x)$, while the dashed line is its third degree approximation. Some further calculations show that the largest difference between the two functions is 0.2554. This is a large number, nevertheless we can be fairly satisfied with our exercise: we have been able to approximate a trigonometric function by means of a simple third degree polynomial.

However, we must be aware that our approximation deteriorates rapidly as we exit from the interval $X = [0, 2\pi]$ (compute $p_3(x)$ at $2.5\pi!$). The rapid deterioration of the approximation outside the interval is a general feature

³⁴Using the jargon, we say that we have chosen four evenly spaced “collocation points” or “nodes”.

of the results obtained by means of this approach. The approximation of a function by means of the collocation method is similar to what one can do with some pins and a flexible plastic stick: one takes the stick (the polynomial) and the pins (the value of the polynomial at the collocation points), and use the pins to fix the stick over the function. With a decent number of pins, one can easily make a good job in the interval inside the first and the last pin; outside this interval, the plastic stick keeps its original shape.

It is now time to leave aside once again paper and pencil. Choosing, for example, eleven collocation points, and hence using a tenth degree polynomial, we can easily build an eleven equations system. We can then use the equation matrix manipulation facilities that are built-in in computation programs such as Matlab or Python to solve the system; with Matlab this should take about 60/100 of a second.³⁵ (You can deal with this problem also by means of Mathematica or Maple).

Figure 8 shows the approximation errors for this exercise. An approximation error is defined as the difference between the actual value for $\sin(x)$ and the value taken by the approximating polynomial, for an $x \in [0, 2\pi]$. The bright side is that we are really making a decent job, the somehow dark side is that the errors are definitely higher than average at the beginning and at the end of the interval. This is usually regarded as a clear sign that we can improve our approximation.

It is now possible to climb a step of the ladder that is bringing us to the solution for the dynamic optimization problem. Up to now, we have shown that we can approximate a function by means of a polynomial. Now consider that the solution for a difference or a differential equation *is* a function (in the example we are about to propose, the solution will be a function of time, which shall play the role of state variable). This suggests that we can approximate the solution for a functional equation by means of our collocation technique.

As before, we choose a simple example, actually, we choose a differential equation that we can easily solve analytically. This will allow us to judge the “goodness of fit” characterizing our exercise.

³⁵A system like (6.3) can be written in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2.09440 & 4.38649 & 9.18704 \\ 1 & 4.18879 & 17.54596 & 73.49636 \\ 1 & 6.28318 & 39.47842 & 248.05021 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.86602 \\ -0.86602 \\ 0 \end{bmatrix},$$

it is then easy to let an appropriate software solve it.

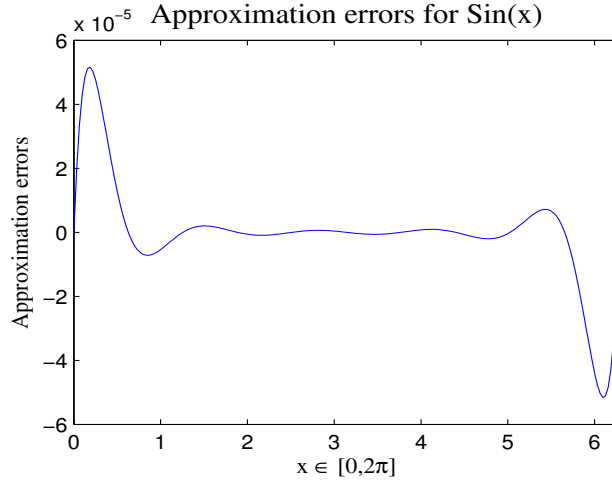


FIGURE 8.

Suppose that we need to solve the first order linear differential equation

$$(6.4) \quad \frac{dx(t)}{dt} = 0.1x(t) + 1, \text{ given } x(0) = 2,$$

in the interval $t \in [0, 4]$. Assume that the solution $x(t)$ must be approximated by a second-degree polynomial: denoting with an upperbar the approximated solution, we have

$$(6.5) \quad \bar{x}(t) = a_0 + a_1t + a_2t^2.$$

A good approximation, behaving “almost like” the true solution $x(t)$, must fulfill equation (6.4). Actually, the true solution must satisfy (6.4) for any t , hence, it is more than reasonable to require that the approximation fulfills (6.4) “at least somewhere”, which means, at the collocation points. There, the approximation must be such that:

$$\frac{d\bar{x}(t)}{dt} = 0.1\bar{x}(t) + 1,$$

which means, exploiting (6.5) in both sides of the last equation, that – at the collocation points – it must be true that:

$$a_1 + 2a_2t = 0.1(a_0 + a_1t + a_2t^2) + 1.$$

To determine the three coefficients we now “collocate” the above equation in *two* points: the third equation for our system is provided by the initial condition $\bar{x}(0) = a_0 = 2$. The two collocation points are the two extrema of

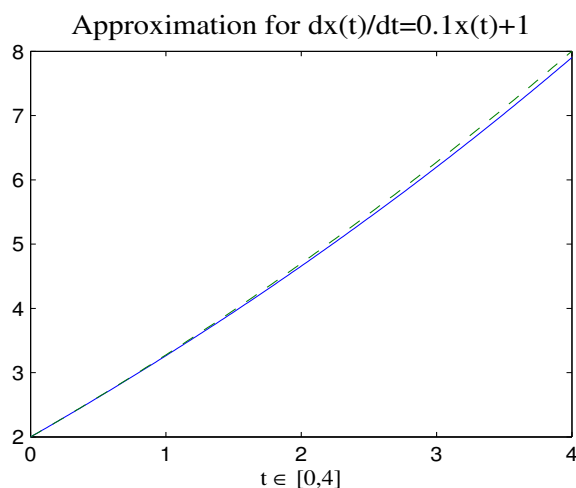


FIGURE 9.

the interval (0 and 4): had we chosen two different points, we would have induced large errors in some portion of the interval of interest.³⁶ Accordingly, the system determining the a_i coefficients is:

$$(6.6) \quad \begin{cases} a_1 = 0.1a_0 + 1 \\ a_1 + 8a_2 = 0.1a_0 + 0.4a_1 + 1.6a_2 + 1 \\ a_0 = 2 \end{cases} .$$

A few calculations suffice to obtain the approximating function:³⁷ $\bar{x}(t) = 2 + 1.2t + 0.075t^2$; in Figure 9 the approximating function is the dashed line, while the continuous line portrays the exact solution (that is: $x(t) = -10 + 12e^{0.1t}$).

We could now compute the approximation errors, as the difference between $x(t)$ and $\bar{x}(t)$. However, if we stop and think for just a second, we realize that the analysis of these errors is of no practical interest: why should we bother with the errors implied by a numerical solution when we can

³⁶We would have large errors for the time (sub-)interval between 0 and the first collocation point, and for that between the second collocation point and 4. There, the “stick” (our second order polynomial in t) would retain its shape.

³⁷A linear system like (6.6) can be written as:

$$\begin{bmatrix} -0.1 & 1 & 0 \\ -0.1 & 0.6 & 6.4 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

and then solved using, for example, a matrix inversion routine.

compute the exact one? This observation brings us close to a crucial point: when we do not know the analytic solution for a functional equation, how can we judge the goodness for the approximation?

The standard answer is grounded again on equation (6.4): because the true function satisfies this equation for any t , a good approximation should do the same. Hence, we compute the “residual function”:

$$R(t) = \frac{d\bar{x}(t)}{dt} - 0.1\bar{x}(t) - 1, \quad t \in [0, 4],$$

and we inspect it. If this function is “almost zero”, the approximation is good.³⁸

In our example, the residual function is given by:

$$R(t) = 1.2 + 0.15t - 0.2 - 0.12t - 0.0075t^2 - 1 = 0.03t - 0.0075t^2.$$

Instead, the approximation errors, computed as the difference between the exact solution and the approximated one, are given by:

$$x(t) - \bar{x}(t) = -10 + 12e^{0.1t} - (2 + 1.2t + 0.075t^2) = 12e^{0.1t} - 12 - 1.2t - 0.075t^2.$$

Figure 10 shows the errors that we make when we let our computer approximate the solution by means of an 8-degree polynomial.³⁹ The continuous line represents the “true” errors, while the residual function is depicted by the dashed line. Errors and residuals are computed for 200 equally spaced points for t in the interval $[0, 4]$.

While the approximation is very good, it can be improved: notice that the residuals are, once again, higher than average at the beginning and especially at the end of the interval, suggesting that some improvement is possible.

Eventually, we are ready to cope with the real problem. We now solve:

$$\begin{aligned} V(k_0) &= \max_{c_0} \{ \ln(c_0) + 0.97V(k_1) \}, \\ \text{s.t. } k_1 &= 3.43643k_0^{0.3} + 0.85k_0 - c_0, \\ &k_0 \text{ given.} \end{aligned}$$

for $k_0 \in [0.5\hat{k}, \hat{k}]$. Notice that this is the same problem we analyzed in Subsection 6.1, but for the fact that the depreciation parameter now takes the

³⁸Obviously, it is important to specify a formal criterion to evaluate the behavior of the residuals. However, at this introductory level of the analysis, we omit the discussion of this point. Nevertheless, be prepared to cope with this issue when dealing with formal research.

³⁹In this exercise, the eight collocation points are equally spaced between 0 and 4.

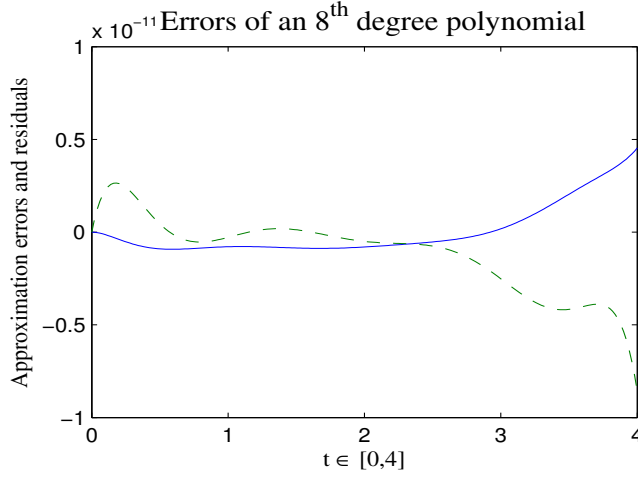


FIGURE 10.

much more sensible value $\delta = 0.15$. Notice also that we are dealing with a quite large interval for the state variable.

Suppose first that the maximum value function is approximated by an ordinary $n - th$ degree polynomial: $V(k) = \sum_{i=0}^n a_i k^i$. Exploiting the first order condition, $1/c_0^* = 0.97V'(k_1)$, one immediately obtains that:

$$c_0^* = \frac{1}{0.97 \sum_{i=1}^n i a_i k_1^{(i-1)}}.$$

Substituting the approximating function and the first order condition above in the original problem, we obtain:

$$\sum_{i=0}^n a_i k_0^i = -\ln(0.97) - \ln \left(\sum_{i=1}^n i a_i k_1^{(i-1)} \right) + 0.97 \sum_{i=0}^n a_i k_1^i.$$

Notice that the unknowns in the equation above are $n + 2$: in fact, these are the $n + 1$ coefficients (the a_i s), and the next-period state, k_1 .

If we couple the equation above with the dynamic constraint, we obtain a two equations system with $n + 2$ unknown, like

$$(6.7) \quad \begin{cases} \sum_{i=0}^n a_i k_0^i = -\ln(0.97) - \ln \left(\sum_{i=1}^n i a_i k_1^{(i-1)} \right) + 0.97 \sum_{i=0}^n a_i k_1^i \\ k_1 = 3.4364k_0^{0.3} + 0.85k_0 - \frac{1}{0.97 \sum_{i=1}^n i a_i k_1^{(i-1)}} \end{cases}.$$

Suppose now to consider *two* level for k_0 (i.e. we consider two collocation points). In this case we have a system of four equations, with $n+3$ unknowns: (the $n+1$ coefficients and two next-period states). Notice that we have moved one step in the direction of the determination of the system: now the number of “missing equation” is $n+3-4=n-1$. Hence, if we choose $n+1$ collocation points (which means $n+1$ values for $k_0 \in [0.5\hat{k}, \hat{k}]$), we have $2n+2$ unknowns (the $n+1$ coefficients and the $n+1$ values for k_1 , one for each collocation point), and we can build a system composed of $2n+2$ equations. In fact, for each of the $n+1$ collocation points we can build a 2-equation system like (6.7), and then we assemble a $2n+2$ system considering all the collocation points. (In other words, system (6.7) is the building block of the large system we need to solve to determine the $n+1$ a_i s coefficients and the $n+1$ k_1 s values).

System (6.7), besides being large, is non linear. To deal with this problem, we can rely on the non-linear equation solver that is embedded into a software like Matlab or Python.⁴⁰

However, a glance at (6.7) is enough to convince oneself that our system is *very* non linear. Hence, one might end up having troubles with the numerical routine.⁴¹

An often used way out is to determine the consumption function starting from the Euler equation, which, in our specific case, readily yields $c_1 =$

⁴⁰If you do not know what a non-linear equation solver is, imagine that – having forgotten the standard formula – you need to solve a second order equation.

In this contingency, you may draw a second order polynomial on a plane, so that the solution(s) you are looking for are the intersection points $f(x) = 0$. To be specific, draw the polynomial upward oriented. This function, $y = f(x)$, intersects the x -axis twice, once, or never. If there are no intersection points, there are no real solutions for the equation $f(x) = 0$. (In this case, you are better drawing another second order polynomial such that it intersects the x -axis.)

Now, pick a value for x and call it x_0 ; this is the “initial condition” for the procedure. Evaluate the function at x_0 , obtaining $y_0 = f(x_0)$.

We can now sketch a naive and oversimplified numerical equation solving procedure. This is made up of a set of instructions like the following ones.

If $f(x_0) > 0$, and $f'(x_0) > 0$, then decrease x_0 by a small amount.

If $f(x_0) > 0$, and $f'(x_0) < 0$, then increase x_0 by a small amount.

If $f(x_0) < 0$, and $f'(x_0) > 0$, then increase x_0 by a small amount.

If $f(x_0) < 0$, and $f'(x_0) < 0$, then decrease x_0 by a small amount.

Notice that, following the instructions above, one moves toward a solution for the second order polynomial. One should apply the above set of instructions over and over again, until a solution $f(x) = 0$ is reached.

Notice also that – whenever the non-linear equation admits more than a solution, which can well happen in our case – the initial condition is crucial to determine which of the possible solutions is selected. Finally, notice that the initial condition is relevant to determine the time needed to converge to the solution. Miranda and Fackler [2002] and Judd [1998] provide compact treatments of techniques used in computational economics to find the solution of a system of nonlinear equations.

⁴¹Although it is possible to solve a system like (6.7), one often needs to feed the non-linear equation solver with a very precise initial condition, and this can be unpleasant.

$c_0 (k_1^{-0.7} + 0.8245)$. Assuming that $c_t = h(k_t) = \sum_{i=0}^n d_i k_t^i$, we have, for each collocation point, a system like:

$$(6.8) \quad \begin{cases} \sum_{i=0}^n d_i k_1^i = \sum_{i=0}^n d_i k_0^i (k_1^{-0.7} + 0.8245) \\ k_1^i = 3.43643k_0^{0.3} + 0.85k_0 - \sum_{i=0}^n d_i k_0^i \end{cases},$$

The system above involves two equations and $n + 2$ unknowns (the $n + 1$ coefficients (the d_i s) and the next-period state, k_1 , which is related to the k_0 characterizing our collocation point).

It is tempting to conclude that system (6.8) is the building block of a $(2n + 2)$ -equations system. In other words, it is tempting to conclude that one can now choose $n + 1$ collocation points, and solve a $(2n + 2)$ -equation system (composed of $n + 1$ blocks like (6.8)) for the $n + 1$ k_1^i s and for the $n + 1$ d_i s. This is *not* the best way to carry on. In fact, in this way we omit to consider an important information, which is the knowledge that, in the steady state,

$$(6.9) \quad \hat{c} \left(= \sum_{i=0}^n d_i \hat{k} \right) = 3.43643 \hat{k}^{0.3} - 0.15 \hat{k}.$$

Loosely speaking, if we do not consider this information, our routine can give us a solution characterizing consumption on one of the (infinite) non optimal paths where the Euler equation is satisfied (refer back to Figure 3). To encompass the piece of information provided by the steady state, we choose n collocation points for a n^{th} degree polynomial. In this way, we have $2n + 1$ unknowns ($n + 1$ coefficient plus n k_1^i s) and $2n$ equations (from system (6.8) we obtain two equations for each of the n collocation point), and the steady state equation above closes the system, which can actually be solved rather easily. Figure 11a and 11b show the consumption function and the residuals.⁴² These have been obtained using an eight degree polynomial; the computation time is about 12/100 of a second.

Our exercises have been based on ordinary polynomials. This choice is usually considered inefficient: ordinary polynomials have some unpleasant characteristics. As already underscored, the residuals are often concentrated in sub-intervals of the approximation interval; more importantly, it may happen that the approximation error rises, rather than falls, with the number of collocation points.

⁴²The residuals have been obtained building a grid of 200 equally spaced points for capital in the interval $[0.5\hat{k}, \hat{k}]$, obtaining for each point on the grid the corresponding next-period value using the second equation in (6.8), and then computing the difference, for each point, between the right- and the left-hand side of the first equation in (6.8)

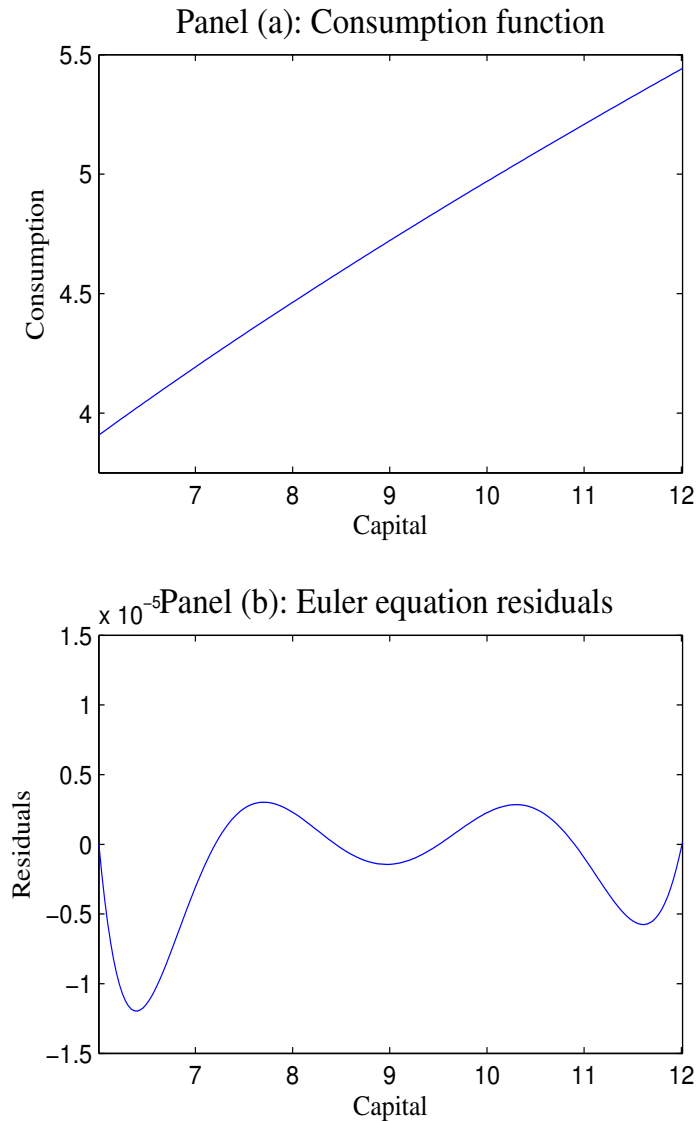


FIGURE 11.

Notice also that we have always chosen evenly spaced collocation nodes. There is no reason to believe that this is the optimal choice for collocation points, nor the Weierstrass theorem provides any guidance on how to determine these points.

