Long-run Heterogeneity in a Lucas' Tree Economy[*]

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Abstract

We consider a Lucas' tree economy with heterogeneous agents and multiple assets and investigate the coupled dynamics of assets' prices and agents' wealth. We assume that agents have heterogeneous beliefs and are generalized Kelly traders, that is, each agent invest on an asset a fraction of wealth proportional to her evaluation of that asset expected dividends. Our main finding is that long-run coexistence of heterogeneous beliefs is a generic outcome of the market dynamics. We provide sufficient conditions for the latter, as well as sufficient conditions for the relative wealth of any given agent converging to zero or to one. Since we use a direct approach that combines the inter-temporal dynamics of wealth and prices via agents' portfolio rules, we can characterize when long-run heterogeneity occurs for both complete and incomplete asset markets.

Keywords: Market Selection Hypothesis; Heterogeneous Beliefs; Evolutionary Finance; Incomplete Markets; Asset Pricing; Generalized Kelly rule.

JEL Classification: C60, D52, D53, G11, G12

1 Introduction

The Market Selection Hypothesis (MSH) applied to financial markets implies that traders' beliefs heterogeneity can be only a short-run phenomenon. In the long-run, the trader with the most accurate beliefs about asset's future dividends should gain all the wealth and price assets accordingly. Indeed, benchmark equilibrium asset pricing models, such as Lucas' model and the CAPM, just dismiss heterogeneity away and assume that all traders have correct beliefs about the assets' returns distribution. Despite these models provide an insightful characterization of the relation between assets equilibrium returns and risk preferences, they have not being validated by the data.[1] In this paper we investigate whether relying on the MSH is a possible source of failure. Are financial markets likely to select for the most accurate beliefs?

The formal investigation of the MSH has started only many years after its formulation by Alchian (1950) and Friedman (1953). The seminal work by Blume and Easley (1992) has lead to two strands of literature, with similar findings. A first group, routed in general equilibrium, assumes that agents are expected utility maximizers, have rational price expectations, but disagree on the dividend process. The main finding is that when markets are complete they do select for a unique trader.[2] Both saving behavior and accuracy of beliefs are important and only the agent who maximizes a given survival index[3] has positive wealth in the long-run. Beliefs heterogeneity is only transient.

Another strand of literature has instead focused on market selection in economies where agents behavior can be modeled directly in terms of saving and portfolio rules, not necessarily coming from expected utility maximization under rational price expectations. These works contend that agents are able to coordinate on having perfect foresight on future prices, especially when they disagree on the dividend process, and prefer to assume that agent's investment behavior is a general adapted process. The question is whether also in this more realistic set-up the market selects for a unique agent. An interesting result is that when saving is homogeneous across agents there exists a portfolio rule that dominates against any

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[1]For a list of puzzles and asset pricing anomalies see e.g. the entries "Financial Market Anomalies" and "Finance (new developments)" in the New Palgrave Dictionary of Economics.

[2]Heterogeneity may be persistent when markets are incomplete, see e.g. the examples in Blume and Easley (2006) and their extension in Beker and Chattopadhyay (2010) and Coury and Sciubba (2012), when agents have recursive preferences Borovicka (2015), or when agents are ambiguity averse Guerdjikova and Sciubba (2015).

[3]The survival index takes into account the trade-off between beliefs accuracy and saving behavior. In economies whose aggregate endowment is bounded both from below and from above only beliefs and discount factors matters, see e.g. Sandroni (2000) and Blume and Easley (2006). In economies where the aggregate endowment is growing at a constant rate, also intertemporal preferences are important, see e.g. Yan (2008).

other combination of adapted rules, the so called generalized Kelly rule, see e.g. Evstigneev et al. (2009) for a survey. In particular Evstigneev et al. (2008) establish the global dominance of the generalized Kelly rule in a Lucas' tree economy where agents can trade multiple long-lived assets. However, a characteristic of the generalized Kelly rule is that it relies on the exact knowledge of the dividend process. On the one hand, when the rule is used by some agents, the market converges to a Lucas' economy with a representative agent that has log-utility and correct beliefs, validating the MSH. On the other hand, it is not known what happens when no agent with correct beliefs is in the market.[4]

In this work we investigate the MSH in a Lucas' tree economy with a finite number of agents having homogeneous discount factors but heterogeneous i.i.d. beliefs. Agents can transfer consumption across time and states by means of long-live assets (trees). The states of the world follow an i.i.d. process and we consider both complete and incomplete markets. Importantly, we assume that agents' saving rates are equal to their discount factors and that they purchase assets according to the generalized Kelly rule of Evstigneev et al. (2008). The rule amounts to investing on each asset a fraction of wealth proportional to its expected dividends, where each agent computes expectations using her beliefs. Thus, due to the heterogeneity of beliefs, agents demands are also heterogeneous. Moreover, since both relative dividends and beliefs are i.i.d., each agent's portfolio corresponds to a constant fraction of wealth to be invested in a given asset. Validating the MSH in this context would imply that, even when no agent knows the truth, only the agent with the most accurate beliefs has positive wealth in the long-run. In this case, Lucas' model would be recovered in the limit and assets would be priced according to the most accurate beliefs. Otherwise, when more agents have positive wealth in the long-run, beliefs heterogeneity cannot be ignored when pricing assets.

