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E *Many* Pluribus Unum:
A Behavioural Macro-Economic Agent Based Model

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Abstract

In this paper we develop a *Hybrid Macroeconomic ABM.*
The economy is populated with firms heterogeneous in term of financial fragility, measured via the *Equity Ratio.* Firms are maximizing profit by choosing capital, which can not be raised on the stock market. Therefore they have to rely on a loan charged by the External Finance Premium that is decreasing in the Equity Ratio; profits are retained as net worth. The economy is populated also with consumers, whom optimize their utility holding individual non-rational expectations; thus they are heterogeneous in their expectations. The expectations formation process is the Heuristic Switching Model: given a fixed set of heuristics, agents choose the one which has performed the best in the past; this model has been extrapolated and validated in several Learn to Forecast Experiments.
Thanks to the Variant Representative Agent approach, there can be built *bottom-up* a macroeconomic system which encapsulates the heterogeneity by considering relevant moments rather than the whole the distributions of the heterogeneous characteristics of the agents. The emergent macro-economic system will be described by an optimized IS, a Taylor Rule and a Phillips Curve; thus the model tries to bridge NK-DSGE with ABM.
The proposed approach result in a macro-economic system with a reduced dimensionality, thus it is analytical tractable and it resumes macroeconomic thinking: the ABM is used to link periods and keep track of the individual distribution, while the macro-economic model is a *frame* that pictures the value of the fundamentals given the distributions.
The model is put at work through a fiscal shock and it shows it capabilities in disentangling the transmission of a shock in its *direct* and *indirect* effect: the former are the ones directly caused by the shocked variable, the latter instead derive from the evolution of the distributions. Moreover the model is also able to distinguish the contributions to the shock of the *Representative Agent Component* opposed to the *Heterogeneous Agent Component.*

JEL classification: E32, E44, E70, D84, D90.

Keywords: Heterogeneity, Financial fragility, Bounded Rationality, Heterogeneous Expectations, Aggregation, Business cycles

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1 Introduction

This paper presents a model which allows heterogeneity and boundedly rationality in a *simple* framework. Over the last decades macroeconomics mainly developed by grounding on two pillars: the *Representative Agent hypothesis* (RA) and *Rational Expectation hypothesis* (RE). RA assumes that is sufficient to study the average agent rather than each single agent, in order to study the aggregate system. RE, proposed by Muth (1961), supposes that agents can not be systematically wrong and hence their expectation are confirmed. Despite the great analytical simplifications these two assumptions bring[1], they are also often not confirmed empirically. Indeed the contribution of the paper is mainly methodological, since we propose a technique to save *simplicity* and *coherence* simultaneously.

Kirman (1992) articulates his critic toward the RA by highlighting four main issues: 1) there is not any direct relation between individual behaviour and collective behaviour, 2) the reactions of the RA to a parameter change or to a shock may not necessarily coincide with the aggregate reaction, hence policy analysis in this framework can be deceiving, 3) the RA is a utility maximizer, but there may be cases in which the preferences of RA do not represent the preferences of the represented individuals, and 4) from an econometric perspective, it is not clear whether one is rejecting any additional hypothesis, or the RA hypothesis itself. Moreover RA prevents from studying distributional issues, which are becoming more and more of interest. Thus any time the differences among agents in the model are *relevant* (i.e. that impact on the second or higher moment of the distributions) and *persistent* (i.e. not bound to disappear), the RA hypothesis should be dismissed in favour of a bottom-up approach, able to consider the individual differences. An ideal technique can be found in Agent Based Models (ABM), which study the *individuals* before considering the *aggregate*. However ABM comes at a cost: it increases exponentially the dimensions of the system, limiting hence the capacity to link causes and consequences at the aggregate level. Thomas (1979) warn of this risk by writing "Science get most of its information by the process of reductionism, exploring the details, then the details of the details, until all the smallest bits of the structure, or the smallest parts of the mechanism, are laid out for counting and scrutiny. Only when this is done can the investigation be extended to encompass the whole organism or the entire system". ABMs thus allow for a complete micro-foundation of the model, enhancing indeed the greatest level of heterogeneity, but they may make hard to retrieve causal relationships of the whole system: there has to be balanced the level of the details with the capability to open the *black-box*; specifically in the case of economics, one has to weight the underlying micro foundations and the emerging macroeconomic system.

The works presented in Assenza and Delli Gatti (2013, 2016), propose an approach which allows to save heterogeneity without losing the capacity to think in macroeconomic terms. They propose a methodology called *Hybrid Macroeconomic* ABM: benefiting from an appropriate stochastic aggregation procedure, they derive the *Modified*-Representative Agent (MRA) "that essentially approximates the evolution over time of the entire distribution of agents' characteristics by means of the dynamics of a finite set of moments of the distribution" (Assenza and Delli Gatti (2013)). The MRA is the *monad* upon whom the emergent macroeconomic model is built. Similarly, the model presented in this paper will be built from the bottom-up thanks to the aforementioned approach: despite the aggregation is done over a *representative agent*, it encapsulates relevant information which depends on the heterogeneity of agents. Firms are characterized by different levels of financial robustness and consumers have heterogeneous expectations. Indeed we will always able to retrieve the moments of the relevant distributions and embed them in the MRA. The introduction of heterogeneous expectations of the consumer is one of the novelty introduced with respect to the original model. Hence, as a consequence of this approach, the emerging macroeconomic model is characterized by an high degree of analytical tractability and it closely resembles a NK-DSGE (e.g Galí (2009)), enriched with heterogeneity and bounded rationality (e.g. Branch and McGough (2009), Hommes and Lustenhouwer (2015), and Kurz

[1]See Fagiolo and Roventini (2017) for a detailed discussion.

et al. (2013)), but it will be agent-based: the paper indeed helps in bridging the gap between DSGE and ABM.

Empirical evidence suggests that RE hypothesis should be relaxed. Mankiw et al. (2004), among other scholars, support the idea of heterogeneity in expectations, which can not be always assumed to be *rational* in the Muthian sense. According to psychology, agents in fact face physical and physiological limits (Simon (1956)) and they are anchored to the past (i.e. *status-quo bias*), preventing them from reaching the Rational Expectation. Since the seminal work of Marimon and Sunder (1994), Learn to Forecast Experiments (LtFE) have become a commonly accepted mean to study how expectations are formed in a controlled environment (see e.g Hommes (2011) for a survey); LtFE allows to study patterns and regularities in order to extrapolate *expectation formation* models empirically based. One of the key results that is often emerging in those experiments (e.g. Anufriev and Hommes (2012), Assenza et al. (2013) and Pfajfar and Zakelj (2014)) is that subjects tend to use past realizations of the variables elaborated in simple rules (i.e. *heuristics* - Tversky and Kahneman (1974)) to predict their future values: subjects are adaptive.

The Heuristic Switching Model (HSM) is a expectation formation model, derived from a series of experiment conducted at Universitat van Amsterdam: out of a set of given heuristics, subjects choose the one that have performed the best in the past in order to form their forecasts. In the model we are presenting, the HSM will be used by the consumers to form boundedly rational and heterogeneous expectations: the model had been proposed theoretically by Brock and Hommes (1997), then experimentally extrapolated by Anufriev and Hommes (2012) and finally validated in Assenza et al. (2013).
Note that, enriching an ABM with pure experimental results goes in the direction of building another bridge: from Experimental Economics to ABM, hence developing those synergies that Duffy (2004) was glimpsing.

Wrapping up, we define the model as a Hybrid Macroeconomic Behavioural Agent Based Model.

Therefore in this paper we develop a model which allows for heterogeneity and boundedly rational expectation without losing analytical capacity, which is crucial in order to spell out the causal relationships and the transmission mechanism of a shock.
The structure of the model can be summarized as follows. There are two types of agents: consumers and firms. Firms optimize profits, by choosing the optimal level of capital. Since they are not always able to finance their activities with internal resources nor by issuing equity (due to *equity rationing*, see Greenwald et al. (1984)), they are forced to ask a loan to banks, which will charge an *External Finance Premium* increasing in the financial fragility of the firm, i.e. the inverse of the *Equity Ratio*[2] (ER). Consumers might be employed or unemployed, earning a wage or receiving a subsidy respectively; they choose their consumption level as the outcome of a utility maximization process, which depends on consumption and inflation expectation.
The model is characterized by three dimensions of heterogeneity: the equity ratio, the individual consumption expectation and the inflation expectation. Thanks to the Variant Representative Agent approach (VRA[3]), presented in Gallegati et al. (2006), we can aggregate individual variables by taking into account (at least) the first moment of the distributions of agents' characteristics[4].
As the macroeconomic fundamentals are computed, we derive the macroeconomic system composed by an optimized IS, a Taylor Rule and a Philipps Curve. The macroeconomic equilibrium is computed at period t given the distributions of characteristics in $t-1$: it is a static system. The underlying ABM studies the evolution of agents, thus updating the distributions of the characteristics subject to

[2]Throughout the paper, the equity ratio is defined as the ratio between net worth (i.e. retained profits net of paid dividends) and working capital

[3]Note that *VRA* is the original labelling for the process used in Assenza and Delli Gatti (2013, 2016) to derive the *MRA*. In this paper we stick to the original labelling.

[4]For the sake of simplicity, we have chosen to consider only the mean and the variance, even though one can theoretically include all the moments considered relevant.

the contemporaneous fundamentals: macroeconomic equilibrium at time t is a frame of a sequence driven by the ABM. The model exhibits *ergodic* properties, therefore it settles on *long-run steady state* values, which depend on the values of parameters and of the emerging individual characteristics.
As the model is shocked, we exploit fully the analytical tractability: indeed we study the transmission of the shock in two dimensions. The first one distinguishes between *direct* and *indirect* effect: the former is the effect solely caused by the shocked variable, the latter is the long term response caused by the endogenous variation of the individual characteristics triggered by the shock. We might define the *direct effect* as the short-term effect, and the *indirect effect* as the long-run effect. The second dimension studies precisely the effect of the *Representative Agent* and of the *Hetereogenous Agent*, by insulating the contribution of the first moments of the distributions from the second moments.
The underling rationale of the paper is not to be found only on the methodological side, but is deeply rooted in the urge of the policy makers to study heterogeneity and boundedly rational expectations. We identify a set of normative research questions, which can be answered thanks to the model developed. In particular, building on Yellen (2016), we study the effect of heterogeneity and of expectations. We believe that our model offers an ideal ground, since we insulate quantitatively the effect of heterogeneity in the transmission of the shock, both in the short-run and in the long-run. Since we introduced in the model also expectations and their explicit formation process, we study the fluctuations of the business cycles as being driven by the emergence of one heuristic over the others, i.e. *weak Animal Spirit* (Franke and Westerhoff, 2016).
Summing up on the normative research questions, we will study which are the consequences of heterogeneity in shaping macroeconomic variable and the dynamics of the system on the one side; on the other the relationship between Animal Spirits and the fluctuations of the Business Cycle (e.g. De Grauwe (2012)).
The model we present is consistent with some economic regularities and intuitions, and shows that heterogeneity plays a significant role in the transmission of a positive fiscal shock, small but detrimental to the economy.

The rest of the paper is organized as follows Section 2 will present the micro-foundations of the model: namely the optimal capital demand as a function of the equity ratio and the optimal consumption driven by the expectations in consumption and inflation; moreover we will also derive the *approximated* optimal values, which are the ones deriving from the VRA. In Section 3 we will present the aggregation of individual variables and derive the analytical macro-economic system. Additionally there will be also presented the structure of the ABM. In Section 4 the model is put at work, by presenting the effect of a positive fiscal shock: we will study its dynamics and transmission mechanisms; finally Section 5 wraps up and presents the results.

2 Micro - Foundations

2.1 Firms

The economy is populated with F firms (indexed by $f = 1, \ldots, F$), which produce a homogeneous good by means of capital and labour in a competitive setting. Firms optimize production by choosing the level of capital to be demanded: if the internal resources are not enough to *cover* the capital demand, firms ask for a loan and thus they might be subject to bankruptcy. The heterogeneity distinguishing the firms is in the degree of financial fragility, which is captured in the *Equity Ratio* (ER)[5], defined as the ratio between net worth ($A_{,f,t}$) and capital ($k_{f,t}$)

$$H_{f,t} := \frac{A_{f,t}}{k_{f,t}} \in [0, 1) \tag{2.1}$$

[5]In the original model, the financial fragility is depending only on the net worth: this definition perhaps does not take into account those firms which are small but solid.

The left side bound is intuitive: a firm can not have a negative net worth or capital; Also the right side is intuitive: as soon as a firm is able to self-finance entirely, i.e $H_{f,t} = 1$, then is not asking any more for a loan.
Banks, when extending loans, take into account the financial fragility so to be backed-up from the bankruptcy risk that firms are facing, thus they charge an individual interest rate, composed by the prevalent interest rate and the component varying with the financial fragility of the firm, which is reflected in the *External Finance Premium* (EFP), that is decreasing in the equity ratio. Therefore firms will be different according to their financial robustness; moreover the financial robustness is depending on the past performances of the firms: this imply that the distribution characterizing the equity ratio is time varying. However it will be always possible to retrieve the cross-sectional moments: the mean indeed will be $\mu_t^H := \frac{1}{F}\sum_{f=1}^{F} H_{f,t}$ and the variance $Var_t^H := \frac{1}{F}\sum_{f=1}^{F}\left(H_{f,t} - \mu_t^H\right)^2$. As the equity ratio is time varying and heterogeneous across firms, also the associated distribution will be time varying: if it eventually settles on a long-run steady state, then the mean and variance will settle too on values denoted by $\widehat{E}^H$ and $\widehat{Var}^H$
Throughout the rest of the paper there we will assume that capital fully depreciates each period. Finally net worth is considered to be the cumulative profits of the firm, net of the paid dividends.