In practice, an often used procedure is to adopt a particular type of polynomials, the Chebychev polynomials, which are associated to non-evenly spaced nodes (the Chebychev collocation points). This procedure allows to cope with both the problems we have mentioned above. Having understood

the logic of the collocation technique, it should not be a big problem to apply it by means of Chebychev nodes and polynomials, following an advanced textbook such as Judd [1998] or Miranda and Fackler [2002].

Also, having understood the collocation technique, it should not be prohibitive to deal with “finite element” methods. While the collocation technique uses as interpolating function a polynomial which can take non-zero values over the entire approximation interval, a finite element method uses interpolating functions that are *defined* to be zero in large part of the approximation interval, while they are assumed to be a (low degree) polynomial in a specific sub-interval of the approximation interval. Refer, again to Judd [1998] or to Miranda and Fackler [2002].

The logic of the collocation methods bears some resemblance also with the one at the basis of the “minimum weighted residuals” technique. In fact, this method is based on the choice of some nodes and of an interpolating function, too. In this case, however, the coefficients of the polynomial are chosen in a way to minimize the (weighted) differences between the value of the function and the polynomial at the nodes. On this, see again Judd, or McGrattan [1999]

7. A LAGRANGIAN APPROACH TO INFINITE-HORIZON STOCHASTIC GROWTH

In our lives, very few things (if any!) can be hold as “completely sure”. Accordingly, in our representative–agent growth model, the consideration of some form of risk is an important step in the direction of realism. Potentially, in our framework, many things can vary stochastically over time: productivity, depreciation rate, preferences.... In this Section, we introduce a single specific element of risk by means of a simple example, that shall be solved through the Lagrangian approach; in the next section we tackle a more general framework using dynamic programming.⁴³

In the example we use to introduce the topic, we consider a Cobb-Douglas production, $A_t k_t^\alpha$, in which the productivity parameter A_t is stochastic. In particular, we assume that A_t can take two values, A_t^H and A_t^L , which occur with probabilities p^H and p^L ($\equiv 1 - p^H$), respectively. Of course, the superscript H stands for high, while L stands for low (hence $A_t^H > A_t^L$). We also assume that the probability for each realization is time independent, and that the two values, A_t^H and A_t^L , do not change over time. This means that the random variable A_t is identically distributed and independent over

⁴³We stick to the standard definition, dating back to the works by Frank Knight in the twenties, according to which there is risk when the realizations for the random variables have measurable (and known) probabilities, while uncertainty implies that the possible outcomes have no measurable probability.

time. Notice, in passing, that although we could take as understood the subscript t , we prefer *not* to simplify the notation, in the attempt to be clearer.

In short, output is obtained by means of the stochastic production function:

$$(7.1) \quad y_t = A_t k_t^\alpha = \begin{cases} A_t^H k_t^\alpha, & \text{with probability } p_H \\ A_t^L k_t^\alpha, & \text{with probability } 1 - p_H \end{cases} .$$

It is important to remark that period t productivity is supposed to be known by the representative agent when she decides upon her period t consumption.

Preferences are logarithmic, and they do not vary over time. Labeling as before the present as period 0 to save on notation, our consumer's preferences can be summarized by the following intertemporal utility function:

$$(7.2) \quad W_0 = \sum_{t=0}^{\infty} \beta^t \ln(c_t).$$

Notice that the representative agent's objective is to maximize the expected value of (7.2), i.e.

$$E_0 [W_0] = E_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right],$$

where by $E_0 [.]$ we denote the expectation conditional on the time 0 information set. Notice that—even in this simple case—writing in details the objective function is challenging. Its first addendum is simply $\ln c_0$; as for period 1 we can write: $\beta (p^H \ln c_1^H + p^L \ln c_1^L)$, where c_1^H and c_1^L are the consumption levels that will be set in place in the next period, depending on the period 1 realization for productivity. At time 2 the situation begins to become fairly complex. In fact, the period 2 consumption levels will depend not only on that period realization for productivity, but also on the productivity level that prevailed at time 1. The latter productivity influences period 1 consumption and, in that way, the period 2 capital level. Accordingly, we should distinguish four cases. Following this line of reasoning, one would get lost fairly soon: the number of cases to be taken into account grows exponentially with the time period!⁴⁴ Indeed, this effort would be not only huge, but also useless: our representative agent's problem is to choose

⁴⁴Bear in mind that we are considering the simplest possible case: usually, one would like to deal with random variables with more than two realization per period. Moreover, productivity is not time-independent, but highly correlated. Taking into account these relevant aspects would make the problem much more intricate.

the c_t s in a “sequential” way: i.e., she chooses a c_t knowing k_t , and having observed the time t realization for productivity. Hence, in every period t , what matters is only the next-period technological risk (which is limited to the random occurrence of the two states A_{t+1}^H , and A_{t+1}^L).⁴⁵

Accordingly, we can write the representative consumer’s problem by forming the present value stochastic Lagrangian in a way that underscores the “sequential” nature for our problem.

$$\begin{aligned}
 L_0^i &= \ln c_0^i - \lambda_0^i (k_1 - A_0^i k_0^\alpha + c_0^i) + \\
 &+ E_0 [\beta \ln c_1^i - \beta \lambda_1^i (k_2 - A_1^i k_1^\alpha + c_1^i) + \\
 &+ E_1 [\beta^2 \ln c_2^i - \beta^2 \lambda_2^i (k_3 - A_2^i k_2^\alpha + c_2^i) + \\
 (7.3) \quad &+ E_2 [\beta^3 \ln c_3^i - \beta^3 \lambda_3^i (k_4 - A_3^i k_3^\alpha + c_3^i) + \\
 &+ \dots \text{ }]]] \\
 &+ \lim_{t \rightarrow \infty} E_0 [\beta^t \lambda_t^i k_t]
 \end{aligned}$$

Some remarks are in order. First, at time 0 the representative agent decides upon her consumption level and upon her period 1 resources (k_1), knowing the level of available resources (i.e. k_0) and the current state of productivity. Hence, the Lagrangian depends on current productivity: one can write a Lagrangian for every productivity level. This is why we have denoted the Lagrangian with a superscript i , where $i = \{H, L\}$.

Second, at time 1, our representative agent decides upon her period 1 consumption level and upon k_2 , knowing the period 1 productivity level and k_1 . *At that time*, the agent maximizes:

$$\begin{aligned}
 &\beta \ln c_1^i - \beta \lambda_1^i (k_2 - A_1^i k_1^\alpha + c_1^i) + \\
 &+ E_1 [\beta^2 \ln c_2^i - \beta^2 \lambda_2^i (k_3 - A_2^i k_2^\alpha + c_2^i) + E_2 [\dots]].
 \end{aligned}$$

Notice that, at time 0 the representative agent is aware of the fact that—in period 1—she will be able to choose a consumption level contingent on the future realization for productivity (and, of course, on k_1). Accordingly, what we shall do is to consider, first, the optimality conditions for period 1 *from the perspective of that period*, and then we shall consider the implications of these first order conditions from the vantage point of period 0. Of course, this line of reasoning is applied also to the subsequent periods.

⁴⁵In the jargon, our problem is of the “closed loop” type and not of the “open loop” kind, as it would have been if the values for the entire sequence(s) of the control variable(s) had been decided upon at time 0. For some details, see Chow [1997], Chapter 2.

Third, notice that from period 1 onward, the Lagrange multipliers become stochastic variables: they represent the marginal evaluations of consumption, that is in itself stochastic, depending on the realization for productivity.

Finally, bear in mind the last line in (7.3) is the stochastic version for the tvc (compare with Eq. (2.6)).⁴⁶

What we do now is to consider, first, the period 0 first order conditions; we then obtain the optimality conditions for period 1 from the perspective of that period, and finally we consider the implications of the period 1 optimality conditions from the perspective of the initial period.

The variables decided upon in period 0 are: c_0^i , k_1 , and λ_0^i . The first order conditions are, respectively:

$$(7.4a) \quad \frac{1}{c_0^{i*}} - \lambda_0^i = 0,$$

$$(7.4b) \quad \lambda_0^i - \alpha\beta (p^H \lambda_1^H A_1^H + p^L \lambda_1^L A_1^L) k_1^{\alpha-1} = 0,$$

$$(7.4c) \quad k_1 - A_0^i k_0^\alpha + c_0^i = 0.$$

Notice, that, in period 0, the productivity realization is known, hence the representative agent computes just *one* set of first order conditions like (7.4), i.e. it computes them for $i = H$ or for $i = L$. The same happens when our agent optimizes at time 1 (with respect to c_1^i , k_2 , and λ_1^i). However, we have to consider the first order conditions for any productivity realization (i.e. for $i = \{H, L\}$). This is because we shall consider the implications of these first order conditions from the vantage point of time 0, when the realization for productivity is still unknown. The period 1 first order conditions are:

$$(7.5a) \quad \frac{1}{c_1^{i*}} - \lambda_1^i = 0,$$

$$(7.5b) \quad \lambda_1^i - \alpha\beta (p^H \lambda_2^H A_2^H + p^L \lambda_2^L A_2^L) k_2^{\alpha-1} = 0,$$

$$(7.5c) \quad k_2 - A_1^i k_1^\alpha + c_1^{i*} = 0,$$

for $i = \{H, L\}$.

⁴⁶In writing Problem (7.3), we have used the law of iterated expectations. This law states that $E_t [E_{t+n} [X_t]] = E_t [X_t]$, where X_t is a random variable and $n \geq 1$. In plain English, the law of iterated expectations tells you that what you expect today about the day after tomorrow, must be equal to what you think today that you are going to expect tomorrow about the day after tomorrow (Otherwise, you should change your current expectation!). Suppose the day you graduate, you decide to accept a job because you expect that your gross yearly income will be 70.000 € in four years time. Clearly, it is not sensible that, the very day of your graduation, you believe that the following year you will expect to earn, say, 50.000 € in three years time.

Let us now consider that, in period 0, the representative agent is aware that in period 1 she will choose her consumption knowing the realization for productivity; moreover, she is also aware that these consumption levels will be decided upon having the same information (apart from productivity) that are available at time 0. In fact, besides being contingent on productivity, the period 1 consumption levels depend on k_1 , which, however is decided upon (and hence known!) at time 0. Hence, in period 0, the unique risky element that the representative agent faces when she decides upon her consumption, is time 1 productivity. Exploiting (7.5a) into (7.4b), and then taking advantage of (7.4a) gives:

$$(7.6) \quad \frac{1}{c_0^{i*}} = \alpha\beta \left(p^H \frac{1}{c_1^{H*}} A_1^H + p^L \frac{1}{c_1^{L*}} A_1^L \right) k_1^{\alpha-1},$$

which can be compactly written as:

$$\frac{1}{c_0^{i*}} = \alpha\beta E_0 \left[\frac{1}{c_1^*} A_1 \right] k_1^{\alpha-1}.$$

This is the period 0 version for the Euler equation, which relates the current marginal utility for consumption to the next period marginal utility, which is stochastic, depending upon productivity.

The first order conditions (7.5a-7.5c) are written from the vantage point of time 1. At the initial time 0 our representative consumers considers that, given the available information, it must be true that:

$$(7.7a) \quad E_0 \left[\frac{1}{c_1^{i*}} - \lambda_1^i \right] = 0,$$

$$(7.7b) \quad E_0 \left[\lambda_1^i - \alpha\beta (p^H \lambda_2^H A_2^H + p^L \lambda_2^L A_2^L) k_2^{\alpha-1} \right] = 0,$$

$$(7.7c) \quad E_0 \left[k_2 - A_1^i k_1^\alpha + c_1^{i*} \right] = 0.$$

In fact, each of the conditions (7.5a-7.5c) must be true for every realization for productivity, and hence *a fortiori*, they must hold in expected value.

We can now substitute the Lagrange multipliers λ_2^H and λ_2^L out of Eq. (7.7b), using the time 2 version of Eq. (7.5a). This gives the stochastic counterpart of Eq. (1.9) for period 1, which is:

$$E_0 \left[\frac{1}{c_1^*} \right] = \alpha\beta E_0 \left[\frac{1}{c_2^*} A_2^i k_2^{\alpha-1} \right].$$

Using this type of reasoning, one obtains that, in general, the Euler equation is:

$$(7.8) \quad E_0 \left[\frac{1}{c_t^*} \right] = \alpha\beta E_0 \left[\frac{1}{c_{t+1}^*} A_{t+1}^i k_{t+1}^{\alpha-1} \right],$$

which tells us that the period t expected marginal utility of consumption must be equal to the discounted expectation of the product between period $t + 1$ marginal utility of consumption and productivity of capital. The fact that the representative agent takes into account the expectation of this product makes economic sense. In fact, for a given current consumption, a high future marginal productivity of capital induces an increase in future consumption (because of the high output level), and hence a reduction in its marginal utility. On the contrary a low future marginal productivity of capital reduces future consumption, and hence increases its marginal utility. These effects must be taken into account by the representative agent when she has to decide her current consumption and hence her savings (which should increase when she is sufficiently risk-averse).

It is easy to see that it must also be true that

$$(7.9) \quad E_0 [k_{t+1}] = E_0 [A_t^i k_t^\alpha] - E_0 [c_t^{i*}],$$

(to obtain the above equation, simply consider the appropriate version of Eq. (7.7c)).

We now work out the details of the explicit solution for the present example.

To determine the consumption levels, it is necessary to guess the consumption function. Our tentative solution is $c_0^{i*} = \chi A_t^i k_0^\alpha$ where χ is an undetermined parameter. Notice that Eq. (7.6) must pin down the *same* χ for any i : this is precisely due to the fact that χ – being a constant – must be independent from the realization for A_t , and therefore from the initial state we pick.

Choosing, for example, Eq. (7.6) with $i = H$, we obtain:

$$\frac{1}{\chi A_0^H k_0^\alpha} = \alpha\beta \left(\frac{p^H A_1^H}{\chi A_1^H k_1^\alpha} + \frac{p^L A_1^L}{\chi A_1^L k_1^\alpha} \right) k_1^{\alpha-1}.$$

Hence, simplifying when possible,

$$\frac{1}{A_0^H k_0^\alpha} = \alpha\beta (p^H + p^L) k_1^{-1}.$$

Notice that, if $c_0^{H*} = \chi A_0^H k_0^\alpha$, then $k_1 = (1 - \chi) A_0^H k_0^\alpha$; hence the above equation reduces to:

$$\frac{1}{A_0^H k_0^\alpha} = \alpha\beta \frac{1}{(1-\chi)A_0^H k_0^\alpha},$$

where we exploit the fact that $p^H + p^L = 1$. The above equation is satisfied for $\chi = (1 - \alpha\beta)$. Therefore, we have an explicit formulation for the consumption function, which is:

$$(7.10) \quad c_t^{i*} = (1 - \alpha\beta)A_t^i k_t^\alpha.$$

The policy function above makes it clear that the optimal consumption is chosen conditionally on the value assumed by the state variables, capital and productivity.⁴⁷

For completeness, we now check that our solution satisfy the stochastic Euler equation (7.8).

Using Eq. (7.10) into (7.8), we obtain:

$$\begin{aligned} E_0 \left[\frac{p^H}{(1-\alpha\beta)A_t^H k_t^\alpha} + \frac{p^L}{(1-\alpha\beta)A_t^L k_t^\alpha} \right] &= \\ = \alpha\beta E_0 \left[p^H \frac{1}{(1-\alpha\beta)A_{t+1}^H k_{t+1}^\alpha} A_{t+1}^H k_{t+1}^{\alpha-1} + p^L \frac{1}{(1-\alpha\beta)A_{t+1}^L k_{t+1}^\alpha} A_{t+1}^L k_{t+1}^{\alpha-1} \right]. \end{aligned}$$

Simplifying where possible, we obtain:

$$E_0 \left[\frac{p^H}{A_t^H k_t^\alpha} + \frac{p^L}{A_t^L k_t^\alpha} \right] = \alpha\beta E_0 \left[\frac{p^H}{k_{t+1}} + \frac{p^L}{k_{t+1}} \right].$$

Because, in any state i , $k_{t+1} = A_t^i k_t^\alpha - c_t^{i*}$, and $c_t^{i*} = (1 - \alpha\beta)A_t^i k_t^\alpha$, then $k_{t+1} = \alpha\beta A_t^i k_t^\alpha$. Hence, the above equation becomes:

$$E_0 \left[\frac{p^H}{A_t^H k_t^\alpha} + \frac{p^L}{A_t^L k_t^\alpha} \right] = \alpha\beta E_0 \left[\frac{p^H}{\alpha\beta A_t^H k_t^\alpha} + \frac{p^L}{\alpha\beta A_t^L k_t^\alpha} \right],$$

which is obviously verified.

Exercise 10. *Modify the example in Section 7, assuming that*

$$W_0 = \sum_{t=0}^3 \beta^t \ln c_t,$$

and find the consumption function.

Exercise 11. *Modify the example in Section 7, assuming that the time $t+1$ realization for productivity depends upon its time t realization. In particular, assume that $\Pr(A_{t+1}^H | A_t^H) = \Pr(A_{t+1}^L | A_t^L) = p$, and that $\Pr(A_{t+1}^H | A_t^L) = \Pr(A_{t+1}^L | A_t^H) = 1 - p$. Find the consumption function.*

⁴⁷You can easily verify that, choosing $i = L$, Eq. (7.6) gives the same value for χ .

Exercise 12. In the example in Section 7, use the guess $c_0^i = \Psi(A_t^i)k_0^\alpha$ (where the functional relation $\Psi(A_t^i)$ between consumption and productivity is assumed to be unknown) and find the consumption function (i.e. show that $\Psi(A_t^i)$ must be linear).

Exercise 13. Assume that: i) the return function is: $c_t^{1-\gamma}/(1-\gamma)$, where $\gamma \in [0, 1) \cup (1, \infty)$; ii) the dynamic constraint is $k_{t+1} = A_t^i k_t - c_t$, in which the structure for the productivity shocks is the one described in Section 7. Find the consumption function.

8. THE BELLMAN FORMULATION FOR THE STOCHASTIC PROBLEM

We now apply the dynamic programming approach to a simple stochastic version of the growth model. As before, we shall consider stochastic the productivity parameter, but our reasoning applies to any form of risk that preserves the additive separability of the objective function.⁴⁸

Assuming that productivity is stochastic, we can formulate the consumer's intertemporal problem as:

$$(8.1) \quad \begin{aligned} V(k_0, A_0) &= \max_{c_0} \{U(c_0) + \beta E_0 [V(k_1, A_1)]\}, \\ \text{s.t. } k_1 &= A_0 f(k_0) + (1 - \delta)k_0 - c_0, \\ &A_0, k_0 \text{ given.} \end{aligned}$$

Problem (8.1) is not fully specified. In fact, we should discuss how productivity evolves over time. More precisely, we should qualify the stochastic process characterizing productivity. However, we postpone this discussion for a while: if you wish, for now you can imagine that the productivity process is independent over time (i.e. that A_{t+1} is independent from A_t for any t).

Importantly, notice that the maximum value function in (8.1) has both capital and random productivity as its argument: as in the deterministic case, the maximum value function depends on the states of the system.

The Bellman equation in (8.1) tells us that, to obtain the period 0 value function $V(k_0, A_0)$, it is necessary to maximize, with respect to c_0 , the expression $U(c_0) + \beta E_0 [V(k_1, A_1)]$. Accordingly, a necessary condition for a

⁴⁸For example, one can deal with stochastic preferences of the type: $W_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, \epsilon_t)$, where ϵ_t is a sequence of random variables. In fact, we may write $W_0 = U(c_0, \epsilon_0) + \beta \sum_{t=1}^{\infty} \beta^t U(c_t, \epsilon_t)$, which forms the basis for the Bellman's formulation.

maximum is obtained differentiating $U(c_0) + \beta E_0 [V(k_1, A_1)]$ with respect to current consumption, which gives:

$$(8.2) \quad U_c(c_0^*) + \beta E_0 [V_k(k_1, A_1)] \frac{\partial k_1}{\partial c_0} = 0,$$

where, $\partial k_1 / \partial c_0 = -1$, from the capital accumulation equation. We can exploit the transition equation in (8.1), the Bellman equation and the first order condition (8.2) to form the system:⁴⁹

$$\left\{ \begin{array}{l} V(k_0, A_0) = U(c_0^*) + \beta E_0 [V(k_1, A_1)], \\ k_1 = A_0 f(k_0) + (1 - \delta)k_0 - c_0^*, \\ U_c(c_0^*) = \beta E_0 [V_k(k_1, A_1)], \\ A_0, k_0 \text{ given.} \end{array} \right.$$

Our task is to find the solution for the above problem. As in the deterministic case there are two ways to proceed: we can (try to) solve the problem “guessing” the solution, or we can use numerical techniques. Whenever useful, in finding a solution we can exploit the envelope theorem, which in this context implies:

$$V_k(k_0, A_0) = \beta E_0 [V_k(k_1, A_1)] \frac{\partial k_1}{\partial k_0}.$$

The proof for this result is trivial, and it is left to the reader, who can also work out how to apply the Benveniste and Scheinkman formula.