We assume that agents choose how to allocate their wealth across assets using a generalized Kelly rule, rather than using a portfolio rules that maximizes their expected utility, for three reasons. First, in a market for long-lived assets, the optimality of a portfolio rule derived from utility maximization relies on agents having perfect foresight on future prices, an hypothesis much stronger than that of agents who do not know the asset dividend process.[5] We view generalized Kelly rules as a first step in relaxing the hypothesis of rational price expectations. In fact, a generalized Kelly is still log-optimal in the limit of the agent using it being alone in the economy. As a result, if an agent with accurate beliefs and very large

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[4]Bektur (2013) show that the agent whose rule is the the closest to the generalized Kelly rule derived using correct beliefs (almost) never vanishes. Bottazzi and Dindo (2014) investigate the same issue in an economy with short-lived assets, finding that the MSH does not generally hold.

[5]In particular, why should (endogenously determined) prices be easier to forecast than (exogenously given) dividends?

wealth fails to dominate, it is not because, conditionally on her beliefs, she uses a non-optimal rule but, rather, because of the non-optimality of the portfolio rules used by her opponents. Indeed, although one can question that 'smart' traders use non-optimal rules, we do not see strong arguments that prevent 'noise' traders to do so.

Second, in an economy of generalized Kelly traders it is already known that the rule derived under correct belies dominates almost surely (Evstigneev et al., 2008). Moreover, provided that markets are complete and assets dividends are negatively correlated, the rule with the most accurate beliefs never vanishes (Bektur, 2013). Finding that the rule with most accurate beliefs (but not correct) dominates would indicate that the MSH, and thus Lucas' asset pricing model, can be used for asset pricing purposes also in financial markets where portfolio rules rely neither on perfect foresight on future prices nor on the exact knowledge of the dividend process.

Third, recent works show that generalized Kelly rules, which are fixed in an i.i.d. economy as ours, provide a realistic characterization of investors behavior. Gigerenzer and Brighton (2009) argue that when agents face uncertainty instead of risk, they tend to use simple heuristics such as fixed portfolio rules. DeMiguel et al. (2009) find that fixed rules do not under-perform with respect to sophisticated optimal rules. In particular, on seven different database of financial prices, the fixed portfolio rule of investing $1/N$ of the wealth in each asset is found to provide no-worst results than investment strategies derived from Mean-Variance optimization.

Working directly with fixed (generalized Kelly) rules has the advantage that the dynamics of wealth and asset prices can be derived from the intertemporal budget constraints and market clearing equations.[6] Although the two dynamics are coupled -since assets are long-lived their payoffs determine the new wealth distribution but the wealth distribution determines, through prices, assets payoffs-we are able to solve them explicitly. Long-run outcomes of the market dynamics can then be studied by means of the Martingale Converge Theorem. In particular, we provide sufficient conditions for a group of agents to have a positive, null, or unitary, fraction of wealth in the long-run. In the simplest case of a 2-agent economy, the sufficient conditions are also necessary (but for hairline cases) and particularly easy to check. Based on these results we are able to characterize when long-run heterogeneity occurs.

Our main finding is that the MSH does not hold. Depending on the initial

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[6]The use of rules has also disadvantages, for example the fact that one cannot rule out arbitrage by relying on agents demand being optimal. However we are still able to give conditions that exclude arbitrage by working directly with the (endogenously determined) payoff matrix. Generalized Kelly rules naturally satisfy these conditions, see Proposition 3.1.

agents' beliefs distribution there exist cases where agents with heterogeneous beliefs have positive wealth, and thus matter for asset pricing, even in the long-run. When this is the case the surviving beliefs wealth distribution changes over time, so that some beliefs become more important in some periods and less important in others. Moreover, we find that these cases occur in all economies, that is, no matter the exact asset structure and the number of agents, and are generic, that is, they do not disappear if agents beliefs are locally perturbed. We explore numerically the occurrence of long-run heterogeneity by analyzing specific examples and find that the areas of beliefs combination where heterogeneity occurs are large. It appears that the survival of different agents is related to portfolios and dividends being anti-correlated: if an agents invests more in the asset that pays more in one state while the other agent invests more in an asset that pays more in another state, then the outcome is long-run heterogeneity. The possibility to work with both complete and incomplete markets lets us study whether the survival of heterogeneous beliefs is more or less likely when markets are completed. We show that both cases occur.

The contrast between our results and those of the general equilibrium literature of Sandroni (2000), Blume and Easley (2006), or Yan (2008) lies in the non-optimality of generalized Kelly rules, even when markets are complete. Consider an economy with two agents, i and j and assume that the beliefs of i are more accurate. When both agents use log-optimal rules, agent i dominates and agent j vanishes. If instead we find that j does not vanish, then it must be that agent j non-optimal portfolio is "better" then the optimal portfolio derived under her beliefs, at least in the limit when agent i has most of the wealth. Thus, the non-optimality of agent j portfolio 'corrects' for the non-optimality of her beliefs, leading to her survival.