The production is carried on by each firm by mean of a Leontief production function, described by

$$Y_{f,t} = min\left[\lambda N_{f,t}, \nu k_{f,t}\right] \tag{2.2}$$

with Y, N and k being the firm-specific production, labour and capital, and λ and ν the economy-wide labour and capital productivity respectively. There can be then defined profit in real terms as

$$\phi_{f,t} = \underbrace{u_{f,t}Y_{f,t}}_{\text{Total Revenues}} \underbrace{- wN_{f,t} + r_{f,t}k_{f,t} + \tau\frac{1}{2}k_{f,t}^2}_{\text{Total Costs}} \Leftrightarrow \tag{2.3}$$

The profit of firm f in real terms is the difference between total real revenues (first parenthesis) and the real total costs (second parenthesis). Total revenues are subject to an idiosyncratic shock

$$u_{f,t} \sim U\left[0, 2\right]$$

Borrowing the assumption from Greenwald and Stiglitz (1987), the shock can be thought as a price shock with $E(u_f, t) = 1$. Total costs are composed by the labour bill ($wN_{f,t}$), the capital bill ($r_{f,t}k_{f,t}$) and the investment costs ($\tau\frac{1}{2}k_{f,t}^2$); investments costs are quadratic as common in the literature.The parameter $0 < \tau < 1$, which can be interpreted as the cost of investment, has to be seen as a scaling parameter, since it allows to *exogenously* scale the capital demand, with respect to other variables in the model, without loss of generality. The real wage w is given and constant, while the cost of capital $r_{f,t}$ is firm specific and will be varying over time according to the financial soundness of the firm.
By assuming that labour is abundant and given the Leontief production function in eq.(2.2), we get

$$Y_{f,t} = \nu k_{f,t}$$

$$N_{f,t} = \frac{\nu}{\lambda}k_{f,t}$$

substituting in definition of real profits in eq.(2.3) and rearranging, we obtain

$$\phi_{f,t} = \left(u_{f,t}\nu - w\frac{\nu}{\lambda} - r_{f,t}\right)k_{f,t} - \tau\frac{1}{2}k_{f,t}^2$$

from which it follows that expect profit in real terms is

$$E(\phi_{f,t}) = \left(\nu - w\frac{\nu}{\lambda} - r_{f,t}\right) k_{f,t} - \tau\frac{1}{2}k_{f,t}^2 \tag{2.4}$$

that represents the objective function of the risk-neutral firm: note that the firm is left to choose only the optimal level of capital, since the employment level will be implicitly determined by the shape of the production function. Therefore each firm solves

$$max_{k_{f,t}} E(\phi_{f,t}) = \left(\nu - w\frac{\nu}{\lambda} - r_{f,t}\right) k_{f,t} - \tau\frac{1}{2}k_{f,t}^2$$

which is maximized at

$$k_{f,t} \quad = \frac{\gamma - r_{f,t}}{\tau} \tag{2.5}$$

with $\gamma := \nu\left(1 - \frac{w}{\lambda}\right)$. It is imposed $\lambda > w$ so that $\gamma > 0$. Optimal capital demand is a function of parameters and the cost of capital.
We are left with specifying a functional form for $r_{f,t}$: following Assenza and Delli Gatti (2013, 2016) it is assumed to be composed by the prevalent risk-free interest rate and the firm-specific EFP, labelled as $f_{f,t}$.

$$r_{f,t} = r_t + f_{f,t} \tag{2.6}$$

As previously introduced, the EFP, in the spirit of Assenza and Delli Gatti (2013), is decreasing in the equity ratio of the previous period, thus there can be specified

$$f_{f,t} = \frac{\alpha}{H_{f,t-1}} \tag{2.7}$$

α is uniform across firms and it represents the exogenous component of the EFP.
If a firm is able to self-finance, that is $A_{f,t} \geq k_{f,t}$, then $H_{f,t} = 1$ and $f_{f,t+1} = 0$ by assumption. On the contrary, if a firm is fragile, $A_{f,t} < k_{f,t} \Leftrightarrow H_{f,t+1} < 1$, then the bank will lend money but at a rate higher than the minimum. As $H_{f,t}$ approaches zero, EFP grows to infinity pushing the firm out of the market, since she would have to bear too high financing costs.
Finally notice that the current EFP is a function of the equity ratio in the previous period: this can be explained by assuming that, in period t the firm walk in a bank to negotiate the loan and shows the books, which are updated until period $t-1$.

Moreover the negativity in capital demand is to be avoided, since it would be illogical in the context of the model, thus we assume:

Assumption 1 (Non-negativity of capital) *In order not to have a negative capital, a firm has to obtain a sufficiently low external finance premium*[6]

$$\Leftrightarrow f_{f,t} < f_t^{max} := \gamma - r_t$$

[6]By plugging eq.(2.6) in eq.(2.5) and imposing positivity:

$$k_{f,t} \quad = \frac{\gamma - r_t - f_{f,t}}{\tau} > 0$$

Since $\tau > 0$ and $\gamma > 0$ by assumption, we obtain

$$k_{f,t} > 0 \Leftarrow \gamma - r_t - f_{f,t} > 0 \Leftrightarrow f_{f,t} < \gamma - r_t$$

The same condition can be written also in terms of the equity ratio[7]

$$\Leftrightarrow H_{f,t-1} > H_t^{min} := \frac{\alpha}{\gamma - r_t}$$

that states that a firm has to be "solid enough" to stay in the market. Observe that $r_t \leq r_{f,t}$ *and that the surviving firms fulfil the condition* $\gamma > r_{f,t}$*, therefore* $H_t^{min} > 0$.
The above condition can be seen as the 'exit condition': any time a firm does not meet the condition, it is forced out of the market[8].

The evolution of firms is described finally by the dynamics of the net worth. We assume, for the sake of simplicity, that a fraction δ of net worth is paid out as dividends every period and the profit is incorporated in the firm's remaining net worth, that is:

$$A_{f,t} = (1-\delta) A_{f,t-1} + \phi_{f,t} \tag{2.8}$$

By plugging eq.(2.3) and the optimal level of capital demand (eq.(2.5)) into eq.(2.8) we obtain:

$$A_{f,t} = (1-\delta) A_{f,t-1} + \left(u_{f,t}\nu - w\frac{\nu}{\lambda} - r_{f,t}\right) k_{f,t} - \tau\frac{1}{2}k_{f,t}^2$$

$$\Leftrightarrow A_{f,t} = (1-\delta) A_{f,t-1} + \nu\left(u_{f,t} - \frac{w}{\lambda}\right)(\gamma - r_{f,t}) + \tau\frac{1}{2}(\gamma - r_{f,t})$$

hence, the evolution of the net worth over time is function of only one endogenous variable at time t, that is $r_{f,t}$; it has to be stressed that net worth is a function also of endogenous variables indexed at time $t-1$, thus they are known at time t and can consequently be treated as parameters.
By imposing the conditions stated in Assumption 1, a lower bound on the net worth required to the firms is endogenously emerging. In fact, the *exit condition* can be written in terms of net worth[9]

$$H_{f,t-1} = \frac{A_{f,t-1}}{k_{t-1}} > \frac{\alpha}{\gamma - r_t} \Rightarrow A_{f,t-1} > A_t^{min} := \frac{\alpha}{\gamma - r_t}k_{f,t-1} > 0$$

which, once more, illustrates a time varying condition on the *size* of the firm. Note that the positivity of the above condition is a consequence of Assumption 1.

If firms were identical[10], than we could have defined the *Representative Agent* characterized by an

[7] Use the definition of $f_{f,t}$ (eq.(2.7)) in the condition previously derived to obtain

$$f_{f,t} < \gamma - r_t \Leftrightarrow \frac{\alpha}{H_{f,t-1}} < \gamma - r_t$$

$$\Leftrightarrow H_{f,t-1} > \frac{\alpha}{\gamma - r_t}$$

[8] Practically, in the agent-based model, the firm will be replaced by a new one. The net worth and the ER of the newly incorporated firm will be randomly drawn from the distributions (arbitrary chosen) out of which were drawn at $t = 0$. They are indeed *small* and *robust*, since the expected net worth (ER) of the new firms is smaller (bigger) than the long-run steady state that is emerging. The *replacement mechanism* it is quite standard in ABM literature for dimensionality issue, despite being an highly unrealistic assumption.

[9] Use the definition of ER (eq.(2.1)) in the exit condition and get

$$H_{f,t-1} > \frac{\alpha}{\gamma - r_t} \Leftrightarrow \frac{A_{f,t-1}}{k_{t-1}} > \frac{\alpha}{\gamma - r_t}$$

$$A_{f,t-1} > \frac{\alpha}{\gamma - r_t}k_{f,t-1}$$

[10] Which obviously imply $\mu_t^H = H_{f,t}$ and $Var_t^H = 0 \; \forall t, f$

equity ratio equal to μ_t^H, resulting hence in a capital demand equal to:

$$k_t^{RA} = \frac{\gamma - r_t - \frac{\alpha}{\mu_{t-1}^H}}{\tau} \tag{2.9}$$

Moreover firms can actually be assumed as heterogeneous in their financial robustness; therefore there can be taken advantage of the *VRA* approach which allows a more detailed aggregation since it gathers information also from higher moments of the distribution. In the followwing, we have chosen to stop at the second moment for the sake of simplicity.
First eq.(2.7) is approximated around the cross-sectional value of individual equity ratio μ_{t-1}^H via a second-order Taylor approximation, which yields

$$f_{f,t}|_{\mu_{t-1}^H} \simeq \frac{\alpha}{\mu_{t-1}^H} - \frac{\alpha}{\left(\mu_{t-1}^H\right)^2}\left(H_{f,t} - \mu_{t-1}^H\right) + \frac{1}{2}\frac{2\alpha}{\left(\mu_{t-1}^H\right)^3}\left(H_{f,t} - \mu_{t-1}^H\right)^2 \tag{2.10}$$

Secondly, the expected value is taken, obtaining

$$f_t = \frac{\alpha}{\mu_{t-1}^H} - \frac{\alpha}{\left(\mu_{t-1}^H\right)^2}E\left(H_{f,t} - \mu_{t-1}^H\right) + \frac{1}{2}\frac{2\alpha}{\left(\mu_{t-1}^H\right)^3}E\left(H_{f,t} - \mu_{t-1}^H\right)^2$$

with $E\left(H_{f,t} - \mu_{t-1}^H\right) = 0$ and $E\left(H_{f,t} - \mu_{t-1}^H\right)^2 = Var_{t-1}^H$. Therefore, by substitution we get the *approximated* EFP:

$$f_t = \frac{\alpha}{\mu_{t-1}^H} + \frac{\alpha}{\left(\mu_{t-1}^H\right)^3}Var_{t-1}^H \tag{2.11}$$

So the *approximated capital demand* in case of heterogeneous agents is:

$$k_t^{HA} = \frac{\gamma - r_t - \frac{\alpha}{\mu_{t-1}^H} - \frac{\alpha}{\left(\mu_{t-1}^H\right)^3}Var_{t-1}^H}{\tau} \tag{2.12}$$

Note that f_t is obviously decreasing in the mean of the equity ratio, in fact as the ER increases firms becomes more solid and thus the EFP is reduced. However, it shall be stressed that also the variance, and so the heterogeneity, plays a role: more specifically, the more dispersed are firms in the *Equity Ratio space*, the higher will be *EFP*, which will be causing then a lower demanded capital. This emerging property of the model can be interpreted as the fact that the *left-tail* of the distribution has a greater impact rather than the *right-tail*: *fragility* matters more than *robustness*. Despite not being explicitly modelled, this fact can be seen as driven by loss aversion of the banks: by charging an higher EFP, firm will have to be more sound, reducing thus the risk of bankruptcy[11].

2.2 Households

The economy is populated also with heterogeneous consumers, which will have to choose the optimal level of consumption, solving a standard maximization problem under a behavioural expectation formation process. Thus agents will behave rationally (i.e. maximizing the utility), with non-rational expectation: one could argue that this fact is an oxymoron, but the issue can be solved by stating that "agent behaves rationally, under his non-rational expectations", in other words, given his expectations, he rationally processes them.

[11] An other interesting results, which has not been explored in the paper, deals with the third moment of the distribution. If we would have expanded eq.(2.7) up to it third order and have token the expectation, then in eq.(2.11) there would also have been the *skewness* in a negative relationship with f_t. On the one side the skewness is positive (i.e. a distribution shifted to the left), then the f_t would be smaller wrt to a *zero-skewed* distribution, on the other a negative skewness would raise f_t: we can interpret this results as the fact,that banks observe that the majority of the firm in the economy is relatively solid (fragile), they would raise (lower) the external finance premium. This result is in the spirit of Stiglitz and Weiss (1981).

Indeed the economy is populated with N households, indexed with $i = 1, \ldots, N$: they are homogeneous in their utility function, that depends only on consumption. Hence, they will be supplying inelastically one unit of labour each period, that allows to explicitly model unemployment: if they happen to be employed, then they will earn a real wage w, if they are unemployed, than the Government will provide an unemployment subsidy σ (in real terms). They have to allocate their budget either in consumption, which yields a positive utility, or in a unit-cost risk-less bond that pays a positive interest rate.
The source of heterogeneity is deriving from the expectation formation process (see section 2.2.1): not only we assume non rational expectations in the *Muthian* sense, but also they will be individual specific. Generally there will be used the following notation when referring to individual expectations formed in t by individual i forecasting variable z at time t: $E_{i,t}z_{t+1}$; when z will be individual specific, than there will be used the notation $E_{i,t}z_{i,t+1}$. The expectation formation process is backward looking: as soon as possible agents collect all the past values of the relevant variable, and, based on those, immediately form their expectation. The solution to maximization problem occurs in the immediately subsequent instant, thus the consumer already knows his expectation.

The consumer problem can be written as

$$\mathcal{P}^c_{i,t} \begin{cases} max_{c_{i,t}} \qquad U_{i,t} := E_{i,t} \displaystyle\sum_{t=0}^{\infty} \beta^t U(c_{i,t}) \\ \qquad\qquad s.t. \qquad p_t c_{i,t} + B_{i,t} \leq (1 + i_{t-1})\, B_{i,t-1} + W_{i,t} \\ \qquad\qquad\quad \forall t = 0, 1, 2, 3, \ldots \end{cases}, \tag{2.13}$$

with p_t being the price level, $B_{i,t-1}$ the individual savings at time $t-1$, i_t the nominal interest rate and $W_{i,t}$ the income level defined as

$$W_{i,t} = \chi_t w_t + (1 - \chi_t)\sigma_t$$

where χ_t is a dichotomous variable (1: employed and 0: unemployed) and β is the standard individual discount factor.
By solving problem (2.13) we obtain the Euler Equation

$$\frac{1}{1 + i_t} = \beta \frac{E(U^{'}_{t+1})}{U^{'}_t} \frac{p_t}{E(p_{t+1})} \tag{2.14}$$

with $U^{'}$ is the first derivative the utility function; by specifying the utility function by means of a log-utility function, i.e.

$$U(c_{i,t}) = log\,(c_{i,t})$$

and using the definition of expected inflation $E_{i,t}(\pi_{t+1}) = \frac{E_{i,t}(p_{t+1}) - p_t}{p_t}$ we can rewrite eq.(2.14) as:

$$\frac{1}{1 + i_t} = \beta \frac{c_{i,t}}{E_{i,t}(c_{i,t+1})} \frac{1}{1 + E_{i,t}\pi_{t+1}}$$

which can be manipulated so to obtain the *individual demand function* of consumption goods:

$$c_{i,t} = \left(\frac{1 + E_{i,t}\pi_{t+1}}{1 + i_t} \right) \frac{1}{\beta} E_{i,t} c_{i,t+1} \tag{2.15}$$

The relationships arising within the Euler Equation are in the direction of consumption smoothing based on individual expectations: if the inflation expectation is bigger than the interest rate, than the agent will increase his current consumption, scaled by the individual discount factor β, so not to incur in the risk of having his savings eroded by the high inflation; on the contrary, if interest rates is bigger

than the expected inflation rate, than the agent will lower his consumption in order to exploit the additional earnings deriving from the interest rate. Finally, in order to smooth consumption, agent will increase (decrease) his consumption if the expected consumption increases (decreases).
Moreover we need to rule out the possibility of negative consumption at the individual level, thus we assume

Assumption 2 (Non-negativity of consumption) *Equation (2.15) has to be non-negative, thus*

$$E_{i,t}c_{i,t+1} \geq 0 \wedge E_{i,t}\pi_{t+1} > -1$$

Following the reasoning in section 2.1, if it is assumed the existence of a *representative consumer*, than eq.(2.15) shrinks to

$$c_t^{RA} = \frac{1}{\beta}\frac{1}{1+i_t}\left[\left(1+E_t\pi_{t+1}\right)E_tc_{t+1}\right] \tag{2.16}$$

while the assumption of heterogeneous consumer and the VRA procedure (see Appendix 6.1) reduce the *approximated consumption* to

$$c_t^{HA} = \frac{1}{\beta}\frac{1}{1+i_t}\left[\left(1+\mu_t^{\pi}\right)\mu_t^c + cov_t^{\pi,c}\right]$$

where μ_t^{π} is the cross sectional mean of inflation expectation, μ_t^c is the cross sectional mean of consumption expectations and $cov_t^{\pi,c}$ is the covariance between inflation expectation and consumption expectation. It is convenient to rewrite the previous equation as

$$c_t^{HA} = \frac{1}{\beta}\frac{1}{1+i_t}\Omega_t \tag{2.17}$$

with $\Omega_t := ((1+\mu_t^{\pi})\mu_t^c + cov_t^{\pi,c})$. Note that the components collected in Ω_t are coming from the *expectation process* and they are pre-determined, therefore they are known to the consumer when he faces his maximization process and can thus be treated as parameters; nevertheless Ω_t is time varying, hence indexed with t.