Before applying the techniques and the results mentioned above to the solution of problem (8.1), we should specify under which conditions the solution for our problem exists and is unique. In stochastic settings, this turns out to be very difficult. In practice, what people often do (in addition to verify that the conditions spelled out in Theorem 1 or in Theorem 2 are fulfilled) is to check that the stochastic process playing a role in their model enjoys the “Markov property”.⁵⁰

Here, we remind what a Markov process is, and then we explain why it is important to restrict our attention to dynamic optimization problems in which the stochastic disturbances belong to this class.

Definition 1. (*Markov process*). *A stochastic process x_t is Markov if for every \bar{x} and for every period $0, 1, 2, \dots, t, \dots$ we have*

$$\Pr \{x_{t+1} \leq \bar{x} | x_t, x_{t-1} \dots x_0\} = \Pr \{x_{t+1} \leq \bar{x} | x_t\}.$$

⁴⁹Alternative formulation exploits the transition equation and/or the first order condition to substitute out c_0^* and/ or k_1 .

⁵⁰For a much more detailed introduction to the issue we refer again to Thompson [2004]. Lucas, Stokey and Prescott [1989] is the key reading.

In words, a Markov process is a random process whose future probabilities are determined by its most recent realization. In fact, the above definition tells us that the probability that the next period value for the process is below a given threshold (i.e. that $x_{t+1} \leq \bar{x}$), depends only on the current realization of the process, x_t . Hence, the past realizations (x_{t-1}, \dots, x_0) are actually irrelevant for the determination of $\Pr \{x_{t+1} \leq \bar{x}\}$. Sometimes it is said that, with Markov processes, “history does not matter” because the current value x_t is all what is needed to compute future probabilities, and it does not matter how variable x got there. Alternatively, we can think to a Markov process as a sequence of random variables for which “memory does not matter”: what we need to know is just x_t , the state of the process, and we do not need to recall the past realizations.

The fact that the random disturbances belong to Markov processes represents an enormous simplification for a dynamic optimization problem. In fact, the maximum value and the policy function include, among their arguments – besides the state variables we would find also in the non-stochastic version of the problem – only the most recent realization for the random variables. In other words, the most recent known realization for the random variables represents the *unique* additional states of the system. The random variables considered in the examples in the previous Section and in the exercises obviously belong to Markov process.⁵¹

9. GUESS AND VERIFY, IN TWO DIMENSIONS

9.1. Logarithmic preferences and stochastic Cobb-Douglas production . In this Sub-section, we incorporate a persistent productivity shock into the version of the Brock and Mirman’s (1972) optimal growth model that we considered in Section 7, and we show how to obtain its closed form solution.

The production function is Cobb-Douglas, characterized by a depreciation parameter as high as unity, while the stochastic productivity A_t evolves according to:

$$(9.1) \quad \ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1},$$

where $\rho \in [0, 1]$ is the “auto-regressive parameter”, and ϵ_{t+1} is a random variable that is time independent and identically distributed over time. This means that ϵ_{t+1} is not influenced by $\epsilon_t, \epsilon_{t-1} \dots$ (hence $\ln A_t$ is a Markov process), and that the characteristics of ϵ_{t+1} do not change over time. If

⁵¹When the random variables in a Markov process can take only a finite number of values, then this process is called a Markov chain. When the random variables in a Markov process are continuous, then the process is known as a Markov sequence. A dynamic optimization problem in which the stochastic variables belong to Markov processes and the payoff enter additively is known as a Markov decision process.

one wishes, one can conceive ϵ_{t+1} as a normal random variable with mean 0 and variance σ_ϵ^2 . Notice that ϵ_{t+1} represents the innovation in the stochastic process (9.1). While some auto-regression in productivity is highly realistic, the structure for A_t postulated in (9.1) may seem rather *ad hoc*: it requires that it is the logarithm of productivity that depends on its past realization. Notice, however, that the particular structure in (9.1) grants that productivity never becomes negative for any realization of the innovation, which makes sense. Moreover, the structure in (9.1) is an essential ingredient to obtain a closed form solution, and hence for now we must live with it.

The consumer's side of the problem is unchanged, hence her preferences are given by (4.1).

As before, the period t productivity is supposed to be known by the representative agent when she takes her period t decisions.

Our problem is now to solve:

$$\begin{aligned} V(k_0, A_0) &= \max_{c_0} \{ \ln c_0 + \beta E_0 [V(k_1, A_1)] \}, \\ \text{s.t. } k_1 &= A_0 k_0^\alpha - c_0, \\ \ln A_1 &= \rho \ln A_0 + \epsilon_1, \\ A_0, k_0 &\text{ given.} \end{aligned}$$

We now introduce the tentative solution, which takes the form:

$$(9.2) \quad V(k_t, A_t) = F + G \ln k_t + H \ln A_t.$$

It is worth emphasizing that because the value function depends on two state variables, it is necessary to specify a solution involving the two states. This simple but important fact will remain true also when we deal with numerical solutions.

The first order condition is $1/c_0^* = \beta E_0 [V_k(k_1, A_1)]$, which rapidly gives: $k_1 = \beta G c_0^*$. This expression can be used to obtain c_0^* from the dynamic constraint:

$$(9.3) \quad c_0^* = \frac{1}{1 + \beta G} A_0 k_0^\alpha.$$

This equation implies that the next period capital stock is:

$$k_1 = \frac{\beta G}{1 + \beta G} A_0 k_0^\alpha.$$

The above equation, Eq. (9.3), and the tentative solution (9.2) must be substituted back into the Bellman equation, which gives:

$$\begin{aligned}
F + G \ln k_0 + H \ln A_0 &= \\
&= \ln \left(\frac{1}{1 + \beta G} A_0 k_0^\alpha \right) + \beta E_0 \left[F + G \ln \left(\frac{\beta G}{1 + \beta G} A_0 k_0^\alpha \right) + H \ln A_1 \right].
\end{aligned}$$

Exploiting the multiplicative property of logarithmic functions, we obtain:

$$\begin{aligned}
F + G \ln k_0 + H \ln A_0 &= \\
&= \ln \left(\frac{1}{1 + \beta G} \right) + \ln(A_0 k_0^\alpha) \\
&\quad + \beta F + \beta G \ln \left(\frac{\beta G}{1 + \beta G} \right) + \beta G \ln(A_0 k_0^\alpha) + \beta H E_0 [\ln A_1],
\end{aligned}$$

and therefore:

$$\begin{aligned}
F + G \ln k_0 + H \ln A_0 &= \\
&= -\ln(1 + \beta G) + \ln A_0 + \alpha \ln k_0 \\
+ \beta F + \beta G \ln \beta G - \beta G \ln(1 + \beta G) + \beta G \ln A_0 + \alpha \beta G \ln k_0 + \beta H E_0 [\ln A_1].
\end{aligned}$$

Notice that we have taken advantage of the assumption according to which the period 0 realization for A_t is known at the time of choosing period 0 consumption. We now exploit (9.1) to substitute, in the equation above, $\beta H \rho \ln A_0$ for $\beta H E_0 [\ln A_1]$.

Having substituted out $E_0 [\ln A_1]$, we recall that the resulting equation must be satisfied for *any* k_0 , for *any* A_0 , and for *any* admissible value of the parameters β and α . Hence, it must be that:

$$\begin{cases} G = \alpha + \alpha \beta G \\ H = 1 + \beta G + \beta H \rho \\ F = -\ln(1 + \beta G) + \beta F + \beta G \ln \beta G - \beta G \ln(1 + \beta G) \end{cases}.$$

From the first equation in the system above we obtain:

$$G = \frac{\alpha}{1 - \alpha \beta}.$$

Exploiting this result in the second equation of the system, we get:

$$H = \frac{1}{(1 - \rho \beta)(1 - \alpha \beta)}.$$

Notice that $G, H > 0$, because $\alpha, \beta \in (0, 1)$ and $\rho \in [0, 1]$. This implies that $V_k(k_t, A_t), V_A(k_t, A_t) > 0$: a wealthier consumer enjoys an higher overall utility, so does a consumer who lives in a more productive economic

environment. Notice also that the higher ρ , the larger is $V_A(k_t, A_t)$: when productivity is more persistent, its increase has a stronger positive impact on the overall value simply because it displays its effects for a longer time span.

Substituting G in the third equation gives:

$$F = \frac{1}{1-\beta} \left(\frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \ln(1-\alpha\beta) \right).$$

Because F , G , and H are independent from capital and productivity, our guess (9.2) is verified, and the maximum value function is:

$$V(k_t, A_t) = \frac{1}{1-\beta} \left(\frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \ln(1-\alpha\beta) \right) + \frac{\alpha}{1-\alpha\beta} \ln(k_t) + \frac{1}{(1-\rho\beta)(1-\alpha\beta)} \ln A_t.$$

The value function above is strikingly similar to the one we obtained in Sub-section (4.1): this is an effect of the functional forms we have chosen to describe preferences, production, and the evolution for productivity.

It is now easy to obtain the consumption function. Exploiting G in equation (9.3), we obtain $k_1 = \alpha\beta A_0 k_0^\alpha$, hence $c_0^* = (1-\alpha\beta)A_0 k_0^\alpha$, as in the non-stochastic case. This is a neat but slightly disappointing result, because optimal consumption turns out to be, again, a linear function of output.

Exercise 14. Find the value function, and the consumption function for the example in Section 7.

Exercise 15. Assume that: i) the return function is: $\ln c_t$; ii) the dynamic constraint is $k_{t+1} = A_t^i k_t^\alpha - c_t$; iii) there are two productivity levels, A_t^H and A_t^L , iv) $\Pr(A_{t+1}^H | A_t^H) = \Pr(A_{t+1}^L | A_t^L) = p$, $\Pr(A_{t+1}^H | A_t^L) = \Pr(A_{t+1}^L | A_t^H) = 1-p$. Find the value function, and the consumption function.

Exercise 16. Assume that: i) the return function is: $c_t^{1-\gamma}/(1-\gamma)$, where $\gamma \in [0, 1) \cup (1, \infty)$; ii) the dynamic constraint is $k_{t+1} = A_t^i k_t - c_t$, in which the structure of the productivity is the one described in Section 7. Find the value function, and the consumption function.

Exercise 17. Assume that i) the single period utility function is: $\ln(c_t) + \gamma \ln(1-l_t)$, where l_t is labour time and $\gamma > 0$; ii) the dynamic constraint is $k_{t+1} = A_t k_t^\alpha l_t^{1-\alpha} - c_t$, where $0 < \alpha < 1$, and A_t is described by Eq. (9.1). Find the value function, and the related policy functions.

Exercise 18. (Cake eating with taste shocks). Assume that i) the return function is: $\ln(z_t c_t)$, where z_t is a random variable that can assume two

values, z^H and z^L with probabilities p^H and $1-p^H$; ii) the dynamic constraint is $y_{t+1} = y_t - c_t$.⁵² Find the value function, and the consumption function.

9.2. Quadratic preferences with a linear constraint. When the return function is quadratic, the time horizon is infinite and the transition function is linear and affected by additive shocks, a closed form solution for the maximum value function exists.

To be specific, assume that preferences are given by Equation (4.8), and that the intertemporal constraint, which is linear in the state variable k_t , is given by

$$(9.4) \quad k_{t+1} = (1+r)k_t - c_t + \epsilon_t,$$

with ϵ_t representing a random variable which is independent and identically distributed over time.

The logic of the solution method is the same we have experienced in Subsections (4.2) (and in 9.1): the tentative solution takes the same functional form of the return function. Since it is quadratic, our tentative solution is a second order polynomial in the state variables (k_t and ϵ_t), namely:

$$V(k_t, \epsilon_t) = F + Gk_t + Hk_t^2 + I\epsilon_t + L\epsilon_t^2 + Mk_t\epsilon_t,$$

where F , G , H , I , L , and M are undetermined coefficients. The equation above suggests that our standard procedure requires the solution of a six-equation system. Hence, this problem is usually dealt with by means of numerical techniques based on matrix manipulations. On this, the standard (and very good) references are Chow [1997] and Sargent and Ljungqvist [2012].

A few more details on the approach to models with a quadratic return function and a linear dynamic constraint are provided in the Appendix, where the solution to Exercise (19) is discussed.

Exercise 19. Consider a Central Bank which can control output (i.e. by means of the interest rate) and aims to minimize:

$$W_0 = E_0 \left[\sum_{t=0}^{\infty} \beta^t (y_t^2 + \delta \pi_t^2) \right],$$

where π_t is inflation and y_t is output. The dynamic constraint the Central Bank faces is a traditional Phillips' curve: $\pi_t = \pi_{t-1} + \gamma y_t + \epsilon_t$, where ϵ_t is

⁵²Interpret y_t as the remaining share of a cake, and c_t as the slice of the cake the consumer decides to cut out at time t . The implicit assumption is that the consumer has free access to a very efficient fridge.

a random variable independent and identically distributed over time. Find the value function and the policy function.

10. NUMERICAL TECHNIQUES AND THE “CURSE OF DIMENSIONALITY”

In this Section, we illustrate two alternative techniques that can be used to approximate numerically the maximum value function in stochastic settings.

First, we extend to the case of stochastic productivity the value function iteration method that we introduced in Sub-section 6.1. We then remark that this technique bitterly suffers from the “curse of dimensionality”. Finally, because the alternative we proposed for the deterministic case, namely the use of collocation techniques, can be troublesome, we discuss a simple application of a more efficient approach, which is the one based on “parameterized expectations”.

10.1. Discretization of the state space. In this Sub-section, we present the stochastic version of the value function iteration method. To favour comparisons, we modify the example presented in Sub-section 6.1, encompassing the simplest possible stochastic process for productivity. In fact, we assume that A_t is independent and identically distributed over time, and that it can take two values, $A_t^L = 3.36770$ and $A_t^H = 3.50515$, both with probability 0.5 (so that $E_{t-1}[A_t]$ is equal to the productivity value used in the non stochastic example: what we are considering here is a mean preserving spread in productivity; notice also that – being A_t time independent – we drop the time suffix⁵³). As before, α is 0.3, and β is 0.97, so that the non-stochastic steady state for capital is normalized to unity.

Accordingly, our problem is to solve:

$$(10.1) \quad \begin{aligned} V(k_0, A) &= \max_{c_0} \left\{ \ln(c_0) + 0.97 E_0 \left[V(k_1, \tilde{A}) \right] \right\}, \\ &\quad s.t. \quad k_1 = A k_0^{0.3} - c_0, \\ &\quad A \in \{3.36770, 3.50515\}, \\ &\quad k_0 \text{ given.} \end{aligned}$$

As in Sub-section 6.1, we assume to be interested in solving the problem for values of capital that are around the non stochastic steady state, and we consider only five possible levels for capital: $\{0.98, 0.99, 1, 1.01, 1.02\}$; accordingly, our state space is composed of ten points, in fact the five capital levels must be coupled with the two possible productivities.

⁵³When productivity is unknown, being random, it is denoted by a twiddle. Accordingly, the absence of the twiddle indicates that productivity has already taken one of its possible values.

The strategy to obtain the maximum value function is not relevantly different from the one we have described for the deterministic case.

To obtain the unknown value function, we need to:

- (Step 1) specify $E_0[V(k_1, \tilde{A})]$ as a set of five arbitrary values, one for each capital level, and denote this set of initial values as $E_0[V_0(k_1, \tilde{A})]$;
- (Step 2) solve Problem (10.1) for each state k_1, A , finding a new (and hence different) maximum value function (in our case this step substantiates in the attainment of a set composed of ten values, denoted by $V_1(k_0, A)$);
- (Step 3) let, for each capital level on the grid, $V_1(k_1, A) = V_1(k_0, A)$, and compute the expected maximum value function $E_0[V_1(k_1, \tilde{A})]$ using the probability distribution over A ;
- (Step 4) using $E_0[V_1(k_1, \tilde{A})]$ as a new arbitrary starting point, obtain $E_0[V_2(k_1, \tilde{A})]$;
- (Step 5) iterate the steps above until $E_0[V_n(k_1, \tilde{A})]$ and $E_0[V_{n+1}(k_1, \tilde{A})]$ are “sufficiently close”.

In the simulations, to stop this iterative procedure, we choose for the set of values representing $E_0[V(k_1, \tilde{A})]$ a convergence criterion which will be analogous to the one chosen in the non-stochastic case.

We now illustrate the above procedure by means of the extremely simplified example that we have already sketched.

We start by choosing the set of values for $E_0[V_0(k_1, \tilde{A})]$. In this example, we choose: $E_0[V_0(k_1, \tilde{A})] = 0$ for any k_1 (Step 1).

As for Step 2, notice that the above choice implies that we face problems of the following type:

$$\begin{aligned} V_1(k_0, A) &= \max_{c_0} \{\ln(c_0)\}, \\ \text{s.t. } k_1 &= Ak_0^{0.3} - c_0, \\ A &\in \{3.36770, 3.50515\}, \\ &k_0 \text{ given.} \end{aligned}$$

We first consider the case $A = A^L$. Notice that, as before, both k_0 and k_1 must take one of the possible values for capital (i.e. $\{0.98, 0.99, 1, 1.01, 1.02\}$): this restricts the set of feasible consumption levels. For example, if $k_0 = 0.98$, for $k_1 = \{0.98, 0.99, 1, 1.01, 1.02\}$, we compute: $c_0 = \{2.36735, 2.35735, 2.34735, 2.33735, 2.32735\}$; the corresponding utilities are $\{0.86177, 0.85754, 0.85329, 0.84902, 0.84473\}$. The highest utility level is obtained for

$c_0 = 2.36735$, which is the consumption level that guarantees that the next period capital is $k_1 = 0.98$.

Hence, the solution for this specific problem is $V_1(k_0 = 0.98, A = A^L) = 0.86177$.

Repeating this reasoning we obtain: $V_1(k_0 = 0.99, A = A^L) = 0.86607$, $V_1(k_0 = 1, A = A^L) = 0.87033$, $V_1(k_0 = 1.01, A = A^L) = 0.87454$, $V_1(k_0 = 1.02, A = A^L) = 0.87870$. As expected, in every problem the solution corresponds to the maximum feasible consumption level, which is the one leading to $k_1 = 0.98$.

We then consider the case $A = A^H$. For example, if $k_0 = 0.98$, for $k_1 = \{0.98, 0.99, 1, 1.01, 1.02\}$, we compute: $c_0 = \{2.50397, 2.49397, 2.48397, 2.47397, 2.46397\}$; the corresponding utilities are $\{0.91788, 0.91388, 0.90986, 0.90583, 0.90178\}$. The highest utility level is reached for $c_0 = 2.50397$, which is, once again, the consumption level that guarantees that the next period capital is $k_1 = 0.98$, and the solution for the problem is $V_1(k_0 = 0.98, A = A^H) = 0.91788$.

As for the other capital levels, we obtain: $V_1(k_0 = 0.99, A = A^H) = 0.92211$, $V_1(k_0 = 1, A = A^H) = 0.92630$, $V_1(k_0 = 1.01, A = A^H) = 0.93044$, $V_1(k_0 = 1.02, A = A^H) = 0.93454$. This completes Step 2 in the procedure.

The third step in the procedure is readily executed: given that $p^H = p^L = 0.5$, $E_0[V_1(k_1, A)] = \{0.88982, 0.89409, 0.89831, 0.90249, 0.90662\}$.