In order to provide an intuition of the interplay between non correct beliefs and non log-optimal rules, we define the "effective" beliefs of an agent as those (time-varying) beliefs such that the generalized Kelly rule derived using the original beliefs coincides with the log-optimal portfolio rule derived using effective beliefs (and rational price expectations). Since a generalized Kelly rule is log-optimal in the limit of the agent using it having all the wealth, effective beliefs and beliefs coincide when an agent has most of the wealth. However, they differ when assets' returns are determined by both agents. In particular, given two agents, the effective beliefs of each agent are a combination of his beliefs with the beliefs of the other agent. The larger the wealth share of one agent, the larger her impact on equilibrium returns, the larger the weight of her beliefs in determining both agents effective beliefs.[7] Long-run heterogeneity occurs when agent i effective beliefs are

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[7]Given that the relative importance of capital gains and dividend yields for assets' returns depends on the market discount rate, the latter plays also a role for how much each agent effective

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more accurate than agent j beliefs when assets' returns are determined by agent j (because she has most of the wealth) and, at the same time, agent j effective beliefs are more accurate than agent i beliefs when assets' returns are determined by agent i. The latter is typically the case when the truth is "in between" agent i and agent j beliefs.

The structure of the paper is as follows. In Section 2 we introduce the model. In Section 3 we characterize the coupled dynamics of asset prices and wealth shares and give conditions that exclude arbitrage. In Section 4 we investigate the long-run behavior of our market economy and provide sufficient conditions for an agent, or a group of agents, to gain all wealth in the long-run. In particular, in Section 4.2 we characterize long-run heterogeneity and show that it is a generic outcome of an economy with long-lived assets. In Section 5 we discuss our results in terms of effective beliefs and log-optimal rules and show, by mean of a numerical exploration of the parameters space, that long-run heterogeneity occurs for a wide range of the economy parameters. Section 6 concludes. All the proofs are collected in the Appendix.

2 The Model

Time is discrete and indexed by $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. At each date $t \in \mathbb{N}$ one of the possible $S = \{1, \ldots, S\}$ states of the world occurs.[8] We assume that each state is drawn from the same distribution π over S, with $\pi(s) = \pi_s$, and that subsequent trials are independent. Without loss of generality $\pi \in \Delta_+^S := \Delta^S \cap \mathbb{R}_{++}^S$.[9] We denote by $s_t \in \{1, \ldots, S\}$ the state that occurs in t, by $\sigma = (s_1, s_2, ..., s_t, ...)$ an entire realization, and by σ_t the partial history up to date t included. The set of all possible realizations is Σ. For each $t \in \mathbb{N}$, the σ-algebra generated by the realizations that share the same partial history till t is $\Im_t$ and $\Im_0 := \{\emptyset, \Sigma\}$. We name $\Im$ the smallest σ algebra that contains all $\Im_t$, so that $\{\Im_t \, ; t = 0, 1, \ldots\}$ is a well-defined filtration of $\Im$. The probability measure P on Σ is obtained as the product of all the measures π on S. The expected value operator $\mathrm{E}\left[\cdot\right]$ integrates with respect to the measure π or P depending on the context. $(\Sigma, \Im, \mathrm{P})$ is the probability space on which we construct our economy. It is understood that all the random variables on $(\Sigma, \Im)$ that we shall introduce (dividends, asset prices, portfolios, wealth...) are adapted to the filtration $\{\Im_t\}$. For this reason we may

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beliefs incorporate the other agent beliefs.

[8]Throughout the paper we use the same capital letter to denote a set and its cardinality, when finite.

[9]Given $\mathbb{R}^S$, Δ^S denotes its simplex, $\mathbb{R}_+^S$ is the subset of vectors with non-negative components (excluding the null vector), and $\mathbb{R}_{++}^S$ is the subset of vectors with all positive components.

use $X_t(\sigma_t)$ in place of $X_t(\sigma)$. Unless otherwise noted, all our statements are true almost surely with respect to the probability measure P.

We consider an exchange economy populated by I agents whose aggregate endowment in each date t is E_t units of the consumption good (apples, the numérarire of the economy). The aggregate endowment dynamics can be any adapted stochastic process on $(\Sigma, \Im, P)$, we only assume that $E_t(\sigma) > 0$, $\forall t \in \mathbb{N}$. Portions of the aggregate endowment can be traded by exchanging K long-lived assets (trees). Asset $k \in K$ traded in $t \in \mathbb{N}_0$ pays a dividend $D_{k,t'}$ for each $t' > t$. Without loss of generality, each asset is in excess unitary supply and the aggregate endowment is the dividend of the market portfolio

D1 $\sum_{k=1}^{K} D_{k,t}(\sigma) = E_t(\sigma), \quad \forall t \in \mathbb{N}_0$.