Notice also that the conditions stated in Assumption 2 are not sufficient to guarantee the non-negativity of consumption also at the approximated level, therefore we are forced to impose an additional constraint to prevent the approximated consumption to be negative i.e.

Assumption 3 (Non-negativity of consumption)

$$c_t^{HA} \geq 0 \Leftrightarrow \Omega_t \geq 0$$

$$\mu_t^c \geq c_t^{min} := \frac{-cov_t^{\pi,c}}{(1+\mu_t^{\pi})}$$

that is the expected consumption should be big enough.

A big role is indeed played by the covariance between consumption expectations and inflation expectations. In fact, if the covariance is positive, than the right-hand side of the equation is negative and the condition is always satisfied, since μ_t^c has to be non-negative (Ass. 2). On the contrary, if the covariance is positive, than Assumption 3 becomes relevant and might become binding. One can argue the covariance to be negative, since from the Euler Equation (see eq.(2.14)) there can be seen that an increase of the level of expected inflation is associated with a decrease of the level of expected consumption to keep consumption constant: if prices are expected to increase, the quantity of goods consumed shall decrease since consumption gets more expensive; conversely it can be argued the covariance to be positive, since an increase of employment[12], is associated to an increase of inflation

[12] Thus an higher probability to be employed, which translates to an higher income, and finally into higher consumption

through the effect of the Philips Curve. However this mechanism might be mitigated, since an increase of the employment level and of inflation, would also cause an increase of the interest rate via the Taylor Rule.
Despite *a-priori* we cannot be confirmed which of the two effects would emerge, simulations exhibit that covariance fluctuates in a small neighbourhood around 0, being thus positive and negative.

2.2.1 Expectation formation process

Consumers in the model will form their expectations following the *Heuristic Switching Model*, presented in Anufriev and Hommes (2012) they are endowed with a palette of backward looking heuristics which are used to produce the forecasts available to consumers, the fitness of each heuristics is then computed via a performance measure and finally the probabilities of choosing the heuristics are updated. In the original version of the model the probability of choosing a rule is equal to the fraction of agents using that specific rule since it is assumed a continuum of agents of mass 1; differently the same reasoning cannot be applied in this context, at least not with certainty, since there are not an infinity of consumers, but it should be kept the *probabilistic* interpretation: each agent will choose only one heuristic with the probability that in the original model was representing the fraction of agent using that heuristic.
One of the aim of the paper is to use experimental results, therefore the calibration, the pay-off function and the heuristics are the same proposed in Anufriev and Hommes (2012) and then replicated in Assenza et al. (2013).

Define Z as the set of heuristics available to the agents and let $y = c_i, \pi$. Therefore each heuristic will produce a forecast of the form

$$E^z_{i,t} y_{t+1} = f^z(y_{i,t-1,t-2,\ldots}) \quad , \quad \forall z \in Z$$

the fitness of the heuristic is then measured via a performance measure

$$U^z_{i,t} = \frac{100}{1 + |y_{i,t} - E^z_{i,t-1} y_{i,t}|} + \eta U^z_{i,t-1} \quad , \quad \forall z \in Z$$

with η being a *memory parameter*: it weights the past realizations of the performance measure. The performance measure is then used to update the impact of the rules, which will be interpreted as the probability for agent i to choose rule z following a *discrete choice model with asynchronous updating*:

$$n^z_{i,t} = \delta^{BH} n^z_{i,t-1} + \left(1 - \delta^{BH}\right) \frac{exp\left(\beta^{BH} U^z_{i,t-1}\right)}{\sum_{z=1}^{H} exp\left(\beta^{BH} U^z_{i,t-1}\right)}$$

as in the original version of the model δ^{BH} is the *inertia in switching* and β^{BH} is the *intensity of switching*: the former reflects that agents have a *status-quo* from which they prefer not to depart, while the latter represents the sensitivity of agents to the differences in heuristics performances .

More in details, the set of proposed heuristics will be

$$\begin{cases} \text{Adaptive Rule} & \rightarrow \quad E^{ada}_{i,t} \pi_{i,t+1} = 0.65 \pi_{t-1} + 0.35 E^{ada}_{i,t-1} \pi_{i,t} \\ \text{Weak Trend} & \rightarrow \quad E^{wtr}_{i,t} \pi_{i,t+1} = \pi_{t-1} + 0.4 \left(\pi_{t-1} - \pi_{t-2}\right) \\ \text{Strong Trend} & \rightarrow \quad E^{str}_{i,t} \pi_{i,t+1} = \pi_{t-1} + 1.3 \left(\pi_{t-1} - \pi_{t-2}\right) \\ \text{Learn and Anchor} & \rightarrow \quad E^{laa}_{i,t} \pi_{i,t+1} = \frac{\left(\pi^{av}_{t-1} + \pi_{t-1}\right)}{2} + \left(\pi_{t-1} - \pi_{t-2}\right) \end{cases}$$

$$
\begin{cases}
\text{Adaptive Rule} & \rightarrow \quad E^{ada}_{i,t} c_{i,t+1} = 0.65 c_{i,t-1} + 0.35 E^{ada}_{i,t-1} c_{i,t} \\
\text{Weak Trend} & \rightarrow \quad E^{wtr}_{i,t} c_{i,t+1} = c_{i,t-1} + 0.4 \left(c_{i,t-1} - c_{i,t-2} \right) \\
\text{Strong Trend} & \rightarrow \quad E^{str}_{i,t} c_{i,t+1} = c_{i,t-1} + 1.3 \left(c_{i,t-1} - c_{i,t-2} \right) \\
\text{Learn and Anchor} & \rightarrow \quad E^{laa}_{i,t} c_{i,t+1} = \frac{\left(c^{av}_{i,t-1} + c_{i,t-1} \right)}{2} + \left(c_{i,t-1} - c_{i,t-2} \right)
\end{cases}
$$

The heuristics are both trend-extrapolative and adaptive, and, according to Assenza et al. (2013), they are sufficient to explain "four different patterns observed in aggregate behaviour of inflation and output, namely convergence to (some) equilibrium level, explosive inflationary or deflationary spirals, dampened oscillations and persistent oscillations", which are patterns which might arise in a macroeconomic setting.

3 Macro Aggregation

In the previous section we have derived the *approximated consumption* and the *approximated capital demand* which can be then used, jointly with the public expenditure, to derive the equilibrium on the goods market.
Therefore we can start from the known relationship, which binds aggregate production Y_t to its main components, assuming a closed economy

$$Y_t = C_t + I_t + G_t$$

with G_t being public expenditure[13] and I_t the aggregate investment level. By assuming total depreciation of capital, thus $I_t = K_t$, the above relation becomes

$$Y_t = C_t + K_t + G_t \tag{3.1}$$

By taking advantage of the VRA aggregation procedure we can easily retrieve the aggregate value of private consumption and capital demand: the former in fact can be defined as $C_t = N \cdot c_t^{HA}$, symmetrically the latter will be $K_t = F \cdot k_t^{HA}$. The total production can be defined as

$$Y_t = \lambda x_t N \tag{3.2}$$

that is the labour productivity multiplied by the employment rate $x_t := N_t/N$, with N_t being the number of employees economy wide.
Finally, by substituting eq.(2.12), eq.(2.17) and eq.(3.2) in eq.(3.1), we derive the *IS-Curve*, which binds employment (and thus production) to the nominal interest rate[14]. Hence the IS curve becomes

$$x_t = \frac{\Omega_t}{\lambda\beta}\frac{1}{1+i_t} - \frac{F}{\lambda\tau N} i_t + \frac{F}{\lambda\tau N}\left(\gamma + \mu_t^{\pi} - f_t\right) + \frac{1}{\lambda N} G_t \tag{IS}$$

What we obtain is a non-linear IS curve.

First of all, note that employment is diminishing in the nominal interest rate, otbe, through two channels: one on the firms' side and one on the consumers' side. On the firms' side, as interest rate goes up, the financing costs grow , thus the capital demand is reduced ; on the consumers' side, an increase in the interest rate, causes consumption to be delayed in order to benefit from the higher

[13] Notice that G will be merely an exogenous parameter in the model, but it will be the varying variable during the fiscal shock (see Section 4), thus it has the time index

[14] Notice that some of the results presented so far are expressed in real interest rate, we use the known relationship

$$r_t = i_t - \mu_t^{\pi}$$

to transform the *real* into *nominal*.

interest rate. The EFP and the employment rate are negatively correlated, since the increase of the former would cause an increase in the financing costs, which results in a lower capital demand, thus lowering employment. Obviously, an increase in public expenditure pushes demand higher, which then drives production and finally employment.
As consumption expectation increases, we observe an increase in current consumption for smoothing purposes (cfr eq.(2.2)) which results in an increase of the employment rate via the increase in Ω_t. Similarly we should expect that as inflation expectation increases also the employment rate will raise; this phenomenon can be explained via two channels: consumers are willing to increase their actual consumption so to prevent from a loss in the purchase power, and firms will increase their capital demand, due to the lowered financing cost.

In order to characterize the fundamentals of a stylized macroeconomic equilibrium we need to specify a monetary policy rule, to define the course of nominal interest rate, and the equation specifying the dynamics of inflation.
Let us start defining there is defined a monetary policy rule: the nominal interest rate is set by means of a Taylor Rule (TR) encompassing the Zero Lower Bound (ZLB), which responds to the contemporaneous value of inflation, as in Assenza and Delli Gatti (2016), and of the output gap[15], $(x_t - x_n)/x_n$ with x_n being the natural level of employment:

$$i_t = \max\left(0, r_n + \pi^T + (1+\alpha_\pi)(\pi_t - \pi^T) + \left(\frac{x_t}{x_n} - 1\right)\alpha_x\right) \tag{TR}$$

where r_n is the natural interest rate and π^T is the inflation target of the Central Bank; moreover, eq.(TR) represents an upward schedule in the (x, i) plane, where it lies the downward IS scheme identified by eq.(IS).
To close the model, we assume that there is a positive feedback relating realized inflation and inflation expectation and that it reacts positively to the output gap. Despite not being micro-funded, the above assumptions result in a standard Phillips Curve:

$$\pi_t = \theta_x(\frac{x_t}{x_n} - 1) + \theta_\pi \mu_t^\pi \tag{PC}$$

with $\theta_x > 0$ and $\theta_\pi = \beta$ as is commonly accepted.

Thus the system fully characterizing the economy consists of eq.(IS), eq.(TR) and eq.(PC) in case of heterogeneous agents, i.e. :

$$\begin{cases} x_t = & \frac{\Omega_t}{\lambda\beta}\frac{1}{1+i_t} - \frac{F}{\lambda\tau N}i_t + \frac{F}{\lambda\tau N}(\gamma + \mu_t^\pi - f_t) + \frac{1}{\lambda N}G_t \\ i_t = & \max\left(0, r_n + \pi^T + (1+\alpha_\pi)(\pi_t - \pi^T) + (\frac{x_t}{x_n} - 1)\alpha_x\right) \\ \pi_t = & \theta_x(\frac{x_t}{x_n} - 1) + \theta_\pi\mu_t^\pi \end{cases} \tag{3.3}$$

The emerging system, even if non-linear and enriched by heterogeneity and non-rationality can be solved analytically and the solution will always be uniquely defined (see Appendix 6.2). Given the shape of the Taylor Rule, there will be two different states of the world (i.e. *ZLB region* when $i_t = 0$, and a *positive region* when $i_t > 0$) yielding two different solutions. We will refer to the *ZLB solution* and the *no-ZLB* solution respectively, i.e.:

ZLB

$$\begin{cases} i_t^* = & 0 \\ x_t^* = & \frac{\Omega_t}{\lambda\beta} + \frac{F}{\lambda\tau N}(\gamma + \mu_t^\pi - f_t) + \frac{1}{\lambda N}G_t \\ \pi_t^* = & \theta_x(\frac{x_t^*}{x_n} - 1) + \theta_\pi\mu_t^\pi \end{cases} \tag{3.4}$$

[15]Since the production function is linear in labour, then we can use the employment rate to compute the output gap.

<u>no-ZLB</u>

$$\begin{cases} i_t^* = & \frac{Z_{3,t}-Z_2+\sqrt{(Z_2+Z_{3,t})^2+4Z_{1,t}Z_2}}{2Z_2} \\ x_t^* = & \frac{\Omega_t}{\lambda\beta}\frac{1}{1+i_t^*} - \frac{F}{\lambda\tau N}i_t^* + \frac{F}{\lambda\tau N}\left(\gamma+\mu_t^\pi - f_t\right) + \frac{1}{\lambda N}G_t \\ \pi_t^* = & \theta_x\left(\frac{x_t^*}{x_n}-1\right)+\theta_\pi\mu_t^\pi \end{cases} \tag{3.5}$$

with $Z_{1,t} := \frac{\Omega_t}{\lambda\beta} > 0$, $Z_2 := \left(\frac{x_n}{(1+\alpha_\pi)\theta_x+a_x} + \frac{F}{\lambda\tau N}\right) > 0$ and $Z_{3,t} := \frac{F}{\lambda\tau N}\left(\gamma+\mu_t^\pi-f_t\right)+\frac{1}{\lambda N}G_t + \frac{x_n}{(1+\alpha_\pi)\theta_x+a_x}\left(r_n+\pi^T-(1+\alpha_\pi)\left(\theta_x-\theta_\pi\mu_t^\pi+\pi^T\right)-\alpha_x\right)$

The system spelled in (3.3) is just a frame of a process, which evolves thanks to other underlying processes. Notice in fact that the economy presented in system (3.3) is a static environment: all the variables are indexed at time t. Nevertheless the model is dynamic: the expectation formation processes and the EFP are the state variables that link periods together, since they contain variables indexed at $t-1$. Figure 3.1 present a schematic representation of the dynamics of the model.