We now use this set of values as a new starting point for the maximum value function (Step 4). Accordingly, we now assume $E_0[V_1(k_1, A)] = \{0.88982, 0.89409, 0.89831, 0.90249, 0.90662\}$, and we proceed with the second iteration, as we did in Sub-section 6.1. Actually, we let our numerical routine to perform Step 5.

The results of the routine that solves the model are reported in Table 2; the last line provides, as a reference point, the exact solution.

TABLE 2

iter. #	$E[V(0.98, \tilde{A})]$	$E[V(0.99, \tilde{A})]$	$E[V(1.00, \tilde{A})]$	$E[V(1.01, \tilde{A})]$	$E[V(1.02, \tilde{A})]$
0	0	0	0	0	0
1	0.889825	0.894094	0.898316	0.902490	0.906619
2	1.753071	1.757366	1.761613	1.765815	1.769976
3	2.590458	2.594755	2.599010	2.603218	2.607381
4	3.402731	3.407029	3.411284	3.415492	3.419656
5	4.190637	4.194935	4.199190	4.203398	4.207563
10	7.789475	7.793774	7.798029	7.802236	7.806402
50	23.197025	23.201323	23.205579	23.209786	23.213952
100	28.255542	28.259841	28.264096	28.268304	28.272469
200	29.599175	29.603474	29.607729	29.611937	29.616102
300	29.663068	29.667367	29.671622	29.675830	29.679995
375	29.665933	29.670232	29.674487	29.678695	29.682860
376	29.665943	29.670242	29.674497	29.678705	29.682870
true	29.666455	29.670751	29.675004	29.679214	29.683383

Notice that the uncertainty about future productivity has an interesting impact on the value function: an increase in the variance for \tilde{A} negatively influences the overall utility. (Compare the last line in Table 2 with the last one in Table 1) This is a consequence of the fact that the preferences (4.1) describe a risk-averse representative consumer.

Exercise 20. Check that the values for $E[V(k, \tilde{A})]$ in the last line of Table 2 are correct.

To provide a more challenging example, we use the program written for the problem above to study what happens for $k \in [0.5, 1.3]$ with 800 grid-points and five equiprobable productivity levels (evenly spaced in the interval $A = [3.36770, 3.50515]$).⁵⁴ Figure 12a plots the expected maximum value function, while Figure 12b shows the differences between the true expected value function and the approximated one; it takes about five minutes to achieve convergence.⁵⁵

Even in this simple example, the required computer time is (relatively) high. Hence, one should consider to adopt a more efficient technique, such as the collocation one or the finite elements method. However, the “curse of dimensionality” never sleeps, and the implementation of these techniques in

⁵⁴Productivity is naturally thought about as being a continuous random variable; in dealing with this class of variables one needs to use “quadrature” techniques. In practice, this amounts to a wise choice of the points used to discretize the continuous random variable. Refer to Judd [1998].

⁵⁵This is, again, exploiting an Intel i7-4600 CPU @ 2.70 Ghz.

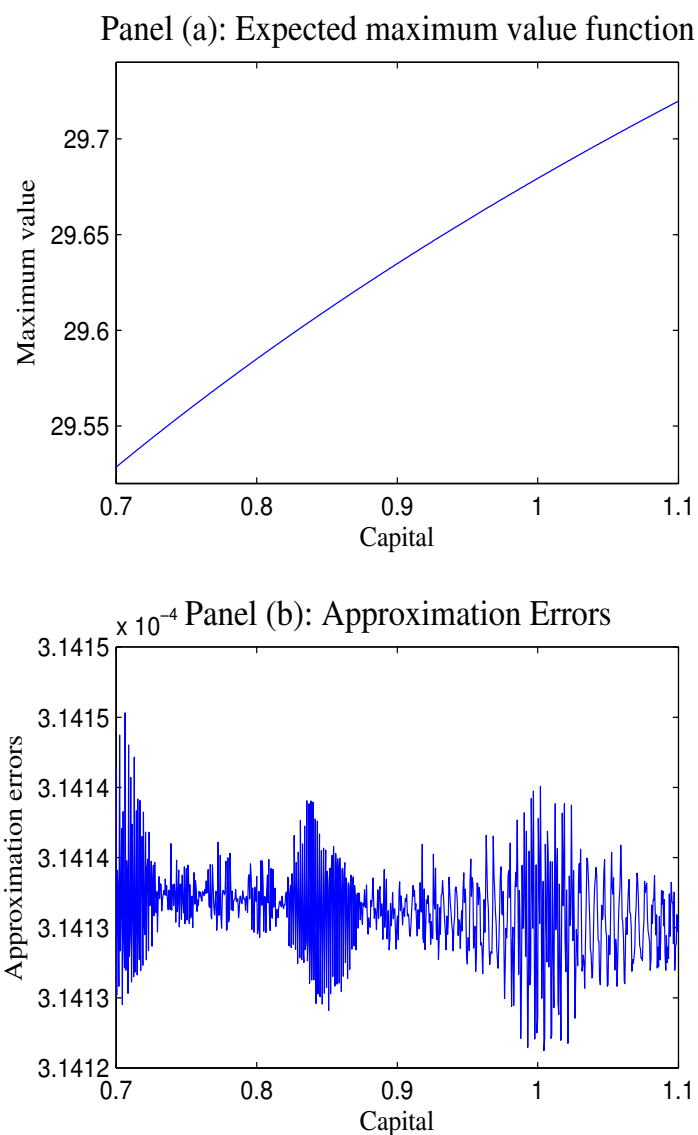


FIGURE 12.

a multidimensional setting is not easy. In fact, the interpolating polynomial must be specified in a number of variables equal to the dimension of the state space, and the number of coefficients to be determined then grows very quickly with the order of the polynomial.⁵⁶ Hence, the number of

⁵⁶Recall that $P_3(x)$ was characterized by four coefficients only, and consider that an order three polynomial in two variables is characterized by ten coefficients. In fact:

$$P_3(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3.$$

equation composing the non-linear system of ordinary equations that comes from the collocation exercise can easily become very large, and therefore hardly manageable for our non-linear equation solver. In practice, when the number of state variables exceeds two, the application of these techniques becomes a fairly tough task. The next section sketches a valid alternative.

10.2. The Parameterized Expectations Approach. As usual, we introduce this new approach by means of an example.

We consider a Ramsey model which is identical to the one discussed in Section (7), but for the fact that capital does not depreciate entirely within one period, and hence $\delta \in (0, 1)$. Accordingly, in our example the stochastic Euler equation (7.8) must be modified to take into account the partial depreciation of capital; from the perspective of period t , (i.e., when the expectation are conditioned upon the time t information), such Euler equation becomes:

$$(10.2) \quad \frac{1}{c_t^*} = \beta \alpha E_t \left[\frac{1}{c_{t+1}^*} (A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \right].$$

Of course, the term on the right hand side in the square bracket is an expectation, conditional on the period t information set.

Notice that the term $\left[\frac{1}{c_{t+1}^*} (A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \right]$ is a function of A_{t+1} and k_{t+1} . In fact, c_{t+1} is a function of the period $t + 1$ realizations for the state variables. Hence $E_t \left[\frac{1}{c_{t+1}^*} (A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \right]$ is the expectation of a function of A_{t+1} and k_{t+1} . These variables must be forecasted on the ground of the period t information set, which means on the ground of A_t and k_t . Hence, these are the variables upon which the expectation on the right hand side of (10.2) must be conditioned.⁵⁷

Because the expectation $E_t \left[\frac{1}{c_{t+1}^*} (A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \right]$ is a function of the state variables, we approximate this conditional expectations by a polynomial in the state variables. Such a polynomial – as any polynomial – is characterized by its parameters, therefore – once we have chosen the degree of the polynomial and the values for the parameters – we have “parameterized” the expectation.

We denote the approximating polynomial by $F(\Psi_0, A_t, k_t)$, where Ψ_0 is the set of coefficients of the polynomial. Notice that, for convenience, $F(\Psi_0, A_t, k_t)$ represents the period t conditional expectation for

$$\alpha E_t \left[\frac{1}{c_{t+1}^*} (A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \right].$$

⁵⁷You may be thinking that k_{t+1} belongs to period t information set. This is true, but consider that k_{t+1} depends on c_t . Because this is the variable we wish to determine, it is inconvenient to condition the expectation on the right hand side of (10.2) on k_{t+1} .

Following Marcat and Lorenzoni (1999), to represent $F(\Psi_0, A_t, k_t)$ we choose an exponentiated polynomial of order one, that is:

$$(10.3) \quad F(\Psi_0, A_t, k_t) = \psi_{0,1} \exp(\psi_{0,2} \ln k_t + \psi_{0,3} \ln A_t).$$

Hence, in this example, Ψ_0 is composed of $\psi_{0,1}$, $\psi_{0,2}$, and $\psi_{0,3}$.

Using an ordinary polynomial might cause problems because it can generate a negative value for $F(\Psi_0, A_t, k_t)$; since, as we shall see in a while, this number is raised to a negative power (refer to Eq. (10.4a)), a numerical error would ensue. Furthermore, we know that the true expectation can take only positive values, and the functional form in (10.3) actually guarantees a positive $F(\Psi_0, A_t, k_t)$ (therefore generating a positive period t consumption level). Increasing the degree of the exponentiated polynomial, we can approximate the conditional expectation with better and better accuracy.

We initialize our procedure attributing to each parameter (i.e. to $\psi_{0,1}$, $\psi_{0,2}$, and $\psi_{0,3}$) an arbitrary value; having chosen these values, knowing the current values for the state variables, A_0 , and k_0 , and letting our software to draw an appropriate sequence of random numbers representing the future realizations for the productivity process, we can easily simulate the model. As it will be commented upon in what follows, what we do is to obtain “artificial” time series for consumption and next-period capital. In fact, in any period $t = \{0, 1, 2, \dots\}$, we have:

$$(10.4a) \quad c_t(\Psi_0) = [\beta F(\Psi_0, A_t, k_t)]^{-1}$$

$$(10.4b) \quad k_{t+1}(\Psi_0) = A_t k_t^\alpha + (1 - \delta)k_t - c_t(\Psi_0).$$

Knowing A_0 and k_0 , we compute $c_0(\Psi_0)$, this value is used, together with A_0 and k_0 , to determine $k_1(\Psi_0)$. Having obtained a random realization for A_1 , we couple this with $k_1(\Psi_0)$, and we iterate the process. Because the values for consumption and for the capital stock obtained by means of (10.4a) and of (10.4b) depend on the vector of parameters, we have denoted these values as $c_t(\Psi_0)$, and $k_{t+1}(\Psi_0)$, respectively.

Notice that this simulation is based on an arbitrary choice for the vector of parameters. Nonetheless, we take it seriously, and we construct a time series for the auxiliary variable w_{t+1} :

$$(10.5) \quad w_{t+1} = \alpha \frac{1}{c_{t+1}(\Psi_0)} (A_{t+1} k_{t+1}(\Psi_0)^{\alpha-1} + 1 - \delta), \quad t = \{0, 1, 2, \dots\}$$

Notice that

$$E_t[w_{t+1}] = E_t \left[\alpha \frac{1}{c_{t+1}(\Psi_0)} (A_{t+1} k_{t+1} (\Psi_0)^{\alpha-1} + 1 - \delta) \right].$$

Hence, β times $E_t[w_{t+1}]$ represents the marginal utility of period t consumption, as determined, following the Euler equation, on the ground of our artificial time series. In our model, the marginal utility $(c_t^*)^{-1}$ is a function of the state variables k_t and A_t . Hence, $E_t[w_{t+1}]$ can be expressed as a function of the state variables; if the parameter vector Ψ_0 and the functional form in (10.3) were correct (i.e. if they were the ones actually satisfying (10.2)), by regressing w_{t+1} on k_t and A_t (plus a constant) we should obtain exactly Ψ_0 . Of course, this regression must be specified in accordance with the functional form (10.3). Because $F(\Psi_0, A_t, k_t)$ is an exponentiated polynomial, we run the regression

$$\log w_{t+1} = \log \psi_{1,1} + \psi_{1,2} \ln k_t + \psi_{1,3} \ln A_t + \xi_t,$$

where ξ_t is a shock, which – under rational expectations – must be independent over time and from the regressors (otherwise the correlations could be exploited to improve the forecast).

We denote the set of regression coefficients as Ψ_1 . If the parameter vector Ψ_0 were correct, then the regression parameters Ψ_1 would confirm Ψ_0 (and hence the polynomial built using Ψ_1 would be equal to the original one, built on Ψ_0). In general, the regression coefficients Ψ_1 can be used to simulate again the model. Hence, we substitute the vector Ψ_1 to the initial arbitrary vector Ψ_0 , and we obtain new values for $c_t(\Psi_1)$, $k_t(\Psi_1)$, and for the related time series w_{t+1} . If we proceed in this way, we can obtain a new estimate Ψ_2 . The crucial aspect is that iterating this reasoning, we obtain better and better result, i.e. we obtain time series for w_{t+1} which are better and better approximations for $(\beta c_t^*)^{-1}$. (See Marcet and Lorenzoni (1999) and the literature quoted there for more details on this convergence result).

We now solve by means of this procedure the log-utility/Cobb Douglas production model.

In the need to specify the parameters' values, we fix, as usual, $\beta = 0.97$, and $\alpha = 0.3$; in coherence with the example in Section 6.2, we pick, for the depreciation parameter, the value $\delta = 0.15$; for the productivity process we choose, as in the previous Sub-section, $A_t^L = 3.36770$, and $A_t^H = 3.50515$, both with probability 0.5. Notice that these parameters value imply a non stochastic steady state for capital as high as 12.01171. The initial condition for the capital stock is $k_0 = 6$: we choose an initial condition quite far from the long-run capital distribution to obtain some information about the “transition” of the system to the stochastic steady state.

Finally, we need to provide the initial values for the parameters in Ψ_0 . Although we have defined as “arbitrary” these values, it is sensible to feed the routine with values that are as close as possible to the “truth”: this speeds up the convergence.⁵⁸ Accordingly, we have chosen for $\psi_{0,i}$, the values that can be computed when $\delta = 1$ (solve Exercise 14). These are: $\psi_{0,1} = 1.4540590$, $\psi_{0,2} = -0.3$, and $\psi_{0,3} = -1$.⁵⁹

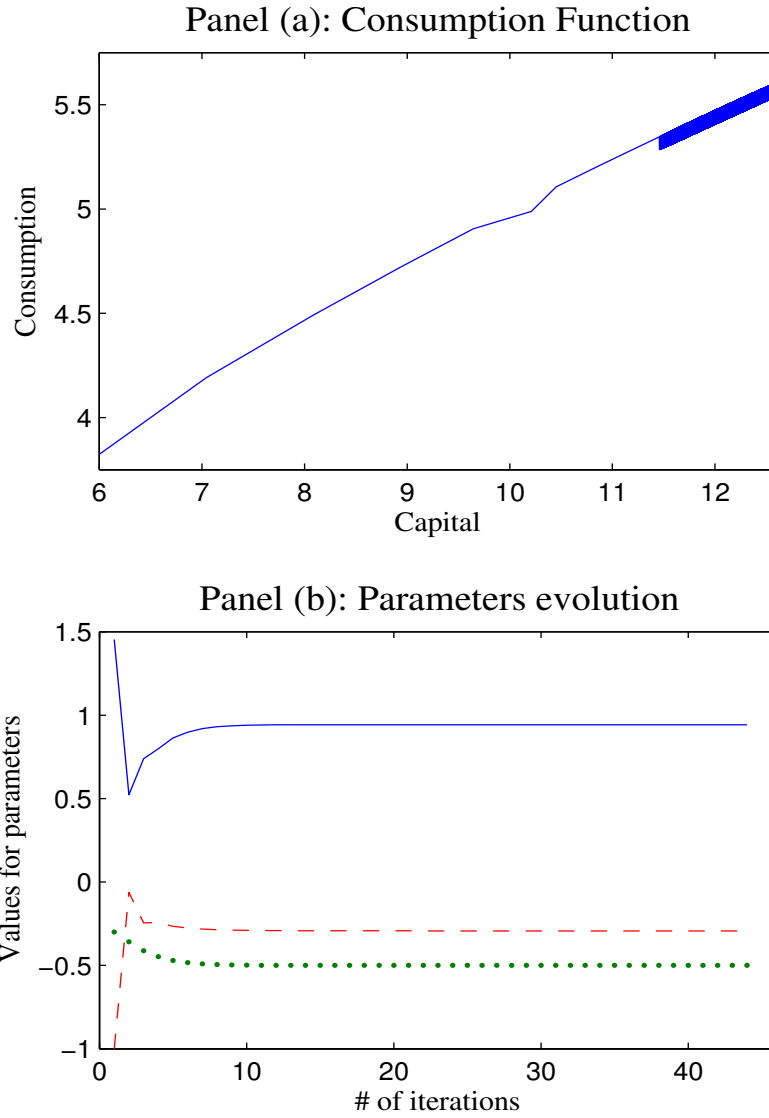
We choose to simulate the model for 100.000 periods; at the end of each iteration we compute, for each parameter, the difference between its initial value, and the estimated one. When the largest difference in absolute value, is lower than $10^{(-9)}$, we assume that convergence is attained. It turns out that this happens in 44 iteration, the required computer time being about 1 minute and one half.⁶⁰

Panel (a) in Figure ?? shows the scatter plot for consumption as a function of capital; Panel (b) displays the evolution for the parameters values: the continuous line represents $\psi_{j,1}$; the dotted line is $\psi_{j,2}$, while the dashed line is $\psi_{j,3}$.

⁵⁸Actually, values that are far from the correct ones may prevent the routine to converge. Also, it may be interesting to underscore that several applications of PEA use an algorithm based on limited modifications of the parameters vector. Denoting by $\bar{\Psi}$ the parameter vector estimated in the least squares section of the procedure, one picks $\Psi_n = \mu\bar{\Psi} + (1 - \mu)\Psi_{n-1}$, with $\mu \in (0, 1)$, instead of choosing $\Psi_n = \bar{\Psi}$.

⁵⁹The difference between $\delta = 1$ and $\delta = 0.15$ is large. Thus, our starting values for the parameters can actually be far from the “true” ones. Accordingly, the initial values may induce instability. Should this happen, it is sensible to carry on as follows. First, run the routine for a δ close to 1, say $\delta = 0.9$, obtaining the “true” values for this case. Second, reduce δ , say to 0.8, using as a starting point for Ψ , the “true” values obtained in the first step. This should allow for the computation of a new set of “correct” values for Ψ . One can progress this way until the desired value ($\delta = 0.15$) is reached.

⁶⁰As already underscored, when the parameter vector Ψ_n is correct, then $E_t[w_{t+1}] = (\beta c_t^*)^{-1}$. Accordingly, we computed the average difference between w_{t+1} and $(\beta c_t^*)^{-1}$, and we found it to be very small: $-5.10 \times 10^{(-7)}$.



From Panel (a) we see that the realized values for consumption and capital in the time series “thicken” around their steady state distribution (consider the “thick” portion of the plot). This happens because k_t takes only a relatively limited number of periods to move from its initial value ($k_0 = 6$) to its long run distribution. Notice in passing that the transition – being short – is affected only by a necessarily limited number of realizations for the productivity shock, and hence that the “transitional part” of the time series we obtain is of limited interest.

The bright side of what we have done is that we have been able to compute a good approximation in a short time. In general, the Parameterized Expectations Approach does quite well. In particular, Christiano and Fisher [2000] argue that it should be the preferred one to solve models with stochastically binding constraint.

In sum, this is a method that it is well worth considering when solving large stochastic models.

The availability of fast and reliable methods for solving stochastic models paves the way to researchers who wish to explore frameworks that are much more complex, and much more interesting, than the one analyzed in these introductory notes.

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APPENDIX A. THE NUMERICAL ROUTINES

In this Appendix, we detail the tasks performed by the Matlab routines we have used in the main text.

The routines have been written for didactic purposes, they aim at being transparent, and they are (very) inefficient. In particular, we have used many times the "for ... end" cycle, which slows down the execution of the routines.