Each agent is endowed in $t = 0$ with a quantity of apples and a portfolio of assets (trees). In every period, agents consume dividends (apples) and trade assets (trees) to transfer future consumption across time and states. Let $h_t^i = (h_{1,t}^i, \ldots, h_{K,t}^i)$ be the holding of the K assets at time t by agent i. . The timing is as follows. At the beginning of period t agent i holds as many assets as those purchased in the previous period $h_{t-1}^i = (h_{1,t-1}^i, \ldots, h_{K,t-1}^i)$. Then a state of the world is realized, s_t, and agent i receives an amount of dividends equal to $h_{0,t}^i = \sum_{k=1}^{K} h_{k,t-1}^i D_{k,t}((\sigma_{t-1}, s_t))$. After that agent i decides about her current consumption and portfolio holding, C_t^i and h_t^i respectively, and trades-in dividends $h_{0,t}^i$ and assets h_{t-1}^i to purchase them.[10] Denoting the vector of date-t asset prices as $P_t = (P_{1,t}, ..., P_{K,t})$, agent i budget constraint in $t \geq 1$ is thus[11]

$$C_t^i + \sum_{k=1}^{K} P_{k,t} h_{k,t}^i = h_{0,t}^i + \sum_{k=1}^{K} P_{k,t} h_{k,t-1}^i . \tag{1}$$

Asset prices are fixed in competitive markets. Having assumed that assets are in unitary supply, date-t asset-k market clearing condition reads

$$\sum_{i=1}^{I} h_{k,t}^i = 1 . \tag{2}$$

The existence and uniqueness of positive clearing prices depends on agents' demands. We postpone to Section 3 the proof that under appropriate assumptions there exists a unique vector of arbitrage free prices such that (2) holds.

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[10]In period t agent i net demand for asset k is thus $h_{k,t}^i - h_{k,t-1}^i$.

[11]The budget constraint in $t = 0$ is similar but dividends and assets' holdings on the right hand side of (1) come from the initial endowment of, respectively, apples and trees.

A central quantity to our analysis is the vector of agents' wealth. We define agent i wealth in t as her pre-consumption net worth

$$W_t^i = \sum_{k=1}^{K} P_{k,t} h_{k,t-1}^i + h_{0,t}^i, \quad \forall\, i \in I \,. \tag{3}$$

$\boldsymbol{W}_t = (W_t^1, \ldots, W_t^I)$ denotes the vector of agents wealth. Eqs. (1-2) can be re-written in terms of W_t and W_{t-1}. For this purpose, it is convenient to express each agent $i \in I$ consumption and portfolio decision in t as a function of her wealth W_t^i. We denote with δ_t^i the fraction of wealth she saves, so that $1 - \delta_t^i$ is the fraction of wealth she consumes, while $x_{k,t}^i$ is the fraction of the saved wealth which is used to purchase asset k. We obtain

$$C_t^i = (1 - \delta_t^i) W_t^i, \quad h_t^i = \frac{\delta_t^i x_{k,t}^i W_t^i}{P_{k,t}} \,. \tag{4}$$

The vector $x_t^i = (x_{1,t}^i, ..., x_{K,t}^i)$ is agent i portfolio rule and the vector $\alpha_t^i = \delta_t^i x_t^i$ is agent i investment rule. Given the budget constraint (1), $\sum_{k=1}^{K} x_{k,t}^i = 1 \ \forall\, i \in I$ and $\forall\, t \in \mathbb{N}_0$. Moreover, δ_t^i must be in $(0,1)$ to guarantee that consumption is positive in every period. These conditions are naturally satisfied given the choice of investment rule that we explicit in Section 2.2.

Using investment rules and agents wealth, budget constraints (1) and market clearing conditions (2) can be re-written for all agents and for all assets as

$$W_t^i = \sum_{k=1}^{K} (P_{k,t} + D_{k,t}) \frac{\delta_{t-1}^i x_{k,t-1}^i W_{t-1}^i}{P_{k,t-1}}, \quad \forall\, i \in I \,, \tag{5}$$

$$P_{k,t} = \sum_{i=1}^{I} \delta_t^i x_{k,t}^i W_t^i, \quad \forall\, k \in K \,. \tag{6}$$

Since assets are long-lived, the dynamics of agents' wealth and assets' prices is coupled. Before solving (5-6), we further characterize assets dividends and agents demands.

2.1 Assets

Together with **D1** we assume that each asset relative dividend process, $D_{k,t}/E_t$, does not depend on partial histories.[12] In other words there exists a $K \times S$ dividend matrix $D = [d_{k,s}]$ such that

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[12]Given our modeling assumption this is also a restriction on initial endowments.

D2 $D_{k,t}((\sigma_{t-1}, s_t)) = d_{k,s_t} E_t((\sigma_{t-1}, s_t)),\ \forall\, k \in K$ and $\forall\, t \in \mathbb{N}_0$.

The vectors d_s and d_k denote, respectively, the s-th column and the k-th row of D. The latter can also be viewed as a random variable on S.[13] By **D1**, $\sum_{k=1}^K d_{k,s} = 1$ for every $s \in S$. We also assume that dividends are non-negative and that every asset pays a positive dividend in at least some states

D3 $d_{k,s} \geq 0\ \forall\, s \in S$ and $\mathrm{E}^\pi[d_k] > 0,\ \forall\, k \in K$;

then, we rule out the existence of redundant assets

D4 $\mathrm{Rank}(D) = K \leq S$.