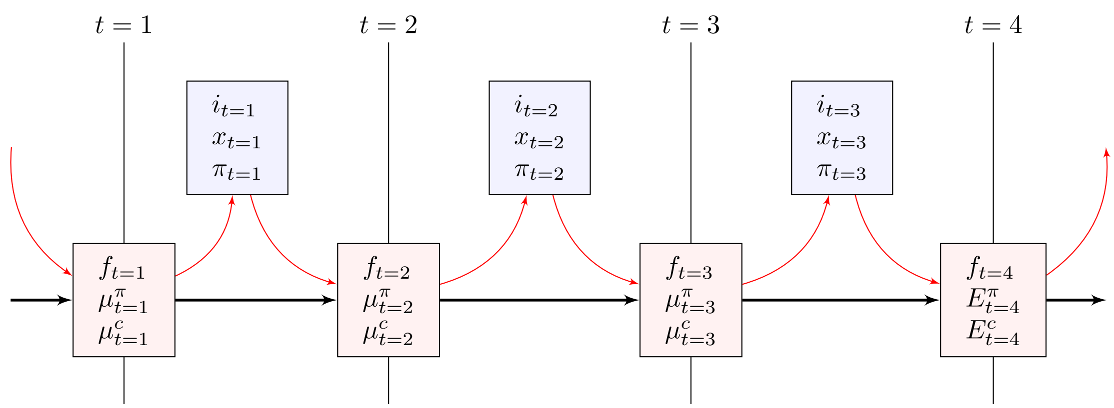


Figure 3.1: Evolution of variables and time-relationships

Despite the economy represented in eq.(3.3) appears as analytically easy, its dynamics can not be studied via standard analytical tool, since it depends on the evolution of each agent, whose behaviour is described by non-linear equations. In the following section we will build an agent-based model which allows to simulate the model and learn the long-run steady state distributions (denoted henceforth by $\hat{\bullet}$) when the model shows its ergodic properties. The long-run steady states are crucial to solve analytically the system in (3.3).

3.1 Agent Based Model

We have assumed a population of $N = 1000$ households and $F = 80$ firms over a time span of $T = 1000$ periods[16], which can be considered as quarters.
The baseline parametrization in Table 3.1, follows as much as possible the calibration proposed in Assenza and Delli Gatti (2016) or some commonly accepted values; the *Heuristic Switching Model* is calibrated following Assenza et al. (2013).

The ABM can be split in three main sections: the initial and the final regards micro-decision (i.e. consumers and firms), while the central one determines the macro-equilibrium. More in details the

[16] A transient phase of 500 periods is always considered

Variable	Symbol	Numerical Value
individual discount (quarterly)	β	0.99
wage	w	2.85
unemployment subsidy	σ	1.65
labour productivity	λ	4.15
capital productivity	ν	0.65
exogenous finance premium	α	0.05
investment cost	τ	0.005
natural employment	x_n	0.95
dividend yield	δ	0.1
natural real interest rate	r_n	3%
central bank sensibility to output gap	α_x	0.5
central bank sensibility to inflation	α_π	0.5
elasticity of inflation to output gap	θ_x	0.04
elasticity of inflation to inflation	θ_π	0.99
inflation target	π^T	0%
public expenditure	G	700
memory in fitness measure	η^{BH}	0.7
intensity of switching	β^{BH}	0.4
inertia in switching	δ^{BH}	0.9

Table 3.1: Baseline parameters configuration

narrative of the ABM in period t is as follows[17]:

1. Micro-decision

Consumers collect all the available past information and form their individual expectations on consumption and inflation, by choosing the forecast produced by the best performing heuristic. As soon as all consumers formed the expectations, we can now compute the cross-sectional means and covariance, i.e. μ_t^c, μ_t^π, and $cov_t^{\pi,c}$.

Firms learn the individual External Finance Premium $f_{f,t}$, since they know the equity ratio $H_{f,t-1}$. Moreover, also the *approximated* EFP f_t is known, since it is a function of μ_{t-1}^H and Var_{t-1}^H.

2. Macro level

The macro-economic equilibrium, which depends on the distributions of individual characteristics, can be computed. Thus we obtain x_t, i_t and π_t

3. Micro-decision

Consumers face an employment process: with probability x_t they will be hired earning the full wage w, else they are subsidized by the government; moreover, given that i_t is now observable, they can compute their optimal demand and consume.

Firms observe the shock $u_{f,t}$ and learn the cost of capital $r_t + f_{f,t}$; they compute the optimal capital demand and realize profits. Net worth is updated: if the equity ratio is too low, the firm goes bankrupt and is replaced by a new one, whose initial net worth is randomly drawn.

[17]In Appendix 6.3, there can be found an explicit time line of the events in the model.

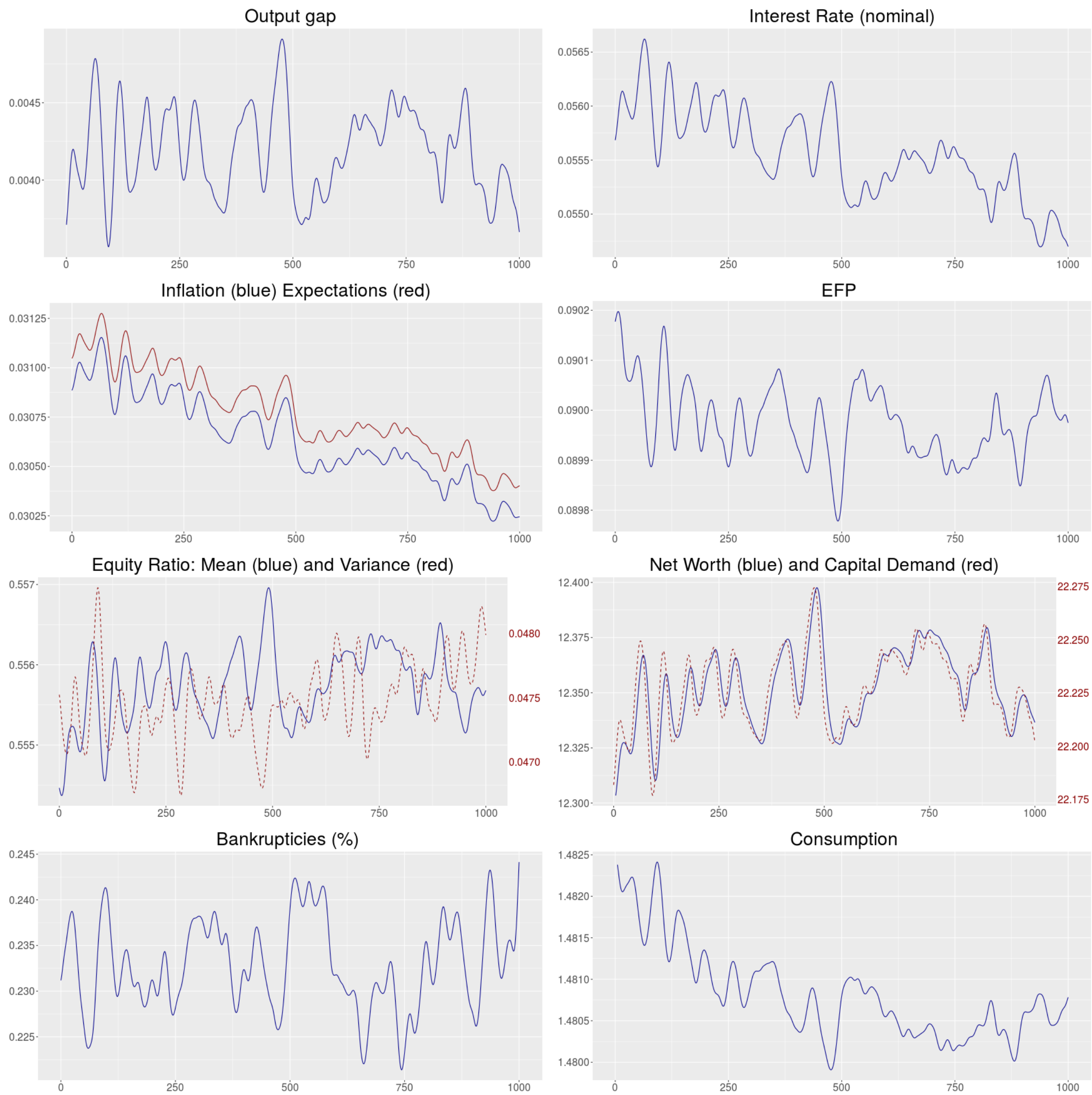


Figure 3.2: Baseline simulation. The individual variables plotted are of the *average* agent.

Figure 3.2 presents the HP filtered time series[18] of the main economic variables over a 50 repetitions Montecarlo procedure: the variables fluctuate around the *emerging long-run mean*, which can be thouhgt as the long-run steady state reported in Table 3.2. There exists a positive relationship between output gap, inflation and interest rate, which is implied by the backbone of the model, namely the Phillips Curve and the Taylor Rule. During the booms of the economy, we observe an increase in the demanded capital, that drives a more than proportional increase of the net worth: the enlarged financial robustness is captured by the higher ER resulting in a lower EFP. It is interesting to observe the negative relationship existing between ER and its variance: as the ER increases, solid firms hit the self-financing condition and the fragile firms too become more solid. We observe a shift of the equity ratio to the right, and so the variance is reduced.

[18] Not showing initial 500 transition periods.

$\widehat{x}$	$\widehat{i}$	$\widehat{\pi}$	$\widehat{E_\pi}$	$\widehat{EFP}$	$\widehat{ER}$	$\widehat{A}$	$\widehat{K}$	$\widehat{BR}$
0.953	0.055	0.03	0.03	0.899	0.555	12.352	22.228	0.232

Table 3.2: Long-run steady state values: mean of the relevant time series

Far more interesting is to observe that as the interest rate goes up, the net worth goes up (third panel): this is apparently not coherent with the system presented in (3.3), since an higher interest rate, would have implied an higher financing cost which would have caused the decrease in net worth. In the simulation, however, we observe a decrease in the EFP and an increase in inflation expectation. These two phenomena are sufficient to offset the increase in nominal interest rate: the net variation in financing costs is negative and so net worth raises.
There exists a positive relationship between the number of bankruptcies and the External Finance Premium: as the EFP increases, financing costs born by the firms increase as well, and those firms which were on the edge of bankruptcy can not sustain the higher costs any more and fail[19].

3.2 Analytical Representation and Solution

From the system presented in (3.3), there can be derived two 2×2 systems: one in the (x, i) plane and one in the (x, π) plane. This approach will be very useful in policy analysis in order to disentangle short term direct effect and long term indirect effect: in fact, it is possible to retrieve the causes and the effects of the variation of the variables.

In the (x, i) plane we can determine the upward augmented-TR (a-TR) by plugging the Phillips curve in the Taylor rule, while the downward scheme is represented by the inverse of the IS[20]. Symmetrically, by inserting the TR in the inverted IS there can be derived the Aggregate Demand (AD), which is the downward sloped and the PC is the upward sloped in the panel (x_t, π_t). Analytically the above two-dimensional systems are represented by

$$\begin{cases} \text{IS:} & i_t = max\left[0, \frac{-IS_{B,t}+\sqrt{IS_{B,t}^2-4IS_AIS_{C,t}}}{2IS_A}\right] \\ \text{aTR:} & i_t = \left(\frac{(1+\alpha_\pi)\theta_x+\alpha_x}{x_n}\right)x_t + r_n + \pi^T + (1+\alpha_\pi)\left(-\theta_x + \theta_\pi\mu_t^\pi - \pi^T\right) - \alpha_x \end{cases}$$
$$\begin{cases} \text{AD: } \pi_t = & \frac{1}{1+\alpha_\pi}\left[-r_n + (1+\alpha_\pi)\pi^T - \frac{\alpha_x}{x_n}x_t + \alpha_x + \frac{-IS_{B,t}+\sqrt{IS_{B,t}^2-4IS_AIS_{C,t}}}{2IS_A}\right] \\ \text{PC: } \pi_t = & \frac{\theta_x}{x_n}x_t - \theta_x + \theta_\pi\mu_t^\pi \end{cases} \tag{3.6}$$

with $IS_A := \frac{F}{\lambda\tau N}$, $IS_{B,t} := \frac{F}{\lambda\tau N}(1-\gamma-\mu_t^\pi+f_t) - \frac{1}{\lambda N}G + x_t$ and $IS_{C,t} := x_t - \frac{\Omega_t}{\lambda\beta} - \frac{F}{\lambda\tau N}(\gamma+\mu_t^\pi-f_t) - \frac{1}{\lambda N}G$.

The above systems are, once more, functions of parameters or those variables which, in case of ergodicity, settle around the long-run steady state values of the system.

As the systems settles, there can be drawn the long term steady state values (i.e. the mean) of the

[19] For sake of clearness, we have explained relationships and movements in *one-direction*; the same relationships, but reversed, can be observed also in the *other-direction*.

[20] Observe that, being the original IS an hyperboloid it has two *branches*, the same will be for the inverse. But one of the two branches will be only defined when $i_t < 0$, thus it is discarded.

Variable	x	i	π
Simulation	0.954	0.055	0.03
Analytical	0.952	0.054	0.030

Table 3.3: Analytical VS Simulated long-run steady state values

time-varying variables, and therefore the systems are reduced to be *two unknowns in two equations*:

$$\begin{cases} \text{IS:} & i = max\left[0, \frac{-\widehat{IS}_B+\sqrt{\widehat{IS}_B^2-4IS_A\widehat{IS}_C}}{2IS_A}\right] \\ \text{aTR:} & i = \left(\frac{(1+\alpha_\pi)\theta_x+\alpha_x}{x_n}\right)\widehat{x} + r_n + \pi^T + (1+\alpha_\pi)\left(-\theta_x + \theta_\pi \widehat{E}^\pi - \pi^T\right) - \alpha_x \end{cases}$$

$$\begin{cases} \text{AD: } \pi = & \frac{1}{1+\alpha_\pi}\left[-r_n + (1+\alpha_\pi)\pi^T - \frac{\alpha_x}{x_n}\widehat{x} + \alpha_x + \frac{-\widehat{IS}_B+\sqrt{\widehat{IS}_B^2-4IS_A\widehat{IS}_B}}{2IS_A}\right] \\ \text{PC: } \pi = & \frac{\theta_x}{x_n}\widehat{x}_t - \theta_x + \theta_\pi \widehat{E}_t^\pi \end{cases}$$

The above system can be then plotted[21] (see Figure 3.3) and also solved analytically for x,i,and π: Table 3.3 compares the long-run mean computed via simulations and the one analytically obtained, they are very close indeed.