We present the routines in the order they have been used in the main text; notice that it should be possible to "cut" the routines from the pdf file, and paste them into the Matlab editor, obtaining working .m files.⁶¹

A.1. **Valfun_iter0.m**. This is the script (i.e. a list of command) that produces Table 1.

```
clear;
time1=clock;
% this stores the initial time.
format('long');
% this let Matlab to display results in the Command Window
% using the "long" format for figures (calculations are unaffected)
% we now declare the parameters values
alpha=0.3;
beta=0.97;
A=1/(alpha*beta);
% this compute the steady state capital stock;
kstar=(alpha*beta*A)^(1/(1-alpha));
states=5;
% this is the number of states we use;
kmin=0.98;
% this is the lowest capital level we use;
kmax=1.02;
% this is the highest capital level we use;
deltak = (kmax-kmin)/(states-1);
kvec=kmin:deltak:kmax;
% these two lines compute the 'grid' for the state variable.
vini=zeros(1,states);
% this is our initial guess for the maximum value function,
% which is 0 for each gridpoint.
vfin=zeros(1,states);
% this creates the maximum value function which will be obtained
% at the end of one optimization "cycle".
cons=zeros(1,states);
% this creates the consumption vector
```

⁶¹For reason that are beyond my understanding, in most versions of the Adobe reader, when you paste the script in the Matlab editor, you need to substitute the character < ' > with the character < / > (which can easily be done automatically – notice that, in the Matlab editor, the character < / > is the very same key, i.e. the apostrophe).

```

opt_choice=zeros(1,states);
% this is the vector used to store the maximum utility, inclusive
% of the discounted value of future utility, in each cycle.
% The maximum value function is made up of the optimal choices
% for every stock of capital.
err=10;
% "err" is the variable used to substantiate the convergence
% criterion. When "err" is below a threshold (specified below),
% the routine has converged.
% Variable "err" is now initialized.
z=0;
% this initializes the counter for the value iterations
while err > 10^(-5);
% convergence criterion.
z=z+1;
% this is the counter for value iterations
for i=1:states;
for j=1:states;
cons(j)=A*kvec(i)^alpha-kvec(j);
% for any given state (i), this computes the consumption compatible
% with the next period state (j).
end;
opt_choice(i)=max((log(cons)+beta*vini)');
% this is optimal choice among consumption values (and their related
% next-period capital stock, summarized by the value in "vini").
% Notice that i indexes the current state
end;
vfin=opt_choice;
% The maximum value function is the optimal choice for every stock
% of capital.
delta=abs(vini-vfin);
% this computes the difference between the maximum value before
% and after the optimization cycle, in absolute value.
err=max(delta');
% the "convergence variable" takes the maximum absolute difference
% between the maximum value before and after the optimization
% cycle.
vini=vfin;
% the new cycle shall start with this new guess.
Table1(z,:)=[z vfin(1) vfin(2) vfin(3) vfin(4) vfin(5)];
% these are the values which appear in Table 1, Section 6.
end;
% this 'closes' the cycle for z.
time2=clock;
% this stores the time at the end of the execution.
time3=time2-time1;
'used_cpu_time'
time3
% this computes the "true" maximum value function (see example
% in Section 6.1)
par =alpha*beta;
cost1=1/(1-beta)*(par/(1-par)*log(par)+log(1-par)+1/(1-par)*log(A));

```

```

cost2=alpha/(1-alpha*beta);
true_val=cost1+cost2*log(kvec);
true_val
% these are the values for the "true" value function.

```

Notice that Table1 contains the values used to produce the corresponding Table in the main text.

A.2. **valfun_iter1.m.** This script is essentially identical to valfun_iter0.m, but for the fact that we consider a grid of 1600 points; the state variable is ranging from 0.7 to 1.1. This routine produces Figure 5. For specific comments, please refer to the previous sub-section.

```

clear;
% for comments, refer to file valfun_iter0.
time1=clock;
format('long');
alpha=0.3;
beta=0.97;
A=1/(alpha*beta);
kstar=(alpha*beta*A)^(1/(1-alpha));
states=1600;
% this changes the number of points in the grid.
kmin=.7;
kmax=1.1;
deltak = (kmax-kmin)/(states-1);
kvec=kmin:deltak:kmax;
vini=zeros(1,states);
vfin=zeros(1,states);
consV=zeros(1,states);
opt_choice=zeros(1,states);
err=10;
z=0;
while err > 10^(-5);
z=z+1;
for i=1:states;
for j=1:states;
consV(j)=A*kvec(i)^alpha-kvec(j);
end;
opt_choice(i)=max((log(consV)+beta*vini)');
end;
vfin=opt_choice;
delta=abs(vini-vfin);
err=max(delta');
vini=vfin;
end;
time2=clock;
time3=time2-time1;
'used_cpu_time'

```

```

time3
% The lines below compute the "true" maximum value function
par=alpha*beta;
cost1=1/(1-beta)*( par/(1-par)*log(par)+log(1-par)+1/(1-par)*log(A));
cost2=alpha/(1-alpha*beta);
true_val=cost1+cost2*log(kvec);
err_approx=true_val-vfin;
% these are the approximation error as difference between true
% values and those approximated by "vfin"
% ===== The lines below produce a two panels figure =====
size=11; % size of font (in points)
subplot1 = subplot(2,1,1,'FontSize',10,'FontName','Times_New_Roman');
plot(kvec,'vfin'); % this builds Panel (a)
xlim([0.7 1.1]);
ylim([29.5 29.75]);
% this creates xlabel
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% this creates ylabel
ylabel('Maximum_value','FontSize',size,...
'FontName','Times_New_Roman');
% this creates title
title('Panel(a):_Maximum_Value_Function','FontSize',size+3,...
'FontName','Times_New_Roman')
subplot2 = subplot(2,1,2,'FontName','Times_New_Roman');
plot(kvec,'err_approx'); % this builds Panel (b)
xlim([0.7 1.1]);
% this creates xlabel
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% this creates ylabel
ylabel('Approximation_errors','FontSize',size,...
'FontName','Times_New_Roman');
% this creates title
title('Panel(b):_Approximation_Errors','FontSize',size+3,...
'FontName','Times_New_Roman')

```

A.3. **prim.m.** The file `colloc_0.m` (to be discussed below) calls for the function “`prim(x,num)`”, which is a function written in a separated file (also called a “function m-file”). Here we discuss the file `prim.m`, as a preliminary for the analysis of `colloc_0.m`.

The function file `prim.m` stores the function `prim(x,num)`, which can be used by other Matlab files. This file produces a row vector of length num , the elements of which take values $x^0, x^1, x^2, \dots, x^{(num-1)}$. For example, if a script file calls `prim(2,5)`, the output of the file `prim.m` will be $[1, 2, 4, 8, 16]$. Notice that if we multiply the output of `prim.m` by a column vector of parameters (of appropriate dimension), we obtain the value of the polynomial of degree $num-1$, characterized by the parameters’ values stored in the column vector,

and computed at x . Notice that, typically, in our exercises, x will be a collocation point.

```
function F=prim(x,num)
i=1;
for i=1:num;
F(i)=x^(i-1);
% this builds the row vector of length num, evaluated at x.
end;
```

A.4. **colloc_0.m.** This script approximates $\sin(x)$ in the interval $[0, 2\pi]$ using first a third, and then a tenth degree polynomial. Figure 7 and 8 are the output of this script. Notice that this script calls for the function “prim(x,num)”.

```
clear;
max_pi=2*pi;
% this sets the approximation range which is [0,2*pi].
num=4;
% this sets number of coefficient of the approximating polynomial.
incr=max_pi/(num-1);
cp=0:incr:max_pi;
% The two line above determine the collocation points (cp), which
% are equally spaced (the distance between two neighboring points
% being "incr").
A=[prim(cp(1),num);prim(cp(2),num);prim(cp(3),num);prim(cp(4),num)];
b=sin(cp)';
% A and b are used to write System (6.3) in matrix form.
z0=A^(-1)*b;
% this solves linear system; z is vector of parameter for the
% approximating polynomial.
% The next lines build the data for Figure 6;
bignum=200;
% this is the number of points where true and approximating functions
% are computed (notice that "bignum" is much larger than "num").
real_line=0:max_pi/(bignum-1):max_pi;
% Vector "real_line" is discretizing the independent variable
% in the interval [0, 2*pi].
sin_x=sin(real_line);
% Vector "sin_x" is discretizing the dependent variable in
% the interval [0, 2*pi].
for i=1:bignum;
approx_funz(i) = prim(real_line(i),num)*z0;
% approximating polynomial in the interval [0, 2*pi].
end;
plot_funz=[sin_x;approx_funz];
% this stores sin(x) and the approximating polynomial in a matrix
```

```

% Below, we repeat the above exercise with a 10th degree polynomial
time1=clock;
num=11;
% number of coefficient in the approximating polynomial is increased
% to 11
incr=max_pi/(num-1);
cp=0:incr:max_pi;
% this creates matrix A (which is 11x11) by concatenation.
% Each of the row represents the values taken by the powers composing
% the polynomial at a collocation point.
A=[prim(cp(1),num)];
% 'prim' is stored in a function m-file;
for i=2:num;
A=[A;prim(cp(i),num)];
end;
% vector b (which is 11x1) stores the values of the function to
% be approximated.
b=sin(cp)';
z=A^(-1)*b;
% z is vector of parameter for the approximating polynomial
time2=clock;
% this stores the time at the end of the execution.
time3=time2-time1;
'used?cpu?time'
time3
% The lines below build the data for Figure 7;
for i=1:bignum;
approx_funz10(i) = prim(real_line(i),num)*z;
% approximating polynomial
end;
err_approx10=sin_x-approx_funz10;
% approximation error as difference between true and approximated
% function.
% The lines below build Figure 6;
size=11; % size of font (in points)
figure;
plot(real_line,plot_funz(1,:),'-',real_line,plot_funz(2,:), '--');
% first series (which is Sin(x) is continuous, second (which is
% the 3rd degree approximation) is dashed.
xlim([0 2*pi])
ylim([-1.25 1.25])
% Create xlabel
xlabel('x?\in?[0,2\pi]', 'FontSize', size, ...
'FontName', 'Times?New?Roman');
title('3^{rd}?degree?approximation?for?Sin(x)', 'FontSize', size+3, ...
'FontName', 'Times?New?Roman')
% The lines below build Figure 7;
size=11; % size of font (in points)
figure;
plot(real_line, err_approx10);
xlim([0 2*pi])
% Create xlabel

```



```
xlabel('x?\in?[0,2\pi]', 'FontSize', size, ...
'FontName', 'Times?New?Roman');
ylabel('Approximation?errors', 'FontSize', size, ...
'FontName', 'Times?New?Roman');
title('Approximation?errors?for?Sin(x)', 'FontSize', size+3, ...
'FontName', 'Times?New?Roman')
```

A.5. **der_1.m.** This file stores the function `der_1(x,num)`. This file produces a row vector of length num , the elements of which take values $0, 1, 2x, \dots, (num - 1) x^{(num-2)}$. For example, if a script file calls `der_1(2,5)`, the output will be $[0, 1, 4, 16, 32]$. Notice that if we multiply the output of `der_1.m` by a column vector of parameters (of appropriate dimension), we obtain the first derivative of the polynomial of degree $num - 1$, characterized by the parameters' values stored in the column vector, and evaluated at x .

```
function F=der_1(x,num)
F(1)=0;
% this is the first element of the row vector of length num.
for i=2:num;
F(i)=(i-1)*x^(i-2);
% this builds the rest of the row vector of length num, evaluated
% at x.
end;
```

A.6. **colloc_1.m.** This script approximates the differential Equation (6.4) for $t \in [0, 4]$ using first a second, and then an eight degree polynomial. It also produces Figures 9 and 10. This script uses the functions “`prim(x,num)`” and “`der_1(x,num)`”.

```
time1=clock;
aa=0.1;
bb=1;
% aa and bb are the parameters of the differential equation.
x0=2;
% initial condition.
tmin=0;
tmax=4;
% these set the approximation interval.
% The lines below solve for a 2nd degree polynomial (three
% coefficients);
num=3;
% number of coefficients
incr=(tmax-tmin)/(num-2);
```

```

vt=tmin:incr:tmax;
% this builds the vector representing calendar time
A=[der_1(vt(1),num)-aa*prim(vt(1),num); ...
der_1(vt(2),num)-aa*prim(vt(2),num)];
% 'prim' and 'der_1' are the functions stored in function m-files.
A=[A;prim(0,num)];
bvec=[bb;bb;x0];
% the line above stores System 6.6 in paper, written in matrix
% notation. Notice that the last line in A and bvec comes from
% the initial condition.
z0=A^(-1)*bvec;
% this solves the linear system; z0 is vector of parameter for
% the approximating polynomial
% The line below build the data for Figure 8;
bignum=200;
% this is the number of points where the true and the approximating
% functions are computed (notice that "bignum" is much larger
% than "num").
check_time=tmin:(tmax-tmin)/(bignum-1):tmax;
% vector of gridpoints for the state variable, which is time in
% the interval [0,4].
true_funz = -bb/aa+(x0+bb/aa)*exp(aa*check_time);
% exact solution for the differential equation.
for i=1:bignum;
approx_funz(i) = prim(check_time(i),num)*z0;
% approximated solution for the differential equation at "bignum"
% points. It exploits again the function m-file "prim.m".
end;
plot_funz=[true_funz;approx_funz];
% this passes the relevant data to the figure.
% The lines below solve for an 8th degree polynomial (which implies
% 9 coefficients).
num=9;
incr=(tmax-tmin)/(num-2);
vt=tmin:incr:tmax;
Anew=[der_1(vt(1),num)-aa*prim(vt(1),num)];
for i=2:num-1;
Dnew=[der_1(vt(i),num)-aa*prim(vt(i),num)];
Anew=[Anew;Dnew];
% Notice that matrix Anew is created by concatenation.
end;
Anew=[Anew;prim(0,num)];
bvecnew=[bb*ones(num-1,1);x0];
% this incorporates condition x0=2.
z=Anew^(-1)*bvecnew;
% this solves linear system; z is vector of parameter for the
% 8th degree approximating polynomial.
for i=1:bignum;
approx_funz8(i) = prim(check_time(i),num)*z;
% this is the approximated solution for the differential equation.
true_check(i)=der_1(check_time(i),num)*...
z-bb-aa*prim(check_time(i),num)*z;

```

```

% approximation error as difference between true and approximated
% function.
end;
fake_check=true_funz-approx_funz8;
% this is the "fake" check, obtained as difference between true
% (analytical) solution and approximated solution.
plot_funz2=[fake_check>true_check];
time2=clock;
% this stores the time at the end of the execution.
time3=time2-time1;
'used_cpu_time'
time3
% The lines below build Figure 8;
size=11; % size of font (in points)
figure;
plot(check_time,plot_funz(1,:),'-',check_time,plot_funz(2,:),'--');
% first series (which is Sin(x) is continuous, second is dashed
xlim([0 tmax])
% Create xlabel
xlabel('t\in[0,4]', 'FontSize',size,...
'FontName', 'TimesNewRoman');
title('Approximation for dx(t)/dt=0.1x(t)+1', 'FontSize',size+3,...
'FontName', 'TimesNewRoman')
% The lines below build Figure 9;
size=11; % size of font (in points)
figure;
plot(check_time,plot_funz2(1,:),'-',check_time,plot_funz2(2,:),'--');
xlim([0 tmax]);
ylim([-10*10^-12 10*10^-12]);
% Create xlabel
xlabel('t\in[0,4]', 'FontSize',size,...
'FontName', 'TimesNewRoman');
ylabel('Approximation errors and residuals', 'FontSize',size,...
'FontName', 'TimesNewRoman');
title('Errors of an 8th degree polynomial', 'FontSize',size+3,...
'FontName', 'TimesNewRoman')

```

A.7. **colloc_2.m.** This file solves system (6.8-6.9). This routine exploits the Matlab built-in nonlinear equation solver. This is done through the command `fsolve(.)`. System (6.8-6.9) is stored in the file `system0.m`; Figure 11 represents the output for this script.

```

clear;
time1=clock;
al=0.3;
de=0.15;
be=0.97;
% this feed the routine with the parameters values.
A=1/(al*be);

```

```

% this is a normalization.
kss=(A*al*be / (1+be*(de-1)) )^(1/(1-al));
% this computes the steady state capital stock.
kmin=0.5*kss;
% this is the lowest capital level we use.
kmax=1.0*kss;
% this is the highest capital level we use.
num=2;
% this is the (initial) number of parameters of the polynomial.
for i=1:num;
% this computes the (equally spaced) collocation points.
ck(i)=kmin+(i-1)/(num)*(kmax-kmin);
% Notice that the steady state capital (kss) is not a collocation
% point.
end;
x1=[-4;.2;.0];
% this is the initial guess for the parameter vector. (not a
% particularly smart one).
options = optimset('display','off','MaxFunEvals',500);
% These are the options for the nonlinear equation solver
% (refer to the Matlab manual).
z=fsolve(@system0,x1,options,num,ck,A,al,be,de,kss);
% this solves the nonlinear system. z is a vector of parameters
%(vector d in system (6.8)). Notice that system0 is a function
% m-file.
for num=3:8;
% this gradually increases the degree of the polynomial.
for i=1:num;
ck(i)=kmin+(i-1)/(num-1)*(kmax-kmin);
end;
% this computes the (equally spaced) collocation points.
x1=[z;0];
% The initial condition is given by the vector of parameters computed
% for the polynomial of degree num-1, and by a 0. This often
% is a fair guess.
options = optimset('Display','iter','MaxFunEvals',500);
z=fsolve(@system0,x1,options,num,ck,A,al,be,de,kss);
% z is vector of parameter d in System (6.8)
end;
time2=clock;
% this stores the time at the end of the execution.
time3=time2-time1;
'used_cpu_time'
time3
% The following lines are useful to build Figure 10;
bignum=200;
% this is the number of points where true and approximating functions
% are computed (notice that "bignum" is much larger than "num").
incr=(kmax-kmin)/(bignum-1);
chk_k=kmin:incr:kmax;
% this is the vector of gridpoints for the state variable
for i=1:bignum;

```

```

approx_cons(i) = prim(chk_k(i),num+1)*z;
% this computes the approximation for consumption.
chk_k1(i)=A*chk_k(i)^al+(1-de)*chk_k(i)-prim(chk_k(i),num+1)*z;
% this computes k(t+1) for every k(t) which is in chk_k
resid(i)=prim(chk_k(i),num+1)*z*be*...
( al*A*chk_k1(i)^(al-1)+(1-de) )-prim(chk_k1(i),num+1)*z;
% this is the 'error' in the Euler equation
end;
% ===== The lines below produce a two panels figure =====
size=11; % size of font (in points)
subplot1 = subplot(2,1,1,'FontSize',10,'FontName','Times_New_Roman');
plot(chk_k,approx_cons); % this builds Panel (a)
xlim([kmin kmax]);
ylim([3.75 5.5]);
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% this creates xlabel
ylabel('Consumption','FontSize',size,...
'FontName','Times_New_Roman');
% this creates ylabel
title('Panel (a): Consumption function','FontSize',size+3,...
'FontName','Times_New_Roman')
% this creates title
subplot2 = subplot(2,1,2,'FontName','Times_New_Roman');
plot(chk_k,resid); % this builds Panel (b)
xlim([kmin kmax]);
ylim([-1.5*10^(-5) 1.5*10^(-5)]);
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% this creates xlabel
ylabel('Residuals','FontSize',size,...
'FontName','Times_New_Roman');
% this creates ylabel
title('Panel (b): Euler equation residuals','FontSize',size+3,...
'FontName','Times_New_Roman')
% this creates title

```

A.7.1. *system0.m*. In this function file, we store seventeen equations, nine being the number of the d_i s coefficients to be determined, and eight being the number of the collocation points (of the k_1^i). Eight equations are of the type $\sum_{i=0}^9 d_i k_1^i = \sum_{i=0}^9 d_i k_0^i (k_1^{-0.7} + 0.8245)$. Because the k_1^i s are endogenous (recall the second equation in (6.8)), each of them is determined as a function of the corresponding k_0^i and c_0^i ($= \sum_{i=0}^{11} d_i k_0^i$), which provides eight further equations. The last equation is the steady state relation $\sum_{i=0}^9 d_i \hat{k}^i = 3.43643 \hat{k}^{0.3} - 0.85 \hat{k}$.

```

function F=system0(x,num,ck,A,al,be,de,kss)
for i=1:num;

```

```

kap(i)= A*ck(i)^al+(1-de)*ck(i)-prim(ck(i),num+1)*x;
% this is second group of equations composing the System (6.8),
% determining the kap(i) as function of the parameters (A,al,
% de), of the k0^(i) (identified by the i-th element of the vector
% ck), and by the vector composed of the parameters d(i),
% represented by vector of unknowns (x).
F(i) = prim(ck(i),num+1)*x*be*( al*A*kap(i)^(al-1)+(1-de) )- ...
prim(kap(i),num+1)*x;
% this is first group of equations composing the System (6.8).
end;
F(num+1) = prim(kss,num+1)*x-A*kss^al+de*kss;
% this exploits the steady state relation (6.9).