As we shall show, the dividend matrix D, rather than the aggregate process $\{E_t\}$, is central to the analysis of agents' relative wealth dynamics. Some examples of dividend matrices follow.

Diagonal Dividends   Assume that there are as many assets as states, $K = S$, and that the dividend of asset k in t is the entire aggregate endowment if and only if state $s_t = k$ is realized. Using our notation, and using $\delta_{i,j}$ for Kronecker's delta, asset k traded in t' pays the dividend

$$D_{k,t} = \delta_{k,s_t} E_t \quad \text{for all} \quad t > t'.$$

The dividend matrix D is just the $S \times S$ identity matrix, $D = I_s$ and **D2** $-$ **D4** are satisfied. Asset k traded in t' is a bet on the occurrence of state $s_t = k$ for all $t > t'$. By construction assets dividends are anti-correlated.

Geometric Random Walk   Here we construct the matrix D that replicates the simplest canonical model of financial markets. Assume that $S = 2$ and that the aggregate endowment follows a geometric random walk:

$$E_t = \begin{cases} g_u E_{t-1} & \text{if } s_t = 1 \\ g_d E_{t-1} & \text{if } s_t = 2 \end{cases},$$

with $g_u > g_d$. Two assets in unitary supply are available. The first, $k = 1$, is risky and when purchased in t' has dividends in all $t > t'$ equal to

$$D_{1,t} = \begin{cases} (g_u - g_d) E_{t-1} & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 2 \end{cases}.$$

The second asset is risk-free and has dividend in all $t > t'$ equal to $g_d E_{t-1}$ independently of the state s_t, like a perpetual bond with a time-varying coupon. Since the

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[13]Thus, under **D2**, $\mathrm{E}^\mathrm{P}[D_{k,t}|\Im_{t-1}] = \mathrm{E}^\pi[d_k]\,\mathrm{E}^\mathrm{P}[E_t|\Im_{t-1}]$ for every $k \in K$ and $t \in \mathbb{N}_0$.

first asset is equivalent to a long position in the aggregate endowment and short position in the second asset, the market is equivalent to one with a risky asset in unitary supply that pays the aggregate endowment as dividends and a risk-free asset in zero supply. The dividend matrix D is found by imposing $\mathbf{D1} - \mathbf{D2}$. Defining $r = g_d/g_u \in (0,1)$ one finds:

$$D = \begin{bmatrix} 1-r & 0 \\ r & 1 \end{bmatrix}.$$

It can be easily checked that also $\mathbf{D3} - \mathbf{D4}$ are satisfied.

Incomplete Markets   In both previous examples, market completeness relies on the full payoff matrix given by the sum of future dividends and prices. Thus, even if D is non-singular, the market might still be incomplete. However, when there are fewer assets than states, $K < S$, we know for sure that asset markets are incomplete. A strength of our approach is that we are able to analyze the long-run outcomes of the economy also for these incomplete markets.

Consider for example an economy as the one just described in the previous paragraph but with $S = 3$ and three possible aggregate endowment growth rates: $g_u \geq g_m > g_d$. Only two assets are traded. As in the previous example the first contract is a long position in the aggregate endowment and a short position in the risk-free asset paying the dividend $g_d E_{t'-1}$ for all $t' > t$. The dividend matrix D is now

$$D = \begin{bmatrix} 1-r_u & 1-r_m & 0 \\ r_u & r_m & 1 \end{bmatrix},$$

where $r_u = g_d/g_u \leq r_m = g_d/g_m$ (also in this case also $\mathbf{D3} - \mathbf{D4}$ are satisfied). Assume now that the first contract is replaced by two contracts that can disentangle the position in the first and second state of the economy. Simple computations show that the dividend matrix is now complete and given by

$$D = \begin{bmatrix} 1-r_u & 0 & 0 \\ 0 & 1-r_m & 0 \\ r_u & r_m & 1 \end{bmatrix}.$$

2.2   Investment Rules

Although one could investigate the market dynamics with general investment rules (x_t^i, δ_t^i), throughout this work we concentrate on a special rule, the so called generalized Kelly rule, by assuming the following[14]

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[14]In so doing, we depart from the standard approach that derives consumption and portfolio decision from the maximization of an objective function subject to beliefs about the distribution of future assets payoffs (both dividend and prices)

R1 Each agent $i \in I$ has discount factor $\delta^i \in (0,1)$, subjective i.i.d. beliefs $\pi^i \in \Delta^S$, and for all $t \in \mathbb{N}_0$ uses an investment rule $(x_t^i; \delta_t^i) = (x^i; \delta^i)$ with $x_k^i = \mathrm{E}^{\pi^i}[d_k]$ for all k.

Moreover, we further assume that each agent believes that all states are possible[15]

R2 $\pi^i \in \Delta_+^S \ \forall\, i \in I$.