It has be stressed that the richness of this approach has not to be intended only in the pure aesthetic of the exercise, but it will be crucial to study the contribution of each single component of the economy to the aggregate dynamic. This approach will be deeply used in the next sections, where the model is going to put at work.

4 An application: Fiscal Shock

In this section the economy will face a positive fiscal shock, namely an increase from $G_0 = 700$ to $G_1 = 750$ at period $t = 500$.

Before getting in the details of the analysis it is convenient to build some common language. As the model settles, we will denote as $\hat{\bullet}$ the long-run steady state values. When considering the variables before the shock, we will index them with a 0 and consider periods $t = [0 : 499]$; on the contrary they are indexed with a 1 and consider periods $t = [500 : 1000]$ when referring to the variables after the shock.
The *direct effect* of the shock is the effect that the variation of G itself invokes on the fundamentals, while the *indirect effect* is the one driven by the changes of the distribution triggered by the change in G (see Fig.4.1). The sum of the two effects is going to be called *total effect*. It can be said, even though not being perfectly rigorous and precise, that the direct effect is the *short run effect*, since it can be observed immediately after the shock; the total effect can be considered as *long-run effect*, because the indirect effect acts on a longer horizon, as it can be observed when the distributions of characteristics settle on the new long-run steady state. The *Heterogeneous Agent (HA) component* is going to be the variation which accounts to the second moment of the distributions, while the *Representative Agent (RA) component* is the one stemming from the first moment.
The dynamics of the shock will be discussed using three approaches, which, not only do not contradict, but they allow to study the shock under slightly different perspectives, thus giving the chance to let different peculiarities to emerge. Initially we will qualitative inspect the time series to confirm

[21] Notice that the *curves* are non-linear despite they appear as straight lines. Indeed, for an easier interpretation of the results, there can be linearised the curves so to be able to interpret the parameters as *shifter* and *rotators*.

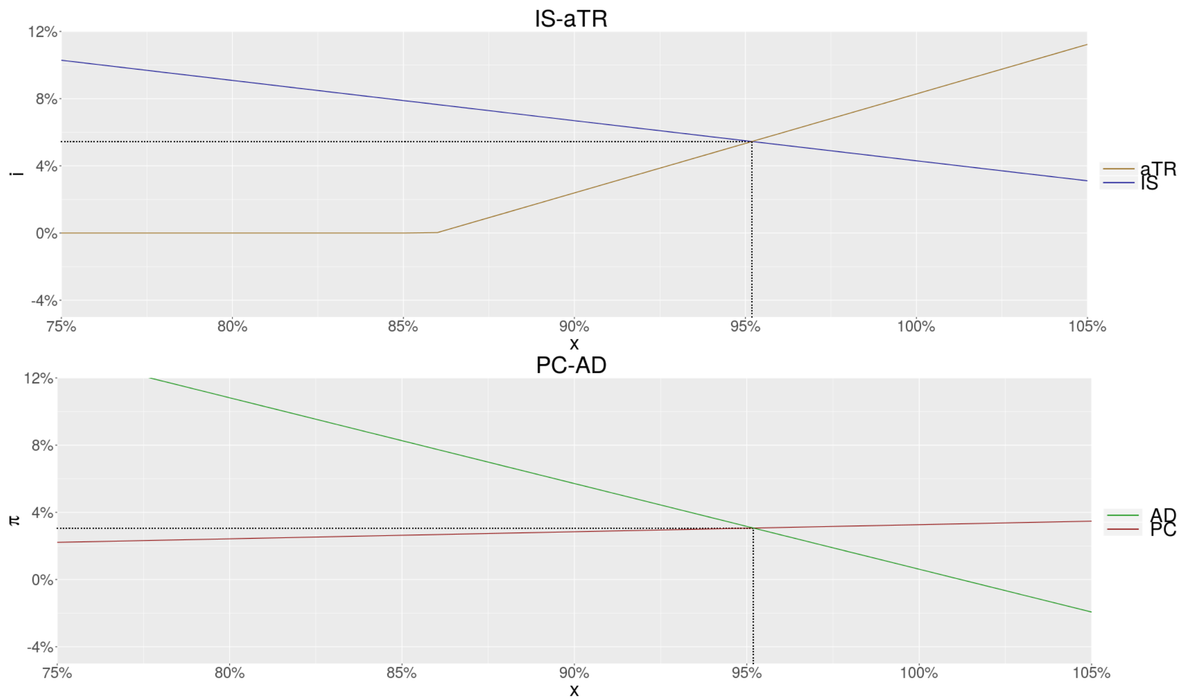


Figure 3.3: Long-Run Equilibrium representation

the general economic intuitions underlying the shock, then we will use a *total-differential* approach applied to the model presented in eq.(3.5) so to disentangle fully the dynamics of the shock and the contribution deriving from the heterogeneity, finally we will use the systems presented in eq.(3.6), so to study simultaneously the *direct* and *indirect* effects on all the fundamentals.

Suppose now that a positive fiscal shock, modelled as an increase of public expenditure, hits at period 500.
First of all let us conjecture how it directly impacts on the economy: according to standard economic intuition, reflected in system (3.3), for what concerns the macroeconomic variables, it will result in an increase of the output gap, due to an increase of the aggregate demand; in turn this will be causing an increase in the inflation rate and in the interest rate as implied by the PC and the aTR curves. Graphically we would expect an upward shift of the IS and of the AD along the aTR and the PC respectively.
For what concerns the indirect effect we would expect an increase of the financing costs of the firm through the effect of the interest rate, causing a decrease in the capital demanded and in the net worth, which will result in a decrease of the ER which will in turn cause an increase of the EFP causing then an increase of bankruptcies, namely of those firms which were more fragile thus lowering the variance, since the *fragile tail* of the distribution will be sensibly reduced. This result is not matched with the work by Assenza and Delli Gatti (2016), but the presented outcome is yet logically coherent. On the consumers side we will expect to observe initially a drop in consumption, since consumers will prefer to delay consumption so to take advantage of the high interest rate, but, as soon as the inflation (and consequently the inflation expectations) will start to raise, then the consumers will start to consume more so not to suffer from too high prices; it can not be said if the HA contribution is going to be negative or positive. We would then expect to see a regime switch in the aTR and in the PC shifting both upwards: this cause a small reduction in the employment level otbe, but the effect is expected to be lower than the direct effect.

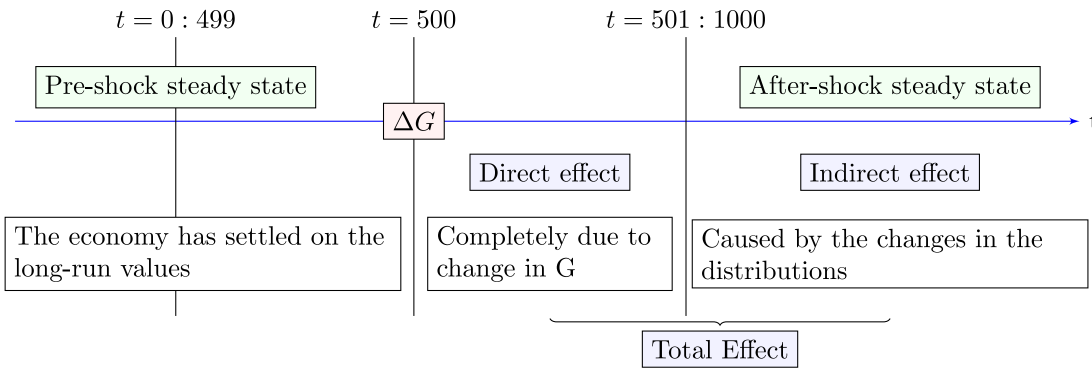


Figure 4.1: Decomposition of a shock in Direct and Indirect effect

Finally, the total expected outcome is a sharp increase in i, π and x, represented by the shift upwards of all the curves in the systems. At a micro level, we expect to see a drop in consumption and capital demanded, joint with an increase in the number of bankruptcies

The relevant HP-filtered time series are presented in Fig. 4.2.

The time series confirm the previous conjectures. As can be seen a positive fiscal shock increases the aggregate demand and thus of the output gap, which results in an increase of the interest rate, via the Taylor Rule, and in an increase of inflation via the Phillips Curve. Moreover the financing costs for the firms increase as the interest rate is increased, thus we observe a reduction in the capital demanded and in the net worth: the reduction in net worth increases the fragility of the firm, thus the ER would decrease, but the reduction in capital demanded, on the contrary, would increases the solidity of the firm resulting in an increase in the ER. Despite the mitigation caused by the reduction in the capital demanded, the reduction in net worth is relatively big and so the ER is reduced. The reduction of the equity ratio leads to higher EFP. The increased financing costs make all the firms which were close to fail, to be actually pushed out of the market (i.e. *crowding out effect*), thus we will observe initially a boom in the number of bankruptcies, which will be then fluctuating on an higher level. Since now it is more easy for the firm to fail (due to the higher financing costs), the variance of the ER is reduced, since only the more robust firms stay in the market. Finally, despite an higher employment level, the increase in inflation is not high enough to offset the increase in interest rate, thus consumer will reduce their consumption so to benefit from the higher interest rate.

The same results, can be explained also by means of the analytical system presented in (3.5) and its evolution: thus we confirm the economic conjectures, with simulations and the analytical systems. The following approach based on the *total-differential*, will prove to be useful in terms of not only insulating (qualitatively and quantitatively) the direct effect from the indirect effect, but it will also be useful in studying also the contribution that spans from heterogeneity.

Before the shock hits the economy, the system presented in (3.5) had settled on the long-run steady state values that is

$$\begin{cases} \widehat{i} = & \frac{\widehat{Z}_3 - Z_2 + \sqrt{\left(Z_2 + \widehat{Z}_3\right)^2 + 4\widehat{Z}_1 Z_2}}{2Z_2} \\ \widehat{x} = & \frac{\widehat{\Omega}}{\lambda\beta}\frac{1}{1+\widehat{i}} - \frac{F}{\lambda\tau N}\widehat{i} + \frac{F}{\lambda\tau N}\left(\gamma + \widehat{E}^\pi - \widehat{f}_t\right) + \frac{1}{\lambda N}G \\ \widehat{\pi}^*_t = & \theta_x\left(\frac{\widehat{x}}{x_n} - 1\right) + \theta_\pi \widehat{E}^\pi \end{cases}$$

with $\widehat{Z}_1 := \frac{\widehat{\Omega}}{\lambda\beta} > 0$, $Z_2 := \left(\frac{x_n}{(1+\alpha_\pi)\theta_x + a_x} + \frac{F}{\lambda\tau N}\right) > 0$ and $\widehat{Z}_3 := \frac{F}{\lambda\tau N}\left(\gamma + \widehat{E}^\pi - \widehat{f}\right) + \frac{1}{\lambda N}G_t + \frac{x_n}{(1+\alpha_\pi)\theta_x + a_x}\left(r_n + \pi^T - (1+\alpha_\pi)\left(\theta_x - \theta_\pi\widehat{E}^\pi + \pi^T\right) - \alpha_x\right)$

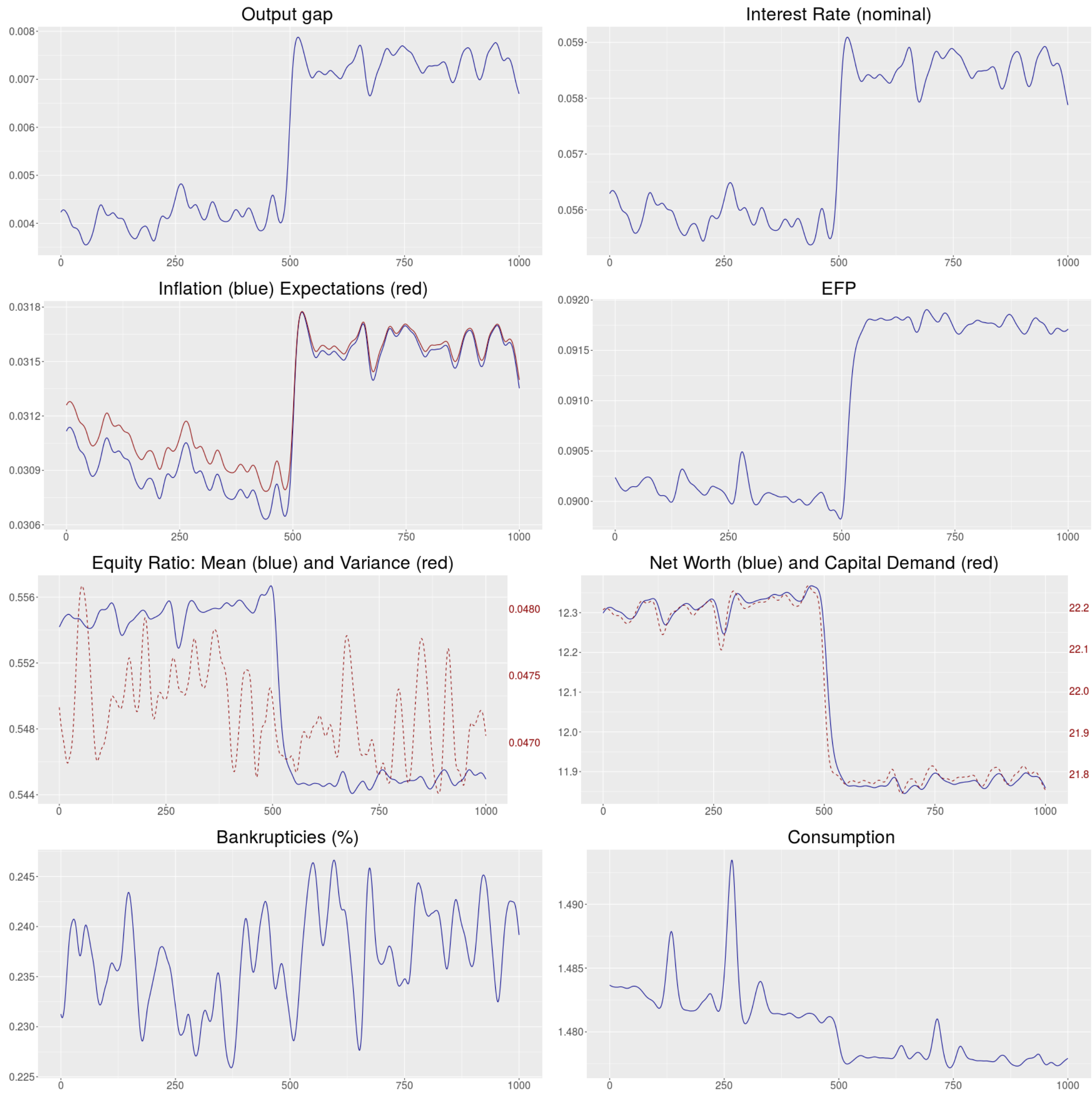


Figure 4.2: Fiscal shock in t=500. The individual variables plotted are of the *average* agent.