```

A.8. **valfun_iter2.m**. This script performs the value function iteration procedure described in Sub-section 10.1. There are 800 gridpoints for capital (ranging again from 0.7 to 1.1) and five equally-spaced and equiprobable productivity levels (going from 3.367697 to 3.505155). This routine yields Figure 12. Table 2 is obtained by changing the number of gridpoints for the capital stock and the number of productivity levels.

```

clear;
time1=clock;
format('long');
% this let Matlab to display results in the Command Window
% using the "long" format for figures (calculations are unaffected)
% Below we declare the parameters values;
alpha=0.3;
beta=0.97;
A=1/(alpha*beta);
kstar=(alpha*beta*A)^(1/(1-alpha));
% this computes the steady state capital stock.
states=800;
% this is the number of gridpoints;
kmin=0.7;
% this is the lowest capital level we use;
kmax=1.1;
% this is the highest capital level we use;
deltak = (kmax-kmin)/(states-1);
kvec=kmin:deltak:kmax;
% these two lines compute the 'grid' for the state variable.
% The next six line describe the productivity process.
prods=5;
% Number of possible productivity levels.
Amin=0.98*A;
Amax=2*A-Amin;
% These implies a 4% productivity differential between best and
% worst case.
deltaA=(Amax-Amin)/(prods-1);

```

```

Avec=Amin:deltaA:Amax;
% this creates the 'grid' for productivity.
p_vec=1/prods*ones(prods,1);
% this introduces equal probability for every productivity level.
vini=zeros(1,states);
% this is the initial guess for expected maximum value function
vfin=zeros(1,states);
% this creates the expected maximum value function which will
% be obtained at the end of one optimization "cycle".
consV=zeros(prods,states);
% this creates the consumption matrix.
opt_choice=zeros(prods,states);
% this is the matrix used to store the maximum utility, inclusive
% of the discounted value of future utility,in each cycle.
% The maximum value function is made up of the optimal choices
% of capital.
err=10;
% "err" is the variable used to substantiate the convergence
% criterion. When "err" is below a threshold (specified below),
% the routine has converged.
% Variable "err" is now initialized.
z=0;
% this initializes the counter for the value iterations.
while err > 10(-5);
% convergence criterion.
z=z+1;
% this is the counter for value iterations.
for h=1:prods;
for i=1:states;
for j=1:states;
consV(h,j)=Avec(h)*kvec(i)alpha-kvec(j);
% For any given capital (i), and for any productivity (h), this
% computes the consumption compatible with next period state (j).
end;
opt_choice(h,i)=max((log(consV(h,:))+ beta*vini)');
% this is the maximum value that can be obtained choosing among
% consumption level, taking into account the related next-period
% capital stock (whose values are in vini). Notice that i
% indexes current capital and h indexes current productivity.
end;
end;
vfin=p_vec'*opt_choice;
% this is the expected maximum value function: for every capital
% level, we compute the expected value of the optimal choice.
delta=abs(vini-vfin);
% this computes the difference between the maximum value before
% and after the optimization cycle, in absolute value.
err=max(delta');
% the "convergence variable" takes the maximum absolute difference
% between the maximum value before and after the optimization
% cycle.
vini=vfin;

```

```

% The new cycle shall start with this new guess.
% Table2(z,:)= [z vfin];
% uncommenting the line above, with some modifications,
% gives the values displayed in Table 2.
end;
% this 'closes' the cycle for z
time2=clock;
% this stores the time at the end of the execution.
time3=time2-time1;
'used_cpu_time'
time3
% The lines below compute the "true" maximum value function (see
% Example in Section <ref>sec: guess_verify2</ref>)
par=alpha*beta;
cost1=1/(1-beta)*( par/(1-par)*log(par)+log(1-par));
cost2=alpha/(1-par);
cost3=1/(1-par)*log(Avec)*p_vec*(1/(1-beta));
true_val=cost1+cost2*log(kvec)+cost3;
err_approx=true_val-vfin;
% these are the approximation errors (with respect to the true
% expected value function)
% ===== The lines below produce a two panels figure =====
size=11; % size of font (in points)
subplot1 = subplot(2,1,1,'FontSize',10,'FontName','Times_New_Roman');
plot(kvec',vfin');
% this builds Panel (a).
xlim([0.7 1.1]);
ylim([29.52 29.74]);
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% Create xlabel.
ylabel('Maximum_value','FontSize',size,...
'FontName','Times_New_Roman');
% Create ylabel.
title('Panel (a):_Expected_maximum_value_function','FontSize',...
size+3,'FontName','Times_New_Roman')
% Create title.
subplot2 = subplot(2,1,2,'FontName','Times_New_Roman');
plot(kvec',err_approx');
% this builds Panel (b).
xlim([0.7 1.1]);
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% Create xlabel.
ylabel('Approximation_errors','FontSize',size,...
'FontName','Times_New_Roman');
% Create ylabel.
title('Panel (b):_Approximation_Errors','FontSize',size+3,...
'FontName','Times_New_Roman')
% Create title.

```


To obtain Table 2 in the main text, set: `states=5`, `kmin=0.98`, `kmax=1.02`, and `prods=2`.

A.9. **pea_rmsy.m**. This builds the example for the Parameterized Expectations Approach described in Sub-section 10.2; it produces Figure ??.

```
clear;
time1=clock;
% we declare below the parameters values;
alfa=0.3;
beta=0.97;
delta=.15;
k0=6;
% this is the initial capital stock
n=100000;
% this is the number of "observations" for the artificial time
% series. The following lines construct the time series for
% productivity.
vecA=rand(n,1);
% this builds a "flat" random variable on (0,1). This is preliminary
% to construct the vector of productivity shocks.
for i=1:n;
% The lines below translate the "flat" variable in a two-states
% i.i.d. process.
if vecA(i)>0.5;
vecA(i)=3.50515;
else
vecA(i)=3.36770;
end;
end;
A_ext=rand(1,1);
% "A_ext" assigns a productivity value for period n+1.
if A_ext<0.5;
A_ext=3.50515;
else
A_ext=3.36770;
end;
% These are the initial values for parameters (refer to the main
% text and to Exercise 14);
fi0=(beta*(1-alfa*beta))^(1);
fi1=-alfa;
fi2=-1;
maxj=200;
% this sets the maximum number of iterations (it the number of
% iteration exceeds maxj, the routines stops even if convergence
% has not been achieved.
err=10;
% "err" is the variable used to substantiate the convergence
% criterion. When "err" is below a threshold (specified below),
```

```

% the routine has converged.
% Variable "err" is initialized.
j=0;
while err>10(-8);
% convergence criterion.
j=j+1;
% The next four variables are used to trace the parameters' evolu-
% tion, which is displayed in the second panel of the Figure.
track1(j)=j;
track2(j)=fi0;
track3(j)=fi1;
track4(j)=fi2;
% Notice that "funct" below represents the functional form F(.),
% refer to Equation (10.3); "consum" is consumption as determined
% in Equation (10.4a); "Kap" is capital at time t+1 (refer to
% Equation 10.4b).
Kap(1)=k0;
funct(1)=fi0*exp( fi1*log(Kap(1))+fi2*log(vecA(1)) );
consum(1)= (beta*funct(1))(-1);
% The lines above yield the first period of the "artificial" time
% series; the cycle below completes it up to period n, and also
% computes w_t+1 as defined in Sub-section 10.2 (which is, W_regr).
for i=2:n;
Kap(i)=vecA(i-1)*Kap(i-1)alfa-consum(i-1)+(1-delta)*Kap(i-1);
funct(i)=fi0*exp( fi1*log(Kap(i))+fi2*log(vecA(i)) );
consum(i)= (beta*funct(i))(-1);
W_regr(i-1)=consum(i)(-1)*(alfa*vecA(i)*Kap(i)(alfa-1)+(1-delta));
end;
% The lines below complete the computation for w_t+1 (computes for
% period n). Notice that A_ext is eventually used. Notice also that
% W_regr is "shifted" one period to facilitate the use of regression
% command
Kap_ext=vecA(n)*Kap(n)alfa-consum(n)+(1-delta)*Kap(n);
funct_ext=fi0*exp( fi1*log(Kap_ext)+fi2*log(A_ext) );
consum_ext= (beta*funct_ext)(-1);
W_regr(n)=consum_ext(-1)*(alfa*A_ext*Kap_ext(alfa-1)+(1-delta));
% Below we transform the variables to run ols.
ln_W=log(W_regr');
ln_K=log(Kap');
ln_A=log(vecA);
XMATR = [ones(size(ln_K)) ln_K ln_A];
% this builds the matrix of independent variables.
n_fi = XMATR\ln_W;
% this command performs the least squares fit, and creates the
% updated parameters vector.
change_fi=[abs(exp(n_fi(1))-fi0);abs(n_fi(2)-fi1);abs(n_fi(3)-fi2)];
% this is the vector of changes in parameters.
err=max(change_fi);
% convergence criterion.
% The following lines update the parameters.
% In case of instability, "mu" has to be reduced.
mu=1;

```

```

fi0=mu*exp(n_fi(1))+(1-mu)*fi0;
fi1=mu*n_fi(2)+(1-mu)*fi1;
fi2=mu*n_fi(3)+(1-mu)*fi2;
end;
% this closes the "while err...." cycle.
coeff=[track2;track3;track4];
% this stores the coefficients (for all the cycles).
time2=clock;
% this stores the time at the end of the execution.
time3=time2-time1;
'used_cpu_time'
time3
% ===== The lines below produce a two panels figure =====
size=11; % size of font (in points)
subplot1 = subplot(2,1,1,'FontSize',10,'FontName','Times_New_Roman');
plot(Kap,consum);
% this builds Panel (a).
xlim([6 12.6]);
ylim([3.75 5.75]);
xlabel('Capital','FontSize',size,...
'FontName','Times_New_Roman');
% Create xlabel
ylabel('Consumption','FontSize',size,...
'FontName','Times_New_Roman');
% Create ylabel.
title('Panel(a):Consumption_Function','FontSize',size+3,...
'FontName','Times_New_Roman')
% Create title.
subplot2 = subplot(2,1,2,'FontName','Times_New_Roman');
plot(track1,coeff(1,:),'-',track1,coeff(2,:),'.',...
track1,coeff(3,:), '--');
% this builds Panel (b).
xlim([0 46]);
xlabel('#of_ iterations','FontSize',size,...
'FontName','Times_New_Roman');
% Create xlabel.
ylabel('Values_for_parameters','FontSize',size,...
'FontName','Times_New_Roman');
% Create ylabel.
title('Panel(b):Parameters_evolution','FontSize',size+3,...
'FontName','Times_New_Roman')
% Create title.

```

APPENDIX B. SOLUTIONS FOR THE EXERCISES

Exercise 1.

Assume that the maximum value function takes the form $V(k_t) = e + f \ln k_t$, so that the Bellman formulation is:

$$e + f \ln k_t = \max_{c_t, l_t} \{ \ln(c_t) + \gamma \ln(1 - l_t) + \beta(e + f \ln k_{t+1}) \}.$$

Since there are two controls, we need to consider two first order conditions, which – exploiting our guess – rapidly give:

$$(B.1) \quad c_t^* = \frac{k_{t+1}}{\beta f}$$

and

$$(B.2) \quad \frac{\gamma}{1 - l_t^*} = \frac{\beta f}{k_{t+1}} (1 - \alpha) A k_t^\alpha l_t^{*(1-\alpha)}.$$

Taking advantage of the dynamic constraint and of the first-order condition (B.1), we obtain a (familiar) relation between the next-period capital and the current output, which is:

$$k_{t+1} = \frac{\beta f}{1 + \beta f} A k_t^\alpha l_t^{1-\alpha}.$$

Hence, the first-order condition (B.2) can be written as:

$$\frac{\gamma}{1 - l_t^*} = \frac{(1 + \beta f)(1 - \alpha)}{l_t^*},$$

which implies that the optimal labour supply is independent from the capital stock (and from the productivity level), so that we can write:

$$l_t^* = l^* = \frac{(1 + \beta f)(1 - \alpha)}{\gamma + (1 + \beta f)(1 - \alpha)} < 1.$$

We now follow a well-known trail: we use the tentative solution into the Bellman equation:

$$\begin{aligned} e + f \ln k_t &= \ln \left(\frac{1}{1 + \beta f} \right) + \ln A + \alpha \ln k_t + (1 - \alpha) \ln l^* + \gamma \ln(1 - l^*) + \\ &\quad + \beta e + \beta f \left(\ln \frac{\beta f}{1 + \beta f} + \ln A + \alpha \ln k_t + (1 - \alpha) \ln l^* \right), \end{aligned}$$

where use has been made of the fact that $c_t^* = \frac{1}{1 + \beta f} A k_t^\alpha l^{*(1-\alpha)}$.

Matching the coefficients for $\ln k_t$ we immediately obtain:

$$f = \frac{\alpha}{1 - \alpha\beta},$$

while the constant is:

$$e = \frac{1}{1-\beta} \left(\frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \ln(1-\alpha\beta) + \frac{1}{1-\alpha\beta} \ln A + \frac{1-\alpha}{1-\alpha\beta} \ln l^* + \gamma \ln(1-l^*) \right).$$

When $\gamma = 0$, it can be verified that e corresponds to (4.7).

Exercise 2.

From the perspective of period t , past consumption, being predetermined, is a state of the system. Let $c_{t-1} = x_t$, and notice that this equation can be interpreted as a transition equation for x_t .

The maximum value function depends upon the state variables, accordingly our tentative functional form is: $V(k_t, x_t) = e + f \ln k_t + g \ln x_t$. The Bellman formulation is now:

$$e + f \ln k_t + g \ln x_t = \max_{c_t} \{ \ln c_t + \gamma \ln x_t + \beta(e + f \ln k_{t+1} + g \ln x_{t+1}) \}.$$

The first order condition with respect to (current) consumption is

$$(B.3) \quad \frac{1}{c_t} + \beta \left(\frac{f}{k_{t+1}} \frac{\partial k_{t+1}}{\partial c_t} + \frac{g}{x_{t+1}} \frac{\partial x_{t+1}}{\partial c_t} \right) = 0$$

and hence, using the dynamic constraints

$$\frac{1}{c_t} = \beta \left(\frac{f}{k_{t+1}} - \frac{g}{c_t} \right),$$

so that we readily obtain $c_t^* = \frac{1+\beta g}{\beta f} k_{t+1}$. The fact that $k_{t+1} = Ak_t^\alpha - c_t$ implies $c_t^* = \frac{1+\beta g}{1+\beta g + \beta f} Ak_t^\alpha$ and $k_{t+1} = \frac{\beta f}{1+\beta g + \beta f} Ak_t^\alpha$. Substituting these results into the Bellman equation, we obtain

$$e + f \ln k_t + g \ln x_t = \ln \frac{1+\beta g}{1+\beta g + \beta f} + \ln A + \alpha \ln k_t + \gamma \ln x_t + \beta \left(e + f \ln \frac{\beta f}{1+\beta g + \beta f} + f \ln A + \alpha f \ln k_t + g \ln \frac{1+\beta g}{1+\beta g + \beta f} + g \ln A + \alpha g \ln k_t \right).$$

Notice that – in the last line of the previous expression – use has been made again of the fact that $x_{t+1} = c_t$.

Matching the coefficients for $\ln x_t$ we obtain that $g = \gamma$; exploiting the terms involving $\ln k_t$ we then obtain that $f = \frac{\alpha(1+\beta\gamma)}{1-\alpha\beta}$. This allows to get the policy function, which is:

$$c_t^* = (1 - \alpha\beta) Ak_t^\alpha.$$

The constant term can then be obtained by exploiting our knowledge for f and g in the following expression:

$$(1 - \beta)e = \ln \frac{1 + \beta g}{1 + \beta g + \beta f} + \ln A + \beta \left(f \ln \frac{\beta f}{1 + \beta g + \beta f} + f \ln A + g \ln \frac{1 + \beta g}{1 + \beta g + \beta f} + g \ln A \right).$$

When $\gamma = 0$, it can be easily verified that e corresponds to (4.7). More interestingly, notice that our specification implies that an increase in present consumption lowers (as usual) the marginal utility of current consumption, and also reduces the next-period utility. In other words, the more the consumer eats today, the hungrier he wakes up tomorrow. In this sense this preferences capture the notion of habit formation: past consumption represents the consumer's "stock of habits" in period t . More precisely, our exercise concerns habits which are labeled "internal", because the current consumption of the (representative) consumer influences *her own* future utility. A popular variant of the habit model considers customs as "external" to the consumer. In this case, our consumer wishes to "keep up with the Joneses", so that it is the *aggregate* past consumption that affects current utility (which would imply a modification of the first-order condition (B.3), since the second addendum in the big round bracket would disappear).

Exercise 3.

Assume that the maximum value function takes the form $V(k_t) = e + f \frac{k_t^{1-\gamma}}{1-\gamma}$;⁶² the Bellman formulation is:

$$e + f \frac{k_t^{1-\gamma}}{1-\gamma} = \max_{c_t} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + \beta \left(e + f \frac{k_{t+1}^{1-\gamma}}{1-\gamma} \right) \right\}.$$

The first order conditions for consumption gives:

$$c_t^{-\gamma} = \beta f k_{t+1}^{-\gamma}.$$

Exploiting, as usual, the dynamic constraint we obtain:

⁶²The practice of proposing a tentative solution which has the same functional form of the return function has a long history in dynamic programming. We exploited the same idea in Sub-Section (4.1). It would however be sensible - after all any tentative is legitimate, we are *guessing* - to start with a generic exponential function, i.e. $V(k_t) = e + f k_t^g$. At the end of the solution for Exercise 3 we will briefly explore this possibility.

$$c_t^* = \frac{(\beta f)^{-1/\gamma}}{1 + (\beta f)^{-1/\gamma}} A k_t,$$

$$k_{t+1} = \frac{1}{1 + (\beta f)^{-1/\gamma}} A k_t.$$

Substituting the above results into the Bellman equation, we get:

$$e + f \frac{k_t^{1-\gamma}}{1-\gamma} = \frac{(\beta f)^{(\gamma-1)/\gamma} A^{1-\gamma} k_t^{1-\gamma}}{(1 + (\beta f)^{-1/\gamma})^{1-\gamma} (1-\gamma)} + \beta \left(e + f \frac{\frac{1}{(1 + (\beta f)^{-1/\gamma})^{1-\gamma}} A^{1-\gamma} k_t^{1-\gamma}}{1-\gamma} \right),$$

which implies that $e = 0$. Moreover, some obvious simplifications give

$$f = \frac{(\beta f)^{(\gamma-1)/\gamma}}{(1 + (\beta f)^{-1/\gamma})^{1-\gamma}} A^{1-\gamma} + \beta f \frac{1}{(1 + (\beta f)^{-1/\gamma})^{1-\gamma}} A^{1-\gamma}.$$

Notice that the equation above allows to conclude that f is independent from the capital stock, and hence that our guess is correct. Some further manipulations yield:

$$f \left(1 + (\beta f)^{-1/\gamma} \right)^{1-\gamma} = \left((\beta f)^{-1/\gamma} + 1 \right) \beta f A^{1-\gamma},$$

or

$$1 + (\beta f)^{-1/\gamma} = \beta^{-1/\gamma} A^{\frac{\gamma-1}{\gamma}},$$

so that

$$(B.4) \quad f = \frac{1}{\beta} \left(\beta^{-1/\gamma} A^{\frac{\gamma-1}{\gamma}} - 1 \right)^{-\gamma}.$$

We now assume that $\beta A^{1-\gamma} < 1$. This restriction guarantees that $f > 1$, which means that the marginal productivity of capital is high enough that a larger stock of capital benefits the representative consumer. Indeed, in an economy in which the marginal productivity of capital does not diminish, the case in which a larger stock of capital is welfare lessening is too sad to be interesting.