By choosing generalized Kelly rules, **R1**, we exclude that rules might depend on market prices or on agents' wealth. Moreover, since beliefs are fixed, rules do not depend on the history of assets' dividend and price processes.[16] Given **R2**, each agent invests at least a positive amount of wealth in all assets. Note that rules allow some form of short selling as long as the aggregate position in the existing assets is positive (see the examples in the previous section). It is particularly important to realize that, given restrictions **R1** − **R2**, the set of consumption allocations that agent i can purchase by trading assets depends critically on D. In particular, given two different dividend matrices D and D', and two sequence of prices $\{P\}$ and $\{P'\}$ such that law of one price holds, there might not exist a pair of portfolio rules x and x' satisfying **R1** − **R2** such that the stream of dividends is the same with x under D and P and with x' under D' and P'. For this reason, the actual choice of D is relevant for the long-run dynamics.

Evstigneev et al. (2008) show that the generalized Kelly rule obtained under correct beliefs gains all the aggregate endowment in the long run when it trades with other generalized Kelly rules, provided all agents have the same discount factor. In particular when $D = I_S$, the rule suggests to bet on the realization of state s proportionally to its underlying probability π_s, see also Kelly (1956); Evstigneev et al. (2002, 2009).

The generalized Kelly portfolio of agent i in **R1** coincides with the portfolio used on an equilibrium path by a representative agent that maximizes a geometrically discounted log-utility with the same discount factor δ^i and beliefs π^i. As a result the generalized Kelly rule of each agent is also optimal in an heterogeneous agent economy in the limit of that agent having all the aggregate endowment.

3 Market Dynamics

In this section, we show that when agents use generalized Kelly rules inter-temporal budget constraints (5) and market clearing conditions (6) can be solved to give

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[15]The same condition is assumed in the market selection general equilibrium literature (see e.g. Axiom 3 in Blume and Easley, 2006) to guarantee existence of a competitive equilibrium.

[16]Our analysis is also informative of the long-run behavior of a market where beliefs are not i.i.d. but are adapted to the information filtration of dividends and prices, provided that beliefs converge to some constant level as more and more information is gathered.

positive and unique market clearing prices P_t and, as a result, a well defined dynamics for agents wealth W_t. While working toward the explicit solution of (5-6) we shall derive an explicit formulation for the payoff matrix, the sum of dividends and next period prices. We shall use the formula to show that equilibrium prices and payoffs do not allow arbitrages.

Without loss of generality, we assume that each agent $i \in I$ starts with some given positive wealth W_0^i.[17]

3.1 Representative agent

We start with the case where agent i posses all the aggregate endowment in $t = 0$, so that $W_0^j = 0$ for all $j \neq i$. Straightforward computations lead to

$$W_t^i = \frac{E_t}{1 - \delta^i}, \quad W_t^j = 0, \, j \neq i \quad \forall t \in \mathbb{N}_0, \tag{7}$$

$$P_{k,t} = \frac{\delta^i}{1 - \delta^i} \, \mathrm{E}^{\pi^i}[d_k] E_t. \tag{8}$$

Asset k is priced as in a log-economy where the representative agent has beliefs π^i and discount factor δ^i.

3.2 Heterogeneous agents

Pricing is more interesting when agents have heterogeneous beliefs. Assume that there exist at least two agents i and j with $W_0^i > 0$ and $W_0^j > 0$ and $\alpha^i \neq \alpha^j$. By substituting (5) in (6) we get

$$\sum_{h=1}^{K} \left(\delta_{k,h} - \sum_{i=1}^{I} \frac{\alpha_k^i \alpha_h^i W_{t-1}^i}{P_{h,t-1}} \right) P_{h,t} = \sum_{h=1}^{K} d_{h,s_t} E_t \sum_{i=1}^{I} \frac{\alpha_k^i \alpha_h^i W_{t-1}^i}{P_{h,t-1}}. \tag{9}$$

The above expression can be conveniently written in matrix form. Consider the vector of price-rescaled investment fractions

$$\beta^i(\boldsymbol{W}; \alpha) = \left(\alpha_1^i / \sum_{j=1}^{I} W^j \alpha_1^j, \dots, \alpha_K^i / \sum_{i=j}^{I} W^j \alpha_K^j \right)$$

and define the positive matrix

$$A(\boldsymbol{W}; \alpha) = \sum_{i=1}^{I} W^i \, \alpha^i \otimes \beta^i(\boldsymbol{W}, \alpha).$$

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[17] This is equivalent to assume that agents start with an initial allocation of assets (trees) and consumption goods (apples).

Then (9) becomes

$$(I_K - A(\boldsymbol{W}_{t-1}; \alpha)) P_t = A(\boldsymbol{W}_{t-1}; \alpha) \, d_{s_t} E_t \tag{10}$$

and one has

Lemma 3.1. *Under the assumption that rules satisfy* **R1 − R3**, *the matrix* $I_K - A(\boldsymbol{W}, \alpha)$ *is invertible for all* $\boldsymbol{W} \in \mathbb{R}_+^I$.