As the shock hits, the above system will be changing driven initially by the direct effect of $\Delta G = G_1 - G_0$, subsequently it will be driven by the changes in the relevant distributions, caused by the increase in the pubic expenditure as is sketched in Fig.4.1. The analysis will be focusing on the interest rate, but, it can be easily extended to any of the other of the fundamentals characterizing the economy.

The total variation of the interest rate can be analytically computed by taking the total differential of i with respect to the endogenous variables, i.e. the *total effect*

$$\Delta i = \left.\frac{\partial i}{\partial \Omega}\right|_{\Omega_0} \Delta\Omega + \left.\frac{\partial i}{\partial f}\right|_{f_0} \Delta f + \left.\frac{\partial i}{\partial \mu^{\pi}}\right|_{\mu_0^{\pi}} \Delta\mu^{\pi} + \left.\frac{\partial i}{\partial G}\right|_{G_0} \Delta G$$

with $\Delta z := z_1 - z_0, \forall z = [\Omega, f, \mu^\pi, G]$. According to the previous discussion, $\frac{\partial i}{\partial G}\Delta G$ can be defined as the direct effect, while the reminder represent the indirect effect net.

Direct Effect

The direct effect is the one implied directly and immediately by the change of the public expenditure, thus it is equal to the differential of the interest rate with respect to the public expenditure

$$\Delta_{DE} i = \frac{\partial i_t}{\partial G}\bigg|_{G_0} \Delta G = \frac{1}{Z_2 \lambda N}\left[1 + \widehat{Z}_3 \left(\left(Z_2 + \widehat{Z}_3\right)^2 - 4\widehat{Z}_1 Z_2\right)^{-\frac{1}{2}}\right]\Delta G$$

By substituting the corresponding values we can compute the analytical value of the direct effect, which gives a good approximation of the simulated one, defined as $i_{t=500} - i_{t=499}$. The analytical result is (error in parenthesis):

$$\Delta_{DE} i = 0.00204(-0.0001)$$

which represents the 86.8% of the total variation of i. It is consistent with the economic intuition: an increase in the public expenditure, via its effect on the aggregate demand, pushes up the interest rate.

Indirect Effect

The indirect effect is the one which links the old system (i.e. with the old distributions) to the new one; it can be explained by the following partial differentials:

$$\frac{\partial i}{\partial \Omega}\bigg|_{\Omega_0} \Delta\Omega = \frac{1}{\lambda}\left[\left(Z_2 + \widehat{Z}_3\right)^2 - 4\widehat{Z}_1 Z_2\right]^{-\frac{1}{2}} \Delta\Omega = 0.000036$$

$$\frac{\partial i}{\partial f}\bigg|_{f_0} \Delta f = \frac{F}{Z_2 \lambda \tau N}\left[1 + \widehat{Z}_3 \left(\left(Z_2 + \widehat{Z}_3\right)^2 - 4\widehat{Z}_1 Z_2\right)^{-\frac{1}{2}}\right]\Delta f = -0.0000707$$

$$\frac{\partial i}{\partial \mu^\pi}\bigg|_{\mu_0^\pi} \Delta\mu^\pi = \frac{1}{Z_2}\frac{F}{\lambda \tau N + \frac{x_n(1+\alpha_x)\theta_\pi}{(1+\alpha_\pi)\theta_x + \alpha_x}} \cdot \left[1 + \widehat{Z}_3 \left(\left(Z_2 + \widehat{Z}_3\right)^2 - 4\widehat{Z}_1 Z_2\right)^{-\frac{1}{2}}\right]\Delta\mu^\pi = 0.000401$$

The first indirect effect is the one which relates how the interest rate moves after a movement in Ω: if Ω increases the interest rate will increase too. In fact Ω might goes up as a consequence of an increase of inflation expectation, or an increase of consumption expectation or the covariance between the two: as spelled out in section 2.2, an increase in consumption expectation cause an immediate increase in consumption for smoothing purposes, an increase in inflation expectation causes too an increase in current consumption because consumer prevent the loss of purchasing power. And obviously, an increase in consumption is reflected in an increase in aggregate demand and thus in the employment level, finally through the Taylor Rule, we will observe an increase the interest rates. Despite we would expect an increase in consumption as a consequence of the increase in Ω, the increase in i is sufficient to offset it, since consumers will prefer to save and exploit the high interest rate rather than increase consumption (see eq. (2.17)).

The second indirect effect projects the variation of the external finance premium on the interest rate: the effect is negative. In fact, the increase in the EFP will cause an increase of the financing cost of the firms, which will then reduce their capital demand and thus the aggregate demand will decrease, resulting in a lower interest rate.

Finally, the last effect analyses how the variation of inflation expectation drives the variation of interest rate. As one may conjecture, this effect is positive and sizeable: as the inflation expectations increase, firms expect to increase their profit while consumers anticipate consumption, so not to lose purchasing

power, thus aggregate demand increases and so it happens to interest rates.
The total indirect effect can be computed by substituting the corresponding values net of the direct effect; and by comparing it with the simulated outcome, $i_{t=500} - \bar{i}_{t=501:1000}$. The analytical result is (deviation from simulation in parenthesis)

$$\Delta_{IE} i = 0.000472(-0.0003)$$

The total variation is the total differential of i wrt G and it can be compared with the simulated total variation , $i_{t=499} - \bar{i}_{t=501:1000}$. The analytical result is indeed close to the simulate one

$$\Delta_i = 0.00251(-0.0001)$$

An additional interesting decomposition that can be studied, is proposed in Assenza and Delli Gatti (2016) by studying how the RA component and the HA component contribute to the dynamics of the shock.
We can further decomposed the indirect effect, studying the indirect effects by insulating the RA contribution from the HA contribution:

$$\frac{\partial i}{\partial \Omega}\Delta\Omega = \frac{\partial i}{\partial \Omega}\left(\Omega_1 - \Omega_0\right)\Delta\Omega = \frac{\partial i}{\partial \Omega}\left[\left(1+\mu_1^{\pi}\right)\mu_1^c + cov_1^{\pi,c} - \left(1+\mu_0^{\pi}\right)\mu_0^c - cov_0^{\pi,c}\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial i}{\partial \Omega}\Delta\Omega = \frac{\partial i}{\partial \Omega}\left[\underbrace{\left(1+\mu_1^{\pi}\right)\mu_1^c - \left(1+\mu_0^{\pi}\right)\mu_0^c}_{\Omega^{RA}\text{contribution}} + \underbrace{cov_1^{\pi,c} - cov_0^{\pi,c}}_{\Omega^{HA}\text{contribution}}\right] = 3.78\cdot 10^{-5} - 1.01\cdot 10^{-8}$$

The RA effect is positive, while the HA is negative and small: both these characteristics can be explained by the covariance. On the one hand it tends to be stable and small along all the simulation, but it appears to be bigger before the shock, rather than after: this is because consumption (and so consumption expectation) decreased after the shock, while inflation (and so inflation expectation) increased.
Similarly we can compute the contribution in the change of the EFP

$$\frac{\partial i}{\partial f}\Delta f = \frac{\partial i}{\partial f}\left(f_1 - f_0\right) = \frac{\partial i}{\partial f}\left(\frac{\alpha}{\mu_1^H} + \frac{\alpha Var_1^H}{\left(\mu_1^H\right)^3} - \frac{\alpha}{\mu_0^H} - \frac{\alpha Var_0^H}{\left(\mu_0^H\right)^3}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial i}{\partial f}\Delta f = \frac{\partial i}{\partial f}\left[\underbrace{\frac{\alpha}{\mu_1^H} - \frac{\alpha}{\mu_0^H}}_{f^{RA}\text{contribution}} + \underbrace{\frac{\alpha Var_1^H}{\left(\mu_1^H\right)^3} - \frac{\alpha Var_0^H}{\left(\mu_0^H\right)^3}}_{f^{HA}\text{contribution}}\right] = -6.94\cdot 10^{-5} - 3.18\cdot 10^{-5}$$

In this decomposition we observe that both the components are negative, in line with the total effect; moreover the two effects are comparable in size.The negative component deriving from the RA is intuitive: after the increase in i, we observe a drop in the cross sectional mean of EFP. The HA component follows a different logic: it decreases not only because of the drop in the mean, but also due to the drop in variance, which is caused by the increase of bankruptcies that hit the most fragile firms.
Summing up, it can be noticed that the direct effect explains the majority of the change; the remainder is due quite totally to the indirect effect caused by the *inflation expectation*. It has to be highlighted any way that the *EFP-channel* is negative and not-negligible, since it tends to lower the interest rate of about 3%, while the *consumption-effect* (i.e. the one depending on Ω) is positive and it values 1.6%. Because of the small values of the two latter components, the HA and RA components are very small in the aggregate.

The last approach that allows us to study the effect of the shock is through the non-linear system and its graphical representation: starting from the systems presented in (3.6) and by changing only those parameters which are needed to recover the *direct effect*, the *indirect effect* and the *total effect*, we can observe and represent the movements of the curves. Note that with the previous methodology we just decomposed the *direct* and *indirect* effects over i but with the possibility to further decompose according to the RA and HA contribution. The methodology that is going to be presented in the next paragraphs, allow to decompose simultaneously all the fundamentals, at the cost of not being able to derive the RA and HA components.

First of all note that the macro-economic environment initially will be characterized by the following systems

$$\begin{cases} \text{IS:} & i_t = max\left[0, \frac{-IS_{B,0}|_{G_0}+\sqrt{IS_{B,0}|^2_{G_0}-4IS_A IS_{C,0}|_{G_0}}}{2IS_A}\right] \\ \text{aTR:} & i_t = \left(\frac{(1+\alpha_\pi)\theta_x+\alpha_x}{x_n}\right)x_t + r_n + \pi^T + (1+\alpha_\pi)\left(-\theta_x+\theta_\pi\mu_0^\pi-\pi^T\right)-\alpha_x \end{cases}$$

$$\begin{cases} \text{AD: } \pi_t = & \frac{1}{1+\alpha_\pi}\left[-r_n+(1+\alpha_\pi)\pi^T-\frac{\alpha_x}{x_n}x_t+\alpha_x+\frac{-IS_{B,0}|_{G_0}+\sqrt{IS_{B,0}|^2_{G_0}-4IS_A IS_{C,0}|_{G_0}}}{2IS_A}\right] \\ \text{PC: } \pi_t = & \frac{\theta_x}{x_n}x_t-\theta_x+\theta_\pi\mu_0^\pi \end{cases}$$

with $IS_A := \frac{F}{\lambda\tau N}$, $IS_{B,0} := \frac{F}{\lambda\tau N}\left(1-\gamma-\mu_0^\pi+f_t\right)-\frac{1}{\lambda N}G_0+x_t$ and $IS_{C,0} := x_t-\frac{\Omega_0}{\lambda\beta}-\frac{F}{\lambda\tau N}\left(\gamma+\mu_0^\pi-f_0\right)-\frac{1}{\lambda N}G_0$. The equilibrium is in point E in Fig.4.3

The *direct effect*, is the change implied only by the change of the public expenditure, before the distributions evolve. Thus, the system defining the state of the economy comprising only the direct effect is

$$\begin{cases} \text{IS:} & i_t = max\left[0, \frac{-IS_{B,0}|_{G_1}+\sqrt{IS_{B,0}|^2_{G_1}-4IS_A IS_{C,0}|_{G_1}}}{2IS_A}\right] \\ \text{aTR:} & i_t = \left(\frac{(1+\alpha_\pi)\theta_x+\alpha_x}{x_n}\right)x_t + r_n + \pi^T + (1+\alpha_\pi)\left(-\theta_x+\theta_\pi\mu_0^\pi-\pi^T\right)-\alpha_x \end{cases}$$

$$\begin{cases} \text{AD: } \pi_t = & \frac{1}{1+\alpha_\pi}\left[-r_n+(1+\alpha_\pi)\pi^T-\frac{\alpha_x}{x_n}x_t+\alpha_x+\frac{-IS_{B,0}|_{G_1}+\sqrt{IS_{B,0}|^2_{G_1}-4IS_A IS_{C,0}|_{G_1}}}{2IS_A}\right] \\ \text{PC: } \pi_t = & \frac{\theta_x}{x_n}x_t-\theta_x+\theta_\pi\mu_0^\pi \end{cases}$$

with $IS_A := \frac{F}{\lambda\tau N}$, $IS_{B,0} := \frac{F}{\lambda\tau N}\left(1-\gamma-\mu_0^\pi+f_t\right)-\frac{1}{\lambda N}G_1+x_t$ and $IS_{C,0} := x_t-\frac{\Omega_0}{\lambda\beta}-\frac{F}{\lambda\tau N}\left(\gamma+\mu_0^\pi-f_0\right)-\frac{1}{\lambda N}G_1$. The direct effect causes a movement only of the IS and the AD curves, leaving the aTR and the PC unchanged: equilibrium moves along the aTR (PC) until it reaches E^{DE} (Fig.4.3). These variations are coherent with the idea that an increase in the level of public expenditure pulls the demand increasing thus production and then the employment level; simultaneously we observe an increase of the interest rate and in the inflation level. On the one side, the Central Bank, via the Taylor Rule, reacts to the increase in production by increasing the interest rate to cool the economy. On the other the underlying relationships through the Phillips Curve, cause the increase of inflation as the consequence of the increase in the employment rate.

We can compute now the *total effect*, which comprises both the variation in the distributions and both the variation in the level of G; it is determined with the new level of G and the new distributions:

$$\begin{cases} \text{IS:} & i_t = max\left[0, \frac{-IS_{B,1}|_{G_1}+\sqrt{IS_{B,1}|^2_{G_1}-4IS_A IS_{C,1}|_{G_1}}}{2IS_A}\right] \\ \text{aTR:} & i_t = \left(\frac{(1+\alpha_\pi)\theta_x+\alpha_x}{x_n}\right)x_t + r_n + \pi^T + (1+\alpha_\pi)\left(-\theta_x + \theta_\pi \mu_1^\pi - \pi^T\right) - \alpha_x \end{cases}$$

$$\begin{cases} \text{AD: } \pi_t = & \frac{1}{1+\alpha_\pi}\left[-r_n + (1+\alpha_\pi)\pi^T - \frac{\alpha_x}{x_n}x_t + \alpha_x + \frac{-IS_{B,1}|_{G_1}+\sqrt{IS_{B,1}|^2_{G_1}-4IS_A IS_{C,1}|_{G_1}}}{2IS_A}\right] \\ \text{PC: } \pi_t = & \frac{\theta_x}{x_n}x_t - \theta_x + \theta_\pi \mu_1^\pi \end{cases}$$

with $IS_A := \frac{F}{\lambda\tau N}$, $IS_{B,1} := \frac{F}{\lambda\tau N}(1-\gamma-\mu_1^\pi+f_t) - \frac{1}{\lambda N}G_1 + x_t$ and $IS_{C,1} := x_t - \frac{\Omega_1}{\lambda\beta} - \frac{F}{\lambda\tau N}(\gamma+\mu_1^\pi - f_1) - \frac{1}{\lambda N}G_1$.
The final equilibrium is E^{New}. The movement from E^{DE} and E^{New} is due to the indirect effect.