From our expression for c_t^* , we find that the policy function is:

$$c_t^* = \frac{\beta^{-1/\gamma} A^{\frac{\gamma-1}{\gamma}} - 1}{\beta^{-1/\gamma} A^{\frac{\gamma-1}{\gamma}}} A k_t.$$

Hence, the proposed restriction guarantees that optimal consumption is a positive function of the stock of capital (a reassuring implication).

Finally, notice that the production function shows constant returns to the accumulable productive factor: hence, it provides a basis for endogenous growth, which makes this structure quite popular.

We now propose an alternative “guess” for the maximum value function, and explore its implications. We assume that our tentative solution takes the form $V(k_t) = e + fk_t^g$, where e , f , and g are unknown parameters. so that the Bellman equation is :

$$(B.5) \quad e + fk_t^g = \max_{c_t} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + \beta (e + fk_{t+1}^g) \right\}.$$

The first order conditions for consumption gives:

$$c_t^{-\gamma} = \beta f k_{t+1}^{g-1},$$

so that

$$(B.6) \quad c_t^* = (\beta f)^{-\frac{1}{\gamma}} k_{t+1}^{\frac{1-g}{\gamma}}.$$

Applying the envelope theorem to (B.5) (refer to Equation (5.1)), we readily obtain

$$k_t = (\beta A)^{\frac{1}{g-1}} k_{t+1}.$$

Exploiting the latter and (B.6) into the Bellman equation (B.5) gives:

$$e + fk_t^g = \frac{1}{1-\gamma} f^{\frac{\gamma-1}{\gamma}} A^{\frac{1-\gamma}{\gamma}} k_t^{\frac{(1-g)(1-\gamma)}{\gamma}} + \beta \left(e + f (\beta A)^{\frac{g}{1-g}} k_t^g \right),$$

from which it is apparent that $e = 0$ and $g = 1 - \gamma$. Notice moreover that it must be true that

$$f = \frac{1}{1-\gamma} f^{\frac{\gamma-1}{\gamma}} A^{\frac{1-\gamma}{\gamma}} + f \beta^{\frac{1}{\gamma}} A^{\frac{1-\gamma}{\gamma}},$$

so that

$$1 = \frac{1}{1-\gamma} f^{-\frac{1}{\gamma}} A^{\frac{1-\gamma}{\gamma}} + \beta^{\frac{1}{\gamma}} A^{\frac{1-\gamma}{\gamma}},$$

which corresponds to (B.4).

Exercise 4.

Recall that $U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, and that $k_{t+1} = Ak_t - c_t$. Notice that the path characterized by the fastest possible growth for capital is obtained by choosing zero consumption at each period of time. Choosing this path, we

have that

$$k_t = A^t k_0.$$

The largest one-period utility is obtained by consuming the entire output that the representative agent can produce in that period:

$$U(c_t) = \frac{(Ak_t)^{1-\gamma}}{1-\gamma}.$$

Therefore, if we follow the policy prescribing to save everything up to period t , and then to consume the entire output, we obtain:

$$U(c_t) = \frac{A^{(1-\gamma)(t+1)} k_0^{1-\gamma}}{1-\gamma}.$$

Imagine counterfactually (as in Section 4.1), that the above policy can be followed in every period. In this case, the lifetime utility for the representative agent would be:

$$\sum_{t=0}^{\infty} \beta^t \frac{A^{(1-\gamma)(t+1)} k_0^{1-\gamma}}{1-\gamma} = \sum_{t=0}^{\infty} (\beta A^{1-\gamma})^t \frac{A^{1-\gamma} k_0^{1-\gamma}}{1-\gamma}.$$

For the above series to converge, we need to assume $\beta A^{1-\gamma} < 1$. When the restriction is fulfilled, the above expression yields

$$\frac{A^{1-\gamma} k_0^{1-\gamma}}{1-\gamma} \frac{1}{1-\beta A^{1-\gamma}}.$$

Hence, when $\beta A^{1-\gamma} < 1$ we obtain a finite upper bound for the maximum value function, so that Theorem 2 applies and the maximum value function is unique. Notice that the condition we have just obtained is the same we have found reasonable to impose when discussing the results in Exercise 3. From the restriction $\beta A^{1-\gamma} < 1$ we readily obtain the required condition on γ , which is

$$\begin{cases} \gamma > 1 + \log_A \beta & \text{if } A > 1, \\ \gamma < 1 + \log_A \beta & \text{if } A < 1. \end{cases}$$

Exercise 5.

To solve this problem, we need to closely refer to Sub-section (4.2).

Preliminarily, we obtain the formulations for k_{t+1} and c_t .

Once the tentative solution (4.11) is considered, from the first order condition (refer to 4.10), we obtain

$$\gamma - \eta[(1+r)k_t - k_{t+1}] = \beta(h + mk_{t+1}).$$

The two coefficients of the maximum value function, h and m , are given by (4.14) and by (4.13), respectively.

Exploiting the expressions for h and m readily gives

$$(B.7) \quad k_{t+1} = \frac{\gamma}{\eta r} \left(1 - \frac{1}{\beta(1+r)} \right) + \frac{1}{\beta(1+r)} k_t.$$

Equation (B.7) is a difference equation that can be solved for the capital stock. Its solution is:

$$(B.8) \quad k_t = \frac{\gamma}{\eta r} + \left(\frac{1}{\beta(1+r)} \right)^t \left(k_0 - \frac{\gamma}{\eta r} \right)$$

Notice that $\frac{1}{\beta(1+r)}$ is the root for the dynamic Equation (B.7) governing capital. When $\beta(1+r) > 1$, capital is bound to decrease over time (and so is consumption), when instead $\beta(1+r) < 1$ the capital stock (and hence consumption) are ever-increasing. In the knife-edge case $\beta(1+r) = 1$, the capital stock is stationary.

Consumption is given by $c_t = (1+r)k_t - k_{t+1}$ (as implied by 4.9), so that we immediately find:

$$(B.9) \quad c_t = \frac{\gamma}{\eta r} \left(\frac{1}{\beta(1+r)} - 1 \right) + \frac{\beta(1+r)^2 - 1}{\beta(1+r)} k_t.$$

Exploiting Equation (B.9), we see that the marginal utility of consumption is

$$(B.10) \quad U'(c_t) = \gamma - \eta c_t = \left(\frac{\beta(1+r)^2 - 1}{\beta(1+r)} \right) \left(\frac{\gamma}{r} - \eta k_t \right).$$

We now recall from Equation (2.6), that the tvc is

$$\lim_{T \rightarrow \infty} \beta^T U'(c_T) k_{T+1} = 0.$$

From Equations (B.10) and (B.8), we immediately see that

$$\begin{aligned} \lim_{T \rightarrow \infty} \beta^T U'(c_T) k_{T+1} &= \\ &= \lim_{T \rightarrow \infty} \beta^T \left(\frac{\beta(1+r)^2 - 1}{\beta(1+r)} \right) \left(\frac{\gamma}{r} - \eta k_T \right) k_{T+1} = \\ &= \lim_{T \rightarrow \infty} \beta^T \left(\frac{\beta(1+r)^2 - 1}{\beta(1+r)} \right) \left(\frac{\gamma}{r} - \eta \left[\frac{\gamma}{\eta r} + \left(\frac{1}{\beta(1+r)} \right)^T \left(k_0 - \frac{\gamma}{\eta r} \right) \right] \right) \times \\ &\quad \times \left[\frac{\gamma}{\eta r} + \left(\frac{1}{\beta(1+r)} \right)^{T+1} \left(k_0 - \frac{\gamma}{\eta r} \right) \right]. \end{aligned}$$

The limit above converges to 0 if (and only if)

$$\begin{cases} \lim_{T \rightarrow \infty} \left(\frac{\beta}{\beta(1+r)} \right)^T = \lim_{T \rightarrow \infty} \left(\frac{1}{1+r} \right)^T = 0 \\ \lim_{T \rightarrow \infty} \left(\frac{1}{1+r} \right)^T \left(\frac{1}{\beta(1+r)} \right)^{T+1} = 0 \end{cases} .$$

The first condition requires $r \geq 0$, while the third involves the additional constraint $\beta(1+r) \geq 1$.

Hence, when (a) $\beta(1+r) > 1$, the tvc is fulfilled and – as remarked above – capital approaches its steady state value, which is $\frac{\gamma}{\eta^r}$ (the discussion of the consumption dynamics is left to the reader). When instead (b) $\beta(1+r) < 1$, the tvc is *not* fulfilled and the capital stock diverges over time.

Exercise 6.

The solution must fulfil the first order condition with respect to consumption, which is

$$U'(c_t^*) + \beta V'(k_{t+1}) \frac{\partial k_{t+1}}{\partial c_0} = 0.$$

Also, the envelope theorem prescribes:

$$V'(k_t) = \beta V'(k_{t+1}) \frac{\partial k_{t+1}}{\partial k_t}.$$

In our case, given the tentative solution $V(k_t) = e + f \ln k_t$, these two conditions become, respectively:

$$\frac{1}{c_t} + \frac{\beta f}{k_{t+1}} (-1) = 0$$

and

$$\frac{f}{k_t} = \frac{\beta f}{k_{t+1}} \alpha A k_t^{\alpha-1}.$$

Exploiting the transition function, one easily obtains $f = \frac{\alpha}{1-\alpha\beta}$.

Exercise 7.

From Sub-section (5.2) we know that in the Ramsey model it is true that:

$$V_k(k_t) = Q_c (f(k_t) - s_t^*) f_k(k_t).$$

The assumptions concerning utility and production functions introduced in Sub-section (1.1) allow to immediately conclude that $V_k(k_t) > 0$.

Consider then:

$$V_{kk}(k_t) = Q_{cc} (f(k_t) - s_t^*) (f_k(k_t))^2 + Q_c (f(k_t) - s_t^*) f_{kk}(k_t).$$

From the assumptions in Sub-section (1.1) it is immediate to conclude that $V_{kk}(k_t) < 0$.

Exercise 8.

Define the control variable $u_t = f(k_t) + (1 - \delta)k_t - c_t$. This definition implies that the next-period state is equal to the current control ($k_{t+1} = u_t$). The Bellman problem is formulated as follows:

$$\begin{aligned} V(k_t) &= \max_{u_t} \{Q(f(k_t) + (1 - \delta)k_t - u_t) + \beta V(k_{t+1})\}, \\ &\text{s.t. } k_{t+1} = u_t, \\ &\quad k_0 \text{ given.} \end{aligned}$$

The first order condition is: $Q_c(f(k_t) + (1 - \delta)k_t - u_t) = \beta V_k(k_{t+1})$. Differentiating the Bellman equation “on the optimal path”, we obtain:

$$\begin{aligned} V_k(k_t)dk_t &= Q_c(f(k_t) + (1 - \delta)k_t - u_t)(df(k_t) + (1 - \delta)dk_t - du_t) + \\ &\quad + \beta V_k(k_{t+1})dk_{t+1}. \end{aligned}$$

Usage of the first order condition gives:

$$\begin{aligned} V_k(k_t)dk_t &= Q_c(f(k_t) + (1 - \delta)k_t - u_t)(df(k_t) + (1 - \delta)dk_t - du_t) + \\ &\quad + Q_c(f(k_t) + (1 - \delta)k_t - u_t)dk_{t+1}. \end{aligned}$$

Because $dk_{t+1}/du_t = 1$, we then obtain:

$$V_k(k_t) = Q_c(f(k_t) + (1 - \delta)k_t - u_t)(f_k(k_t) + 1 - \delta).$$

Exercise 9.

From our calculations in Section (4.1), we are already aware of the fact that:

$$V(k) = \frac{1}{1 - \beta} \left(\frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta) + \ln(1 - \alpha\beta) + \frac{1}{1 - \alpha\beta} \ln A \right) + \frac{\alpha}{1 - \alpha\beta} k.$$

Substituting the values for the coefficients and for productivity proposed in the main text, and the values for k in the numerical exercise, it is trivial to verify the claim.

Exercise 10.

One can verify, setting up the constrained maximization problem, that $k_4 = 0$ (an obvious result, given that our consumer lives for three periods). Accordingly, period 3 consumption is, for each state i ,

$$(B.11) \quad c_3^{i*} = A^i k_3^\alpha.$$

(We have omitted the time-subscript, since productivity is time-independent.)

From the Lagrangian, one can easily obtain that, from the perspective of period 2, it must be true that:

$$\frac{1}{c_2^{i*}} = \alpha\beta \left(p^H \frac{1}{c_3^{H*}} A^H + p^L \frac{1}{c_3^{L*}} A^L \right) k_3^{\alpha-1}.$$

Exploiting Equation (B.11) we obtain:

$$\frac{1}{c_2^{i*}} = \alpha\beta \left(\frac{p^H + p^L}{k_3} \right).$$

Hence, we have that $k_3 = \alpha\beta c_2^{i*}$ (for $i = L, H$); moreover the intertemporal budget constraint guarantees that $k_3 = A^i k_2^\alpha - c_2^{i*}$. Accordingly, we can obtain the period 2 policy function:

$$(B.12) \quad c_2^{i*} = \frac{1}{(1 + \alpha\beta)} A^i k_2^\alpha,$$

for $i = L, H$.

We then consider the Euler equation, specified from the perspective of period 1, which implies:

$$\frac{1}{c_1^{i*}} = \alpha\beta \left(p^H \frac{1}{c_2^{H*}} A^H + p^L \frac{1}{c_2^{L*}} A^L \right) k_2^{\alpha-1}.$$

Exploiting Equation (B.12) we obtain:

$$\frac{1}{c_1^{i*}} = \alpha\beta(1 + \alpha\beta) \left(\frac{p^H + p^L}{k_2} \right).$$

Hence, we have that $k_2 = \alpha\beta(1 + \alpha\beta)c_1^{i*}$ (for $i = L, H$); from the intertemporal budget constraint we have, as usual, that $k_2 = A^i k_1^\alpha - c_1^{i*}$. So, we can conclude that $c_1^{i*} = \frac{1}{1 + \alpha\beta + (\alpha\beta)^2} A^i k_1^\alpha$ for $i = L, H$.

Following the steps outlined above, the reader can show that the initial period policy function is: $c_0^{i*} = \frac{1}{1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3} A^i k_1^\alpha$ for $i = L, H$.

Notice that, as we move farther and farther away from the final period, the policy function gets closer and closer (actually, converges) to the one prescribed for the infinite horizon case. In fact, we have that

$$\sum_{i=0}^n (\alpha\beta)^i = \frac{1 - (\alpha\beta)^{n+1}}{1 - \alpha\beta}.$$

Exercise 11.

The fact that the probability for the realization of state j at time $t + 1$ depends upon the time t productivity level implies that the Euler equation

(that can be obtained from the Lagrangian) is affected by the time t state. Hence, we have

$$(B.13) \quad \frac{1}{c_0^{H*}} = \alpha\beta \left(p \frac{1}{c_1^{H*}} A^H + (1-p) \frac{1}{c_1^{L*}} A^L \right) k_1^{\alpha-1}$$

when the period 0 productivity level is A^H , and

$$(B.14) \quad \frac{1}{c_0^{L*}} = \alpha\beta \left((1-p) \frac{1}{c_1^{H*}} A^H + p \frac{1}{c_1^{L*}} A^L \right) k_1^{\alpha-1}$$

when productivity is A^L in the initial period.

We now assume that the policy function takes the form

$$(B.15) \quad c_0^{i*} = \chi A^i k_0^\alpha$$

in which χ is an undetermined coefficient. Notice that we are assuming that the propensity to consume is state-independent (as it was in the version of the model discussed in Section 7).

Taking advantage of the guess for the policy function, substitute the consumption levels in the Euler equation (B.13), to obtain:

$$\frac{1}{\chi A^H k_0^\alpha} = \alpha\beta \left(p \frac{1}{\chi A^H k_1^\alpha} A^H + (1-p) \frac{1}{\chi A^L k_1^\alpha} A^L \right) k_1^{\alpha-1},$$

which rapidly reduces to:

$$k_1 = \alpha\beta A^H k_0^\alpha.$$

Since our guess (B.15) implies, for $i = H$, that $k_1 = (1 - \chi) A^H k_0^\alpha$, we have that $\chi = 1 - \alpha\beta$. To complete the solution, one should check that the consumption function (B.15) applies also when the initial productivity level is low. This can be done starting from the Euler equation (B.14).

Hence, the probability structure considered in this Exercise does not affect the consumption function. This, however, does not imply that such a probability structure is completely void of consequences. On this, please refer to Exercise 15.

Exercise 12.

The Euler equation (that can be – as always – obtained from the Lagrangian) is:

$$\frac{1}{c_0^{i*}} = \alpha\beta \left(p \frac{1}{c_1^{H*}} A_t^H + (1-p) \frac{1}{c_1^{L*}} A_t^L \right) k_1^{\alpha-1}.$$

Exploiting the guess $c_0^i = \Psi(A_t^i)k_0^\alpha$ in the above equation, we obtain:

$$(B.16) \quad \frac{1}{\Psi(A_t^i)k_0^\alpha} = \alpha\beta \left(p^H \frac{1}{\Psi(A_t^H)k_1^\alpha} A_t^H + p^L \frac{1}{\Psi(A_t^L)k_1^\alpha} A_t^L \right) k_1^{\alpha-1}.$$

Notice that our tentative solution implies $k_1 = (A^i - \Psi(A^i))k_0^\alpha$, so that Equation (B.16) becomes, after some simplifications:

$$(B.17) \quad \frac{1}{\Psi(A_t^i)} = \alpha\beta \left(p^H \frac{1}{\Psi(A_t^H)} A_t^H + p^L \frac{1}{\Psi(A_t^L)} A_t^L \right) \frac{1}{A_t^i - \Psi(A_t^i)}.$$

Since Equation (B.17) must be valid for $i = L, H$, we can write:

$$\frac{A_t^H - \Psi(A_t^H)}{\Psi(A_t^H)} = \alpha\beta \left(p^H \frac{1}{\Psi(A_t^H)} A_t^H + p^L \frac{1}{\Psi(A_t^L)} A_t^L \right) = \frac{A_t^L - \Psi(A_t^L)}{\Psi(A_t^L)}.$$

Accordingly, we have that

$$\frac{A_t^H}{\Psi(A_t^H)} = \frac{A_t^L}{\Psi(A_t^L)},$$

which implies $\Psi(A_t^i) = \psi A_t^i$.

Exercise 13.

With the proposed utility function, the Euler equation takes the form:

$$c_0^{i*(-\gamma)} = \beta \left[p c_1^{H*(-\gamma)} A^H + (1-p) c_1^{L*(-\gamma)} A^L \right].$$

We now assume that the consumption function is

$$(B.18) \quad c_0^{i*} = \chi A^i k_0$$

in which χ is an undetermined coefficient. We exploit our guess to obtain from the Euler equation

$$(\chi A^i k_0)^{-\gamma} = \beta \left[p (\chi A^H k_1)^{-\gamma} A^H + (1-p) (\chi A^L k_1)^{-\gamma} A^L \right].$$

Notice that Equation (B.18) implies $k_1 = (1-\chi)A^i k_0$, so that the equation above, after some simplifications, becomes

$$(A^i k_0)^{-\gamma} = \beta \left[p (A^H)^{1-\gamma} + (1-p) (A^L)^{1-\gamma} \right] ((1-\chi)A^i k_0)^{-\gamma},$$

so that

$$(1-\chi) = \left(\beta \left[p (A^H)^{1-\gamma} + (1-p) (A^L)^{1-\gamma} \right] \right)^{\frac{1}{\gamma}}.$$

Notice that the coefficient χ can be rewritten in a more compact way as:

$$\chi = 1 - \left(\beta E \left[\tilde{A}^{1-\gamma} \right] \right)^{\frac{1}{\gamma}}.$$

This formulation reveals that, whenever $\gamma > 1$, a mean-preserving increase in the variance for A reduces consumption, which implies the presence of precautionary savings. In fact, the parameter γ can be interpreted as a measure of the representative agent's (relative) risk aversion.