From the previous Lemma and from (10) it follows that market clearing prices are uniquely defined for every $t \in \mathbb{N}_0$

$$P_t(\sigma_{t-1}, s_t) = (I_K - A(\boldsymbol{W}_{t-1}; \alpha))^{-1} A(\boldsymbol{W}_{t-1}; \alpha) \, d_{s_t} E_t(\sigma_{t-1}, s_t) =$$

$$\sum_{n=1}^{\infty} A^n(\boldsymbol{W}_{t-1}(\sigma_{t-1}); \alpha) d_{s_t} E_t(\sigma_{t-1}, s_t). \tag{11}$$

Given $\boldsymbol{W}_{t-1}(\sigma_{t-1})$ and investment rules α for all i, there is a period t price vector P_t for every realization of the dividend process s_t. This is because all assets (with positive dividend in s_t) contribute to the next period wealth distribution, which in turns determines prices. As a result, given a dividend matrix D and rules α, for every $\boldsymbol{W}$ there exists a matrix $P(\boldsymbol{W}; \alpha, D)$, with the same dimension of D, such that $P_{k,t}(\sigma_{t-1}, s_t) = P_{k,s_t}(\boldsymbol{W}_{t-1}; \alpha, DE_t)$. Since, by **D3** and **R2 − R3**, $A(\boldsymbol{W}; \alpha)$ is stricly positive and D is positive, the equation above shows that $P(\boldsymbol{W}; \alpha, D)$ is strictly positive.[18] When the wealth distribution is degenerate, in that only agent j has positive wealth, it is $A = \delta^j x^j \otimes \boldsymbol{1}$ and (8) is recovered.

Long-lived assets prices and dividends D determine the payoff matrix

$$R(\boldsymbol{W}; \alpha, D) = P(\boldsymbol{W}; \alpha, D) + D =$$

$$(I_K - A(\boldsymbol{W}; \alpha))^{-1} D = \sum_{n=0}^{\infty} A^n(\boldsymbol{W}; \alpha) D. \tag{12}$$

Since the payoff matrix depends, through prices, also on the wealth distribution $\boldsymbol{W}$, it keeps changing as the wealth distribution evolves. Its rows $R_k(\boldsymbol{W}; \alpha, D)$, with $k \in K$, are strictly positive random variables on S given by the sum of the two random variables $P_k(\boldsymbol{W}; \alpha, D)$ and d_k. By substituting (12) in (5) one obtains the explicit evolution of the wealth distribution. By construction it is adapted to the information filtration. We can summarize the result of this section in the following proposition[19].

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[18]However, it could still be the case that asset prices admits arbitrage. We rule out that arbitrage exists in equilibrium in Proposition 3.1.

[19]It is straightforward to see that the same proposition holds even when beliefs π_t^i are adapted to the information filtration generated by s_τ and $P_{\tau-1}$ for all $\tau \leq t$. Evstigneev et al. (2006) provides a different proof of the same result. In particular they do not explicitly characterize the payoff matrix R.

Proposition 3.1. *Consider an exchange economy where I agents using rules obeying* **R1** − **R2** *are trading K assets satisfying* **D1** − **D3**. *If $\boldsymbol{W}_0 \in \mathbb{R}_{++}^I$ then for all $t \geq 1$ the process $\{\boldsymbol{W}_t\}$ is $\mathbb{R}_{++}^I$, it is adapted to $\{\Im_t\}$, and evolves according to*

$$W_t^i(\sigma_{t-1}, s_t) = W_{t-1}^i(\sigma_{t-1}) \sum_{k=1}^{K} \beta_k^i(\boldsymbol{W}_{t-1}(\sigma_{t-1}); \alpha) \, R_{k,s_t}(\boldsymbol{W}_{t-1}(\sigma_{t-1}); \alpha, DE_t), \quad \forall \, i \in I \,.$$

Moreover the sequence of wealth distributions $\{\boldsymbol{W}_t\}$ is such that for all $t \geq 1$ market equilibrium prices $P_t = P_{s_t}(\boldsymbol{W}_{t-1}; \alpha, DE_t)$ and payoffs $R_{t+1} = R(\boldsymbol{W}_t; \alpha, DE_{t+1})$ do not admit arbitrages.

The proposition proves that the coupled price-wealth dynamics is not only well defined but also prevents arbitage opportunities to arise in equilibrium, so that state prices are always positive. The result is non-trivial beacuse, with long-lived assets, the dynamics of the wealth distribution determines both date t prices and date $t + 1$ payoffs. Under the standard utility maximization approach arbitrage never occurs in equilibrium because asset holdings are unconstrained. In our model, however, asset holdings are constrained and arbitrage might thus occur. A sufficient condition to avoid arbitarges turns out to be that the vector of portfolio rules is in the interior of the cone generated by the S column of the matrix D (see the proof of Prop. 3.1 for more details). A condition that, given **R3**, is naturally satisfied by generalized Kelly rules.