These movements are caused by a *regime switch*: in fact we allow the relevant distribution to move and to set on the new long-run steady state values. We can observe that, since inflation expectations have increased, the aTR and the PC shifts upwards: this would cause a lower employment level due to the decreased capital demand driven by the high financing costs (i.e. the interest rate). The higher inflation expectation drives up current inflation (shift of the PC), which in turns, via the Taylor Rule, reinforces the increase in the interest rate. Simultaneously, the new distributions shift the IS and the AD to the right: an increase in the expected inflation rate reduces the real financing costs of the firm and increase current consumption, enough to overcome the negative effect caused by the increase in the EFP; this effect calmierate the decrease in x implied by the shifts of aTr and PC. Such a decomposition allows to fully understand which are the movement caused by the changes in the distribution, and thus that are also accountable to the heterogeneity.

The general effects of an increase of the public expenditure are an increase in the employment level, in the interest rate and in the inflation rate, but we have observed that the direct effect of the manoeuvre is positive across all the fundamental, while the cross effect is detrimental to the employment level. Thus the direct effect prevails.
Also this methodology is consistent with the simulations and with the analytical results studied by means of the *total-differential* in the three-equations setting, as Table 4.1

	x	i	π
Pre-Shock			
Simulation	0.954	0.056	0.031
Analytical	0.952	0.054	0.031
Direct-Effect			
Simulation	0.0031	0.0020	0.00013
Analytical	0.0034	0.0020	0.00014
Indirect-Effect			
Simulation	-0.0011	0.0004	0.0007
Analytical	-0.0039	0.0008	0.0007
After-Shock			
Simulation	0.957	0.058	0.0316
Analytical	0.955	0.057	0.0315

Table 4.1: Decomposition of the shock

Before concluding this section, it is worth to focus on the expectations, the evolution of the fraction of agents using a certain heuristic and the business cycles (i.e. *weak animal spirits*). The baseline

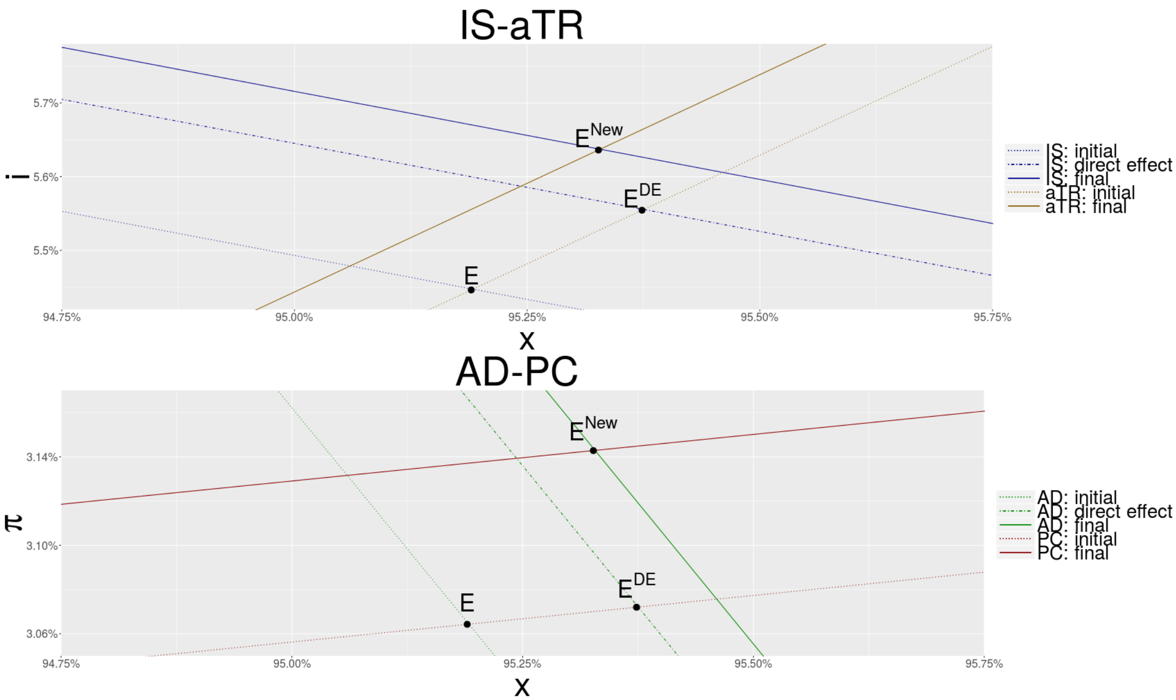


Figure 4.3: Long-Run Equilibrium representation

model, being ergodic, is too stable to exhibit significant patterns. The shock provides an interesting trigger: in fact the change in G should provide a shock big enough to have an impact on the chosen heuristic: this proves to be true for what concerns inflation expectations, but unfortunately it is not for consumption expectations. Inflation expectations heuristics (see Fig.4.4) changes significantly after the shock moving from a *stable* heuristic (such as the Learn and Anchor (LAA)) to heuristics which are less influenced by the past level of inflation: LAA in fact anchors expectation to the mean of inflation observed until period t, and, after the shock, we have observed a shift in π, therefore the anchor is not any more reliable and agents prefer to switch to more *volatile* heuristics. For what concern consumption we do not observe such a movement, because consumption is very stable[22] and also the shock is not causing such a significant movement.

Wrapping up, the model allows to analyse the shock using different methodologies, each of which gives interesting insights on the dynamics and the transmission of the shock. Despite the shock, consumption is too stable to cause significant variation in the consumption expectation heuristics while we do observe a significant variation in the inflation expectation heuristics.

5 Conclusions

The model presented in this paper presents an extension of the *Hybrid Macroeconomic* ABM, along the lines originally proposed by Assenza and Delli Gatti: the main novelties are the introduction of N consumers endowed with non-rational and heterogeneous expectations, namely the formed Heuristic Switching Model, and the introduction of different and more precise way to measure financial fragility, reminding the approach proposed in Assenza and Delli Gatti (2013). Heterogeneity is present both

[22]Stability in consumption is also due to the employment process: given that it is random and that the employment rate is close to 95%, it is very unlikely that an agent is unemployed for a time span long enough to make him suffer from a significant loss in income.

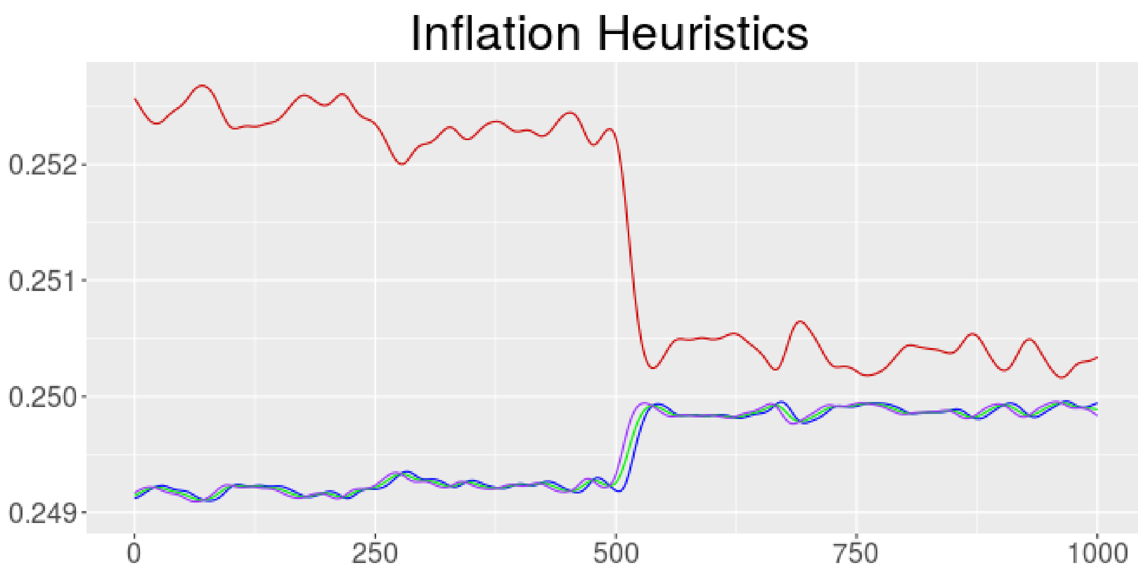


Figure 4.4: Fraction of agents using Adaptive Heuristic (blue), Weak Trend Heuristic (green), Strong Trend Heuristic (purple) and Learn and Anchor Heuristic (red).

the firms' population and in the consumers' population: the former have to chose the optimal capital demand, as a function of their equity ratio (i.e. proxy for financial fragility), the latter have to choose their optimal consumption based on their expectations. Thanks to the VRA approach, the emerging macro-economic system is enriched with the second moments of the relevant distributions which allows to take track of the heterogeneity; nonetheless the macro-economic model is still *analytically tractable*: in fact it can be reduced to a *three-equations* model composed by a *optimized IS*, a *Taylor Rule* and a *Phillips Curve* resembling closely a NK-DSGE; moreover it can be reduced to two 2-d systems, one in the (x, i) and one in the (x, π) plains. The macro-economic system is backward looking, while the underlying ABM links the subsequent periods: once it settles after a transition phase, the long-run steady state equilibrium values emerge and they are susceptible to the distributions. Thus any change that affects the distributions them selves (i.e. a regime switch), is going to change the long-run state to a long run steady state value.

On the micro level, despite not being explicitly modelled nor assumed, in the model an interesting property is automatically emerging along the lines of *loss-aversion*: when extending loan to the firms, uncertainty makes the bank to increase the price of the loan. As presented in eq.(2.12), we can observe that the approximated EFP charged by the banks is increasing in the dispersion (i.e. the variance) of the equity ratio.

One of the virtues of the H-ABM has to be found in the fact that it allows to disentangle completely the dynamics of a shock and its transmission mechanism: on the one side we are able to distinguish the *direct effect* and the *indirect effect*, on the other we can identify the contribution due to the *representative agent* component and to the *heterogeneous agent* component. It has been run an experiment taking the form of a positive fiscal shock: the concern is not the manoeuvre per-se, but rather the capacity of the model to study the dynamics in its constitutive elements.
The *direct effect*, i.e. the one caused only by the variation in the public expenditure, increases production (and thus inflation an the interest rate) through the increase of aggregate demand. On the contrary, the *indirect effect* will suffer from the increase in interest rate, lowering capital demand and consumption trough the changes in the distribution. Anyway, the direct effect offset the indirect effect,

thus the *total effect* will bring the economy to a situation characterized by an higher employment, interest and inflation rates wrt to the pre-shock situation. Graphically we have observed that the shift in the IS and in the AD curves caused by the direct effect are big enough to completely hinder the long-run movements, caused by the changes in the distributions, of the PC and the TR.
Since heterogeneity is found both in consumers and in firms, we identify two channels through which we can analyse the HA and RA contributions: a *consumer-based* and a *firm-based*. The total effect of the former (i.e. HA + RA) is positive, but we shall stress that the heterogeneous component is negative: as the heterogeneity in expectations among agents increases, the approximated consumption demand decreases; on the contrary the representative agent component is positive: as the mean of expectations goes up, the same is true also for consumption. The latter is always negative. In fact as the cross sectional ER (i.e. RA) is diminished, the EFP will be higher, since firms are generally more fragile; similarly the reduction of variance, caused by the exit from the market of the firms on the verge of bankruptcy due to the higher financing costs, cause an increase in the EFP because also the remainder of the firms will be more fragile wrt to the pre-shock situation.

On the one hand, the model presented so far leaves space to answer quantitatively the relationship existing between business cycles and *Animal Spirits* in a deeper way, since the results presented so far can be considered preliminary. On the other there are some open modelling issues which can be addressed: the most relevant is the micro-foundation of the Phillips Curve.

6 Appendix

6.1 VRA on consumers

Recall the consumption demand

$$c_{i,t} = \left(\frac{1 + E_{i,t}\pi_{t+1}}{1 + i_t}\right)\frac{1}{\beta}E_{i,t}c_{i,t+1}$$

Definitions Lets define, for notation convenience:
Cross sectional means : $\mu_t^c =< E_{i,t}(c_{i,t+1}) >$, $\mu_t^\pi =< E_{i,t}(\pi_{t+1}) >$ and $\mu_t^i =< i_t >$
Distances from means : $\Delta_c = E_{i,t}(c_{i,t+1}) - \mu_t^c$ and $\Delta_\pi = E_{i,t}(\pi_{t+1}) - \mu_t^\pi$ and $\Delta_i = i_t - \mu_t^i$
Derivatives first order generic derivative of variable x with respect to y: x_y; second order generic derivative of variable x with respect to y and z: x_{yz}
Drop time indexes wlog

1. Taylor Expansion up to the second order

$$\begin{aligned}
c_{i,t}|_{\mu^c,\mu^\pi,\mu^i} &\simeq \frac{1+\mu_t^\pi}{\beta}\frac{1}{1+\mu_t^i}\mu_t^c \quad + \\
&+ \begin{bmatrix} c_\pi & c_c & c_i \end{bmatrix}\begin{bmatrix} \Delta_\pi \\ \Delta_c \\ \Delta_i \end{bmatrix} \quad + \\
&+ \frac{1}{2}\begin{bmatrix} \Delta_\pi & \Delta_c & \Delta_i \end{bmatrix}\begin{bmatrix} c_{\pi\pi} & c_{\pi c} & c_{\pi i} \\ c_{c\pi} & c_{cc} & c_{ci} \\ c_{i\pi} & c_{ic} & c_{ii} \end{bmatrix}\begin{bmatrix} \Delta_\pi \\ \Delta_c \\ \Delta_i \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow c_{i,t}|_{\mu^c,\mu^\pi,\mu^i} &\simeq \frac{1}{\beta}\frac{1+\mu_t^\pi}{1+\mu_t^i}\mu_t^c + \\
&+ c_\pi\Delta_\pi + c_c\Delta_c + c_i\Delta_i + \\
&+ \frac{1}{2}\left(c_{\pi\pi}\Delta_\pi^2 + c_{cc}\Delta_c^2 + c_{ii}\Delta_i^2\right) + c_{c\pi}\Delta_\pi\Delta_c + c_{\pi i}\Delta_\pi\Delta_i + c_{ic}\Delta_c\Delta_i
\end{aligned}$$

Observe that $c_{\pi\pi} = c_{cc} = 0$ and that $< i_t >= \frac{1}{N}\sum_{n=1}^N i_t = i_t \Leftrightarrow \Delta_i = i_t - i_t = 0$, hence the previous equation can be simplified to

$$\begin{aligned}
\Leftrightarrow c_{n,t} &\simeq \frac{1}{\beta}\frac{1+\mu_t^\pi}{1+i_t}\mu_t^c + \\
&+ c_\pi\Delta_\pi + c_c\Delta_c \\
&+ c_{c\pi}\Delta_\pi\Delta_c
\end{aligned}$$