Exercise 14.

The maximum value function depends upon the two state variables, capital and productivity. Accordingly our tentative functional form is: $V(k_t, A_t) = F + G \ln k_t + H \ln A_t$. The Bellman formulation is:

$$F + G \ln k_t + H \ln A_t = \max_{c_t} \{ \ln c_t + \beta E_t [F + G \ln k_{t+1} + H \ln A_{t+1}] \}.$$

The first order condition with respect to (current) consumption is

$$\frac{1}{c_t} = \beta \frac{G}{k_{t+1}}.$$

The expectation operator $E_t[\cdot]$ “disappears” because the capital stock at time $t + 1$ is known when consumption is decided upon.

Exploiting the transition function we readily obtain

$$k_{t+1} = \frac{\beta G}{1 + \beta G} A_t k_t^\alpha,$$

and hence

$$c_t^* = \frac{1}{1 + \beta G} A_t k_t^\alpha.$$

We now take advantage of the Bellman equation, obtaining

$$\begin{aligned} F + G \ln k_t + H \ln A_t &= \ln \frac{1}{1 + \beta G} + \ln A_t + \alpha \ln k_t + \\ &+ \beta E_t \left[F + G \ln \frac{\beta G}{1 + \beta G} + G \ln A_t + G \alpha \ln k_t + H \ln A_{t+1} \right]. \end{aligned}$$

Matching the coefficients for $\ln k_t$ we get that $G = \frac{\alpha}{1 - \alpha\beta}$; this allows to obtain the policy function, which is akin to the one for the non-stochastic case (which is our standard “disappointing” result):

$$c_t^* = (1 - \alpha\beta) A_t k_t^\alpha.$$

Considering the coefficients for $\ln A_t$, we then obtain that $H = \frac{1}{1 - \alpha\beta}$; finally, we match the constant terms, which gives

$$F = \ln \frac{1}{1 + \beta G} + \beta \left(F + G \ln \frac{\beta G}{1 + \beta G} + H E_t [\ln A_{t+1}] \right).$$

Exploiting G and H , we readily get:

$$F = \frac{1}{1 - \beta} \left(\ln(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \ln(\alpha\beta) + \frac{\beta}{1 - \alpha\beta} E_t [\ln A_{t+1}] \right).$$

Compare the expression for F with expression (4.7), and notice that, although the variability of productivity does not affect the consumption decision, it does affect welfare: a mean preserving spread in A reduces $E_t [\ln A_{t+1}]$, and therefore the overall utility of the representative consumer.

Exercise 15.

In this application, the current productivity level influences the future one. Notice, in particular, that if p is close to 1, the current productivity is going to influence production for a long time in the future. This must bear some relevant consequence for the overall utility for the representative agent. Nonetheless, we stick to our standard assumption:

$$V(k_t, A_t) = F + G \ln k_t + H \ln A_t,$$

The Bellman formulation is, for the generic realization i :

$$\begin{aligned} \text{(B.19)} \quad F + G \ln k_t + H \ln A_t^i &= \\ &= \max_{c_t} \left\{ \ln c_t^i + \beta E_t [F + G \ln k_{t+1} + H \ln A_{t+1} | A_t = A^i] \right\}. \end{aligned}$$

The first order condition with respect to the current consumption is

$$\frac{1}{c_t^i} = \beta \frac{G}{k_{t+1}}.$$

The expectation operator $E_t [\cdot]$ “disappears” because the capital stock at time $t + 1$ is known when consumption is decided upon.

Exploiting the transition function we readily obtain

$$k_{t+1} = \frac{\beta G}{1 + \beta G} A_t k_t^\alpha,$$

and hence

$$c_t = \frac{1}{1 + \beta G} A_t k_t^\alpha.$$

Inspecting Equation (B.19) one might be tempted to conclude that the guess is not verified, since $E_t [\ln A_{t+1} | A_t = A^i] = p \ln A^i + (1 - p) \ln A^j$ so that the right hand side of the Equation also depends on the state A^j . However, this conclusion concerning the guess would be wrong. Notice, in

fact, that the conditional expectation $E_t [\ln A_{t+1} | A_t = A^i]$ can be expressed as a function of the *unconditional* expectation $E [\ln \tilde{A}]$, which is a constant, and of the state A^i :

$$\begin{aligned} E_t [\ln A_{t+1} | A_t = A^i] &\equiv p \ln A^i + (1-p) \ln A^j = \\ &= 2(1-p) \left(\frac{\ln A^i + \ln A^j}{2} \right) + (2p-1) \ln A^i = \\ &= 2(1-p) E [\ln \tilde{A}] + (2p-1) \ln A^i. \end{aligned}$$

Exploiting the Equation above and our result concerning k_{t+1} in Equation (B.19) we obtain (omitting the time index for productivity)

$$\begin{aligned} F + G \ln k_t + H \ln A^i &= \\ &= \ln \frac{1}{1+\beta G} + \ln A^i + \alpha \ln k_t + \beta F + \beta G \ln \frac{\beta G}{1+\beta G} + \\ &+ \beta G \ln A^i + \beta G \alpha \ln k_t + \beta H \left(2(1-p) E [\ln \tilde{A}] + (2p-1) \ln A^i \right). \end{aligned}$$

Matching the coefficients for $\ln k_t$ we obtain

$$G = \frac{\alpha}{1-\alpha\beta}.$$

This allows to get the policy function, which is not affected by the persistence probability p :

$$c_t^{i*} = (1-\alpha\beta) A^i k_t^\alpha.$$

We then consider the coefficients for $\ln A^i$, obtaining $H = 1 + \beta G + \beta H (2p-1)$, and hence

$$H = \frac{1}{(1-\alpha\beta)(1+\beta(1-2p))}.$$

This shows that the current productivity level does bear relevant consequence on the overall utility for the representative agent; this effect is stronger the higher is the persistence parameter p . In fact we have that $\partial H / \partial p = 2H\beta / (1+\beta(1-2p))$.

Finally, we determine F . Matching the left and the right hand side of the Bellman Equation we obtain, once G and H have been substituted out:

$$F = \ln(1-\alpha\beta) + \beta F + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \frac{2\beta(1-p)}{(1-\alpha\beta)(1+\beta(1-2p))} E [\ln \tilde{A}],$$

and therefore:

$$F = \frac{1}{1-\beta} \left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \frac{2\beta(1-p)E[\ln \tilde{A}]}{(1-\alpha\beta)(1+\beta(1-2p))} \right).$$

Notice that, when $p = 1$ (so that the problem becomes deterministic), the maximum value function becomes:

$$V(k_t, A^i) = \frac{1}{1-\beta} \left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \frac{\ln A^i}{1-\alpha\beta} \right) + \frac{\alpha}{1-\alpha\beta} \ln k_t,$$

which matches what has been obtained in Sub-section 4.1.

Finally, notice that, when $p = 1/2$, there is no persistence, and hence the maximum value function should correspond to the one obtained solving Exercise (14). This is exactly what happens. In fact, we have:

$$\begin{aligned} V(k_t, A^i) = \frac{1}{1-\beta} \left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \frac{\beta}{1-\alpha\beta} E[\ln \tilde{A}] \right) + \\ + \frac{\alpha}{1-\alpha\beta} \ln k_t + \frac{1}{1-\alpha\beta} \ln A^i. \end{aligned}$$

Exercise 16.

As in the previous examples, the maximum value function depends upon the two state variables, capital and productivity. As a tentative maximum value function we choose: $V(k_t, A_t^i) = F + G \frac{(A_t^i k_t)^{1-\gamma}}{1-\gamma}$, so that the Bellman formulation is:

$$F + G \frac{(A_t^i k_t)^{1-\gamma}}{1-\gamma} = \max_{c_t} \left\{ \frac{c_t^{1-\gamma}}{1-\gamma} + \beta E_t \left[F + G \frac{(A_{t+1}^i k_{t+1})^{1-\gamma}}{1-\gamma} \right] \right\}.$$

The first order condition with respect to current consumption is

$$c_t^{-\gamma} = \beta G E_t \left[A_{t+1}^{i(1-\gamma)} \right] k_{t+1}^{-\gamma},$$

so that

$$c_t = (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} k_{t+1},$$

Exploiting the intertemporal constraint, we readily obtain

$$k_{t+1} = \frac{1}{1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}}} A_t^i k_t,$$

and hence

$$c_t^* = \frac{(\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}}}{1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}}} A_t k_t.$$

The Bellman equation is now used to pin down the coefficients. Our starting point is:

$$\begin{aligned} F + G \frac{(A_t^i k_t)^{1-\gamma}}{1-\gamma} &= \frac{1}{1-\gamma} \frac{(\beta G)^{\frac{\gamma-1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{\frac{\gamma-1}{\gamma}}}{\left(1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} \right)^{1-\gamma}} (A_t^i k_t)^{1-\gamma} + \\ &+ \beta E_t \left[F + G \frac{A_{t+1}^{i(1-\gamma)}}{1-\gamma} \frac{1}{\left(1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} \right)^{1-\gamma}} (A_t^i k_t)^{1-\gamma} \right], \end{aligned}$$

from which it is immediate to see that $F = 0$, and that G is determined by:

$$\begin{aligned} G &= \\ &= \frac{(\beta G)^{\frac{\gamma-1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{\frac{\gamma-1}{\gamma}}}{\left(1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} \right)^{1-\gamma}} + \beta G \frac{E_t \left[A_{t+1}^{i(1-\gamma)} \right]}{\left(1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} \right)^{1-\gamma}}. \end{aligned}$$

The above expression – scaring as it may be – guarantees that our guess is correct. In fact, it is apparent that the coefficient G does not depend on capital, moreover it does not depend on productivity as well. In fact our assumptions guarantee that $E_t \left[A_{t+1}^i \right]$ is time and state independent.

Moreover notice that, grouping terms, it is easy to obtain:

$$G = \beta G E_t \left[A_{t+1}^{i(1-\gamma)} \right] \frac{1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}}}{\left(1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} \right)^{1-\gamma}},$$

and hence

$$1 = \beta E_t \left[A_{t+1}^{i(1-\gamma)} \right] \frac{1}{\left(1 + (\beta G)^{-\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{-\frac{1}{\gamma}} \right)^{-\gamma}},$$

so that

$$G = \left(1 - \beta^{\frac{1}{\gamma}} E_t \left[A_{t+1}^{i(1-\gamma)} \right]^{\frac{1}{\gamma}} \right)^{-\gamma}.$$

Exercise 17.

In this example, there are two states and two controls. We assume that the maximum value function takes the form $V(k_t, A_t) = F + G \ln k_t + H \ln A_t$, so that the Bellman formulation is:

$$\begin{aligned} F + G \ln k_t + H \ln A_t &= \max_{c_t, l_t} \{ \ln(c_t) + \gamma \ln(1 - l_t) + \\ &\quad + \beta E_t [F + G \ln k_{t+1} + H \ln A_{t+1}] \}, \end{aligned}$$

with the two first order conditions quickly leading to:

$$(B.20) \quad c_t^* = \frac{k_{t+1}}{\beta G}$$

and to

$$(B.21) \quad \frac{\gamma}{1 - l_t^*} = \frac{\beta G}{k_{t+1}} (1 - \alpha) A k_t^\alpha l_t^{*(-\alpha)}.$$

Exploiting the dynamic constraint and the first-order condition (B.20) we obtain a (familiar) relation between the future state and current output, which is:

$$k_{t+1} = \frac{\beta G}{1 + \beta G} A k_t^\alpha l_t^{*1-\alpha},$$

so that the first-order condition (B.21) can be written as:

$$\frac{\gamma}{1 - l_t^*} = \frac{(1 + \beta G)(1 - \alpha)}{l_t^*},$$

which implies that the optimal labour supply is independent from the capital stock (and from the productivity level):

$$l_t^* = l^* = \frac{(1 + \beta G)(1 - \alpha)}{\gamma + (1 + \beta G)(1 - \alpha)} < 1.$$

We now follow our usual procedure: we use the tentative solution into the Bellman equation, which gives

$$\begin{aligned}
F + G \ln k_t + H \ln A_t &= \\
&= \ln \left(\frac{1}{1 + \beta G} \right) + \ln A_t + \alpha \ln k_t + (1 - \alpha) \ln l^* + \gamma \ln (1 - l^*) + \\
&+ \beta F + \beta G \left(\ln \frac{\beta G}{1 + \beta G} + \ln A_t + \alpha \ln k_t + (1 - \alpha) \ln l^* \right) + \beta H E_t [\ln A_{t+1}].
\end{aligned}$$

Since the evolution for productivity is described by (9.1), we notice that the above equation can be rewritten as:

$$\begin{aligned}
F + G \ln k_t + H \ln A_t &= \\
&= \ln \left(\frac{1}{1 + \beta G} \right) + \ln A_t + \alpha \ln k_t + (1 - \alpha) \ln l^* + \gamma \ln (1 - l^*) + \\
&+ \beta F + \beta G \left(\ln \frac{\beta G}{1 + \beta G} + \ln A_t + \alpha \ln k_t + (1 - \alpha) \ln l^* \right) + \beta H \rho \ln A_t.
\end{aligned}$$

Matching the coefficients for $\ln k_t$ and for $\ln A_t$ we obtain, respectively:

$$\begin{aligned}
G &= \frac{\alpha}{1 - \alpha\beta}, \\
H &= \frac{1}{(1 - \rho\beta)(1 - \alpha\beta)}.
\end{aligned}$$

It is now trivial to compute F .

The disappointment we expressed in the main text (Sub-section 9.1) extends to the version of the model with endogenous labour supply: with logarithmic preferences, the role for productivity shocks (even when they are persistent!) appears to be limited.

Exercise 18.

The maximum value function depends upon two state variables: the remaining part of the cake (y_t) and the taste shock (z_t). Our guess for the value function is : $V(y_t, z_t) = F + G \ln y_t + H \ln z_t$, so that the Bellman formulation becomes:

$$F + G \ln y_t + H \ln z_t = \max_{c_t} \{ \ln(z_t c_t) + \beta E_t [F + G \ln y_{t+1} + H \ln z_{t+1}] \}.$$

The first order condition with respect to current consumption is

$$\frac{1}{c_t} = \beta \frac{G}{y_{t+1}}.$$

The expectation operator $E_t[\cdot]$ disappears because the time $t + 1$ size of the cake is known when current consumption is decided upon.

Exploiting the transition function, we readily obtain

$$y_{t+1} = \frac{\beta G}{1 + \beta G} y_t,$$

and hence

$$c_t^* = \frac{1}{1 + \beta G} y_t.$$

We now substitute the above results into the Bellman equation, which gives

$$\begin{aligned} F + G \ln y_t + H \ln z_t &= \ln z_t + \ln \frac{1}{1 + \beta G} + \ln y_t + \\ &+ \beta E_t \left[F + G \ln \left(\frac{\beta G}{1 + \beta G} \right) + G \ln y_t + H \ln z_{t+1} \right]. \end{aligned}$$

Matching the coefficients for $\ln z_t$ we immediately obtain that $H = 1$, while the coefficient for $\ln y_t$ turns out to be $G = \frac{1}{1-\beta}$. Accordingly, the consumption function is $c_t^* = (1 - \beta) y_t$. Notice that the higher is β , the more patient is the consumer and hence the smaller is the slice of the cake she decides to consume today.

Finally, one can verify that

$$F = \frac{1}{1 - \beta} \left(\ln(1 - \beta) + \frac{\beta}{1 - \beta} \ln \beta + \beta E_t [\ln z_{t+1}] \right).$$

Exercise 19.

In this old-fashion economy, past inflation is a state variable, as the shock is. Since the target function is quadratic, our tentative solution is a second order polynomial in π_{t-1} and ϵ_t : $V(\pi_{t-1}, \epsilon_t) = F + G\pi_{t-1} + H\pi_{t-1}^2 + I\epsilon_t + L\epsilon_t^2 + M\pi_{t-1}\epsilon_t$. We assume, with no loss of generality, that $E_{t-1}[\epsilon_t] = 0$, $Var[\epsilon_t | I_{t-1}] = \sigma^2$, and we introduce the Bellman equation:

$$\begin{aligned} F + G\pi_{t-1} + H\pi_{t-1}^2 + I\epsilon_t + L\epsilon_t^2 + M\pi_{t-1}\epsilon_t &= \max_{y_t} \{ y_t^2 + \delta\pi_t^2 + \\ &+ \beta E_t [F + G\pi_t + H\pi_t^2 + I\epsilon_{t+1} + L\epsilon_{t+1}^2 + M\pi_t\epsilon_{t+1}] \}. \end{aligned}$$

The first order condition with respect to current output is

$$2y_t + \beta E_t \left[(G + 2H\pi_t + M\epsilon_{t+1}) \frac{\partial \pi_t}{\partial y_t} \right] = 0,$$

which gives:

$$2y_t = -\beta\gamma (G + 2H(\pi_{t-1} + \gamma y_t + \epsilon_t)).$$

Accordingly, the optimal decision for a policy maker who decides the output level having observed the realization for the shock is:

$$(B.22) \quad y_t^* = -\frac{\beta\gamma G}{2(1+\beta\gamma^2 H)} - \frac{\beta\gamma H}{1+\beta\gamma^2 H} (\pi_{t-1} + \epsilon_t),$$

so that the transition equation becomes:

$$(B.23) \quad \pi_t = -\frac{\beta\gamma^2 G}{2(1+\beta\gamma^2 H)} + \frac{1}{1+\beta\gamma^2 H} (\pi_{t-1} + \epsilon_t).$$

Exploiting Equations (B.22) and (B.23), and our assumptions about ϵ_t into the Bellman equation, we obtain:

$$\begin{aligned} F + G\pi_{t-1} + H\pi_{t-1}^2 + I\epsilon_t + L\epsilon_t^2 + M\pi_{t-1}\epsilon_t &= \\ &= \left(-\frac{\beta\gamma G}{2(1+\beta\gamma^2 H)} - \frac{\beta\gamma H}{1+\beta\gamma^2 H} (\pi_{t-1} + \epsilon_t) \right)^2 + \\ &+ (\delta + \beta H) \left(-\frac{\beta\gamma^2 G}{2(1+\beta\gamma^2 H)} + \frac{1}{1+\beta\gamma^2 H} (\pi_{t-1} + \epsilon_t) \right)^2 + \\ &+ \beta F - \frac{(\beta\gamma G)^2}{2(1+\beta\gamma^2 H)} + \frac{\beta G}{1+\beta\gamma^2 H} (\pi_{t-1} + \epsilon_t) + \beta L\sigma^2. \end{aligned}$$

From the above expression, it is apparent that our guess is correct; it is also clear that – at this stage – the most sensible thing to do is to feed our computer with values for β , δ , γ , and σ^2 , and let it solve for the coefficients.

Exercise 20.

The problem is analogous to the one presented in Exercise (14). Hence, the maximum value function is:

$$\begin{aligned} V(k_t, A_t) &= \frac{1}{1-\beta} \left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \frac{\beta}{1-\alpha\beta} E_t[\ln A_{t+1}] \right) + \\ &+ \frac{\alpha}{1-\alpha\beta} \ln k_t + \frac{1}{1-\alpha\beta} \ln A_t, \end{aligned}$$

so that

$$\begin{aligned} E[V(k_t, \tilde{A})] &= \\ &= \frac{1}{1-\beta} \left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \frac{1}{1-\alpha\beta} E_t[\ln \tilde{A}] \right) + \frac{\alpha}{1-\alpha\beta} \ln k_t. \end{aligned}$$

Substituting the values for the coefficients and the values for k_t , and using the productivity levels and their probabilities to compute the expected value, it is trivial to verify the claim.

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