2. Apply *E(·)* and observe that
$\rightarrow E(\Delta_\pi) = 0$
$\rightarrow E(\Delta_c) = 0$
$\rightarrow E\left(\Delta_\pi\Delta_c\right)) = E\left(E(E_{n,t}(\pi_{t+1}) - \mu_t^\pi)E(E_{n,t}(c_{t+1}) - \mu_t^c)\right) = cov_t^{\pi c}$, by definition of covariance

So we can take advantage of the above results and obtain

$$\begin{aligned} c_t^{HA} &= \frac{1}{\beta}\frac{1+\mu_t^\pi}{1+i_t}\mu_t^c + \frac{1}{\beta}\frac{1}{1+i_t}cov_t^{\pi,c} \Leftrightarrow \\ \Leftrightarrow c_t^{HA} &= \frac{1}{\beta}\frac{1}{1+i_t}\left[(1+\mu_t^\pi)\mu_t^c + cov_t^{\pi,c}\right] \end{aligned}$$

6.2 Solving the macroeconomic system

Recall the system characterizing the macroeconomic equilibrium

$$\begin{cases} \text{IS: } x_t = & \frac{\Omega_t}{\lambda\beta}\frac{1}{1+i_t} - \frac{F}{\lambda\tau N}i_t + \frac{F}{\lambda\tau N}\left(\gamma + \mu_t^\pi - f_t\right) + \frac{1}{\lambda N}G_t \\ \text{TR: } i_t = & \max\Big(0, r_n + \pi^T + (1+\alpha_\pi)(\pi_t - \pi^T) + (\frac{x_t}{x_n} - 1)\alpha_x\Big) \\ \text{PC: } \pi_t = & \theta_x(\frac{x_t}{x_n} - 1) + \theta_\pi\mu_t^\pi \end{cases}$$

Notice that the piecewise linear Taylor Rule identifies two regions: a *positive interest rate region* which is characterizing "normal times", and a *zero lower bound region*, which might emerge during times of crisis.
Due to the non linearities stemming from the IS-curve, there might the case of the solution being not-defined or non-uniquely defined; but as there will be proven, the solution will be always one.

Assume initially that the zero lower bound is not binding.

1. Plug the Philips Curve in the Taylor Rule

$$\begin{aligned} i_t = &\quad r_n + \pi^T + (1+\alpha_\pi)\left[\left(\theta_x(\frac{x_t}{x_n} - 1) + \theta_\pi\mu_t^\pi\right) - \pi^T\right] + (\frac{x_t}{x_n} - 1)\alpha_x \Leftrightarrow \\ \Leftrightarrow i_t = &\quad r_n + \pi^T - (1+\alpha_\pi)\theta_x + \frac{(1+\alpha_\pi)\theta_x}{x_n}x_t + (1+\alpha_\pi)\theta_\pi\mu_t^\pi - (1+\alpha_\pi)\pi^T + \frac{a_x}{x_n}x_t - a_x \Leftrightarrow \\ \Leftrightarrow i_t = &\quad r_n + \pi^T - (1+\alpha_\pi)\theta_x + (1+\alpha_\pi)\theta_\pi\mu_t^\pi - (1+\alpha_\pi)\pi^T + \left(\frac{(1+\alpha_\pi)\theta_x + a_x}{x_n}\right)x_t - a_x \Leftrightarrow \\ x_t = &\quad \frac{x_n}{(1+\alpha_\pi)\theta_x + a_x}\left(i_t - r_n - \pi^T + (1+\alpha_\pi)\theta_x - (1+\alpha_\pi)\theta_\pi\mu_t^\pi + (1+\alpha_\pi)\pi^T + a_x\right) \qquad (6.1) \end{aligned}$$

2. Equate eq.(6.1) and eq.(IS)

$$\begin{cases} x_t = & \frac{x_n}{(1+\alpha_\pi)\theta_x + a_x}\left(i_t - r_n - \pi^T + (1+\alpha_\pi)\theta_x - (1+\alpha_\pi)\theta_\pi\mu_t^\pi + (1+\alpha_\pi)\pi^T + a_x\right) \\ x_t = & \frac{\Omega_t}{\lambda\beta}\frac{1}{1+i_t} - \frac{F}{\lambda\tau N}i_t + \frac{F}{\lambda\tau N}\left(\gamma + \mu_t^\pi - f_t\right) + \frac{1}{\lambda N}G_t \end{cases}$$

$$\begin{aligned} \Leftrightarrow \quad & \frac{1}{1+i_t}\frac{\Omega_t}{\lambda\beta} \\ & - i_t\left(\frac{x_n}{(1+\alpha_\pi)\theta_x + \alpha_x} + \frac{F}{\lambda\tau N}\right) \\ & + \frac{F}{\lambda\tau N}\left(\gamma + \mu_t^\pi - f_t\right) + \frac{1}{\lambda N}G_t + \frac{x_n}{(1+\alpha_\pi)\theta_x + a_x}\left(r_n + \pi^T - (1+\alpha_\pi)\theta_x + (1+\alpha_\pi)\theta_\pi\mu_t^\pi - (1+\alpha_\pi)\pi^T - a_x\right) = 0 \end{aligned} \qquad (6.2)$$

Define and observe the sign:
$Z_{1,t} := \frac{\Omega_t}{\lambda\beta} > 0$

$Z_2 := \left(\frac{x_n}{(1+\alpha_\pi)\theta_x + a_x} + \frac{F}{\lambda \tau N} \right) > 0$
$Z_{3,t} := \frac{F}{\lambda \tau N} \left(\gamma + \mu_t^\pi - f_t \right) + \frac{1}{\lambda N} G_t + \frac{x_n}{(1+\alpha_\pi)\theta_x + a_x} \left(r_n + \pi^T - (1+\alpha_\pi) \left(\theta_x - \theta_\pi \mu_t^\pi + \pi^T \right) - \alpha_x \right) \lesseqgtr 0$

Notice also that $Z_{1,t}$ and $Z_{3,t}$ are time varying.

It is possible to rewrite eq.(6.2) in a more compact way:

$$\frac{1}{1+i} Z_{1,t} - i Z_2 + Z_{3,t} = 0 \Leftrightarrow$$

$$Z_2 i^2 + (Z_2 - Z_{3,t})\, i - (Z_{1,t} + Z_{3,t}) = 0 \tag{6.3}$$

It cannot stated *a-priori* if the solution will have zero, one or two solution. It will be proved that indeed there exists only one admissable solution.

3. Existence and uniqueness of one admissible equilibrium

Claim 1: there is always at least one solution to eq.(6.3)
Proof: let Δ be the discriminant of eq.6.3, then in order to have at least one solution the following must hold

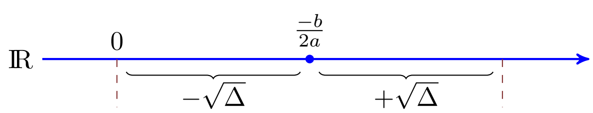
$$\Delta \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (Z_2 - Z_{3,t})^2 + 4 Z_2 (Z_{1,t} + Z_{3,t}) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (Z_2 + Z_{3,t})^2 + 4 Z_{1,t} Z_2 \geq 0$$

Since it is a sum between a square and a positive number, the discriminant is always greater than zero, $\therefore$ there exists two different solutions.

■ *Qed*

Claim 2: At least one solution is always negative, and thus not admissible in the context of the model
Proof : SUPPOSE NOT, then both solutions are positive.
Let, from eq.(6.2): $a := Z_2$, $b := Z_2 - Z_{3,t}$ and $c := -Z_{3,t} - Z_{1,t}$. Two positive solution implies

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} > 0$$



Thus there has to be satisfied the following condition

$$\text{C.1} \qquad -\frac{b}{2a} > 0 \Rightarrow Z_{3,t} > Z_2$$

Moreover if the minimum solution is positive, so it will be for the maximum. Thus also the following must hold

$$min \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right) > 0 \Leftrightarrow -b \; > \sqrt{b^2 - 4ac}$$

both sides of the dis-equation are positive, so we can square them, which leads to condition

$$\text{C.2} \qquad b^2 > b^2 - 4ac \Leftrightarrow ac > 0 \Leftrightarrow -Z_{1,t} > Z_{3,t}$$

Now collect and study conditions C.1 and C.2 :

$$-Z_{1,t} > Z_{3,t} \;\; \& \;\; Z_{3,t} > Z_2 \Rightarrow -Z_{1,t} > Z_{3,t} > Z_2 \Rightarrow -Z_{1,t} > Z_2 \bot$$

$$\because Z_{1,t} > 0 \text{ and } Z_2 > 0$$

$\therefore$ *Claim 2* is confirmed

■ *Qed*

Thus there will always be a negative solution, which, surely, is going to be smaller one. Therefore the bigger solution *might* be positive: the solution will then be

$$i_t = \frac{Z_{3,t} - Z_2 + \sqrt{\left(Z_2 + Z_{3,t}\right)^2 + 4Z_{1,t}Z_2}}{2Z_2}$$

Note that the proposed solution is not necessarily positive, but the ZLB will prevent a negative interest rate to emerge, granting the existence of only one solution.

Thus the solution always exists and it is uniquely defined.

6.3 Sequence of events of the ABM Models

1. Consumers compute their heuristics on consumption and inflation, based on information collected:

$$\begin{cases} \text{Adaptive Rule} & \rightarrow & E^{ada}_{i,t}\pi_{i,t+1} = 0.65\pi_{t-1} + 0.35E^{ada}_{i,t-1}\pi_{i,t} \\ \text{Weak Trend} & \rightarrow & E^{wtr}_{i,t}\pi_{i,t+1} = \pi_{t-1} + 0.4\left(\pi_{t-1} - \pi_{t-2}\right) \\ \text{Strong Trend} & \rightarrow & E^{str}_{i,t}\pi_{i,t+1} = \pi_{t-1} + 1.3\left(\pi_{t-1} - \pi_{t-2}\right) \\ \text{Learn and Anchor} & \rightarrow & E^{laa}_{i,t}\pi_{i,t+1} = \frac{\left(\pi^{av}_{t-1} + \pi_{t-1}\right)}{2} + \left(\pi_{t-1} - \pi_{t-2}\right) \end{cases}$$

$$\begin{cases} \text{Adaptive Rule} & \rightarrow & E^{ada}_{i,t}c_{i,t+1} = 0.65c_{i,t-1} + 0.35E^{ada}_{i,t-1}c_{i,t} \\ \text{Weak Trend} & \rightarrow & E^{wtr}_{i,t}c_{i,t+1} = c_{i,t-1} + 0.4\left(c_{i,t-1} - c_{i,t-2}\right) \\ \text{Strong Trend} & \rightarrow & E^{str}_{i,t}c_{i,t+1} = c_{i,t-1} + 1.3\left(c_{i,t-1} - c_{i,t-2}\right) \\ \text{Learn and Anchor} & \rightarrow & E^{laa}_{i,t}c_{i,t+1} = \frac{\left(c^{av}_{i,t-1} + c_{i,t-1}\right)}{2} + \left(c_{i,t-1} - c_{i,t-2}\right) \end{cases}$$

Then they compute the fitness, for each heuristic, both for $c_{n,t+1}$ and for π_{t+1}

$$U^z_{i,t} = \frac{100}{1 + |y_{i,t} - E^z_{i,t-1}y_{i,t}|} + \eta U^z_{i,t-1} \quad , \quad \forall z \in Z$$

and then it is updated the probability to choose rule z

$$n^z_{i,t} = \delta^{BH}n^z_{i,t-1} + \left(1 - \delta^{BH}\right)\frac{exp\left(\beta^{BH}U^z_{i,t-1}\right)}{\sum\limits_{z=1}^{H} exp\left(\beta^{BH}U^z_{i,t-1}\right)}$$

Once all the consumers have gone through the **expectation formation process,** we can compute:

$$\text{Cross-sectional inflation mean} \longrightarrow \mu^\pi_t = \quad < E^z_{n,t}\pi_{t+1} >$$

$$\text{Cross-sectional consupmtion mean} \longrightarrow \mu^c_t = \quad < E^z_{n,t}c_{t+1} >$$

$$\text{Cross-sectional covariance} \longrightarrow cov^{\pi,c}_t$$

Simultaneously, **firms bargain the EFP** on their loans:

$$f_{f,t} \quad = \quad \frac{\alpha}{H_{f,t-1}}$$

and the average EFP can be computed

$$f_t = \frac{\alpha}{\mu^H_{t-1}} + \frac{\alpha}{\left(\mu^H_{t-1}\right)^3}Var^H_{t-1}$$

2. The **macroeconomic equilibrium** can be obtained

$$\begin{cases} x_t = & \frac{\Omega_t}{\lambda\beta}\frac{1}{1+i_t} - \frac{F}{\lambda\tau N}i_t + \frac{F}{\lambda\tau N}\left(\gamma + \mu^\pi_t - f_t\right) + \frac{1}{\lambda N}G_t \\ i_t = & \max\left(0, r_n + \pi^T + (1+\alpha_\pi)(\pi_t - \pi^T) + (\frac{x_t}{x_n} - 1)\alpha_x\right) \\ \pi_t = & \theta_x(\frac{x_t}{x_n} - 1) + \theta_\pi\mu^\pi_t \end{cases}$$

3. Firms now compute their **optimal capital demand**

$$K_{f,t} = \frac{\gamma - r_t - f_{f,t}}{\tau}$$

which shapes **profit**

$$\phi_{f,t} = \left(u_{f,t}\nu - w\frac{\nu}{\lambda} - (r_t + f_{f,t})\right) K_{f,t} - \tau\frac{1}{2}K_{f,t}^2$$

and finally update the **net worth**

$$A_{f,t} = (1-\delta)\, A_{f,t-1} + \phi_{f,t}$$

if the **exit condition** is met

$$H_{f,t-1} \leq H_t^{min} := \frac{\alpha}{\gamma - r_t}$$

the firm gets out from the market and is replaced by a new one.

At the same time, consumers learn whether they are **employed** or not by participating to a lottery in which they *win the job* with probability x_t, and form their **good demand**

$$c_{n,t} = \left(\frac{1 + E_{n,t}\left(\pi_{t+1}\right)}{\beta}\right) \frac{1}{1+i_t} E_{n,t}\left(c_{n,t+1}\right)$$

Finally we can compute the **average capital demand** and the **average consumption demand**

$$K_t^{HA} = \frac{\gamma - r_t - \frac{\alpha}{\mu_{t-1}^H} - \frac{\alpha}{\left(\mu_{t-1}^H\right)^3} Var_{t-1}^H}{\tau}$$

$$c_t^{HA} = \frac{1}{\beta}\frac{1}{1+i_t}\Omega_t$$

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