3.3 Relative wealth dynamics

If agents have a different saving rate, the agent who saves more is advantaged in terms of long-run wealth. If, for example, there are only two agents and they use the same portfolio rule, $x^1 = x^2$, but $\delta^1 > \delta^2$, then agent 1 relative wealth grows geometrically at the rate δ^1/δ^2. When agents have different portfolios there is a trade-off between having a higher saving rate and a "better" portfolio. Although the trade-off is certainly interesting, here we concentrate on the heterogeneity of portfolio rules and assume homogeneous saving rates:

R3 $\delta^i = \delta \,, \; \forall \, i \in I \,.$

Under this assumption the relative wealth dynamics does not depend on the aggregate endowment process. Introducing normalized wealth and price

$$w_t^i = \frac{1-\delta}{E_t} \, W_t^i \quad \text{and} \quad p_{k,t} = \frac{1-\delta}{\delta \, E_t} \, P_{k,t} \,, \tag{13}$$

such that at any t its is $\sum_{i=1}^{I} w_t^i = 1$ and $\sum_{k=1}^{K} p_{k,t} = 1$, one can still apply Lemma 3.1 and Proposition 3.1 to normalized variables, provided $\boldsymbol{w}_0 \in \Delta_+^I$. At

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this purpose the payoff matrix R defined in (12) should be replaced with

$$r(\boldsymbol{w}; x, \delta, D) = [r_{k,s}] = (1 - \delta)(I_K - \delta A(\boldsymbol{w}; x))^{-1} D.$$

In the definition of matrix A the normalized wealth replaces the original wealth, and portfolio shares of agent i x^i replaces her investment shares α^i. When δ is close to zero normalized prices get very small and the (normalized) payoff matrix r is close to the dividend matrix d. Conversely when δ is close to one, normalized prices are much larger than dividends. When $\delta \to 1$, the payoff matrix becomes singular. An advantage of working with the normalized variables is that, since both states of the world and agents' beliefs are i.i.d., the relative wealth dynamics is a Markov process.

Corollary 3.1. *Under the assumptions of Proposition 3.1, the normalized wealth $\boldsymbol{w}_t$ follows a Markov Process on Δ_+^I such that for every $t \geq 1$ with probability π_s the relative wealth vector $\boldsymbol{w}_{t-1}$ evolves into*

$$w_t^i = w_{t-1}^i \sum_{k=1}^K \beta_k^i(\boldsymbol{w}_{t-1}; x) \, r_{k,s}(\boldsymbol{w}_{t-1}; x, \delta, D) \quad \forall \, i \in I. \tag{14}$$

4 Market Selection and Long-run Heterogeneity

In the rest of the paper, we characterize the long-run behavior of the relative wealth dynamics (14). In particular, we focus our attention on the long-run performance of groups of agents. For any proper subset $J \subset I$, we denote the sum of period t wealth of agents in J as w_t^J, so that $1 - w_t^J$ is the sum period t wealth of agents in $I \setminus J$. The aggregate portfolio rule of the two groups of agents at time t are, respectively,

$$x_k^J(\boldsymbol{w}_t; x) = \sum_{j \in J} x_k^j \frac{w_t^j}{w_t^J} \quad \text{and} \quad x_k^{-J}(\boldsymbol{w}_t; x) = \sum_{j \in I \setminus J} x_k^j \frac{w_t^j}{1 - w_t^J}. \tag{15}$$

With usual notation we set $\beta^J = x^J/p = (x_1^J/p_1, \ldots, x_K^J/p_K)$ and $\beta^{-J} = x^{-J}/p = (x_1^{-J}/p_1, \ldots, x_K^{-J}/p_K)$. In this section, we provide sufficient conditions for the survival or dominance of a generic group J of agents. The next definition makes precise what we mean by dominance, survival, and vanishing.

Definition 4.1. We say that group J dominates on a realization σ if

$$\lim_{t \to \infty} w_t^J(\sigma) = 1. \tag{16}$$

Group J survives on a realization σ if

$$\limsup_{t \to \infty} w_t^J(\sigma) > 0 \,. \tag{17}$$

If a group does not survive on σ, we say that it vanishes on that realization. We say that group J dominates or survives if (16) or (17) hold P-a.s.. Group J vanishes if it survives on a set of realizations with P-measure zero.

If a group composed by only one agent dominates, heterogeneity is only a transient property and the economy converges with probability one to a representative agent limit. If instead more than one agent survives, then the economy exhibits long-run heterogeneity.

Definition 4.2. An I-agent asset market economy exhibits long-run heterogeneity if there exists a proper subset of traders $J \subset I$ such that both the group J and the group $I \setminus J$ survive P-a.s..

Dominance of a single agent is the unique possible long-run outcome when market are complete, when all agents maximize an expected log-utility given their beliefs and discount factors, see e.g. Sandroni (2000); Blume and Easley (2006); Yan (2008).[20] Conversely, as we shall see below, long-run survival of more than one agent is a generic outcome of our model with 'quasi-optimal' log-rules.

To derive convergence results we shall need to assume that agents J aggregate rule x^J cannot be replicated by a combination of other agents rules. We assume the following.

R4 Agents beliefs $\{\pi^i, i \in I\}$ are such that there exists an hyperplane in R^K which separates the generalized Kelly rules of agents in J from the generalized Kelly rules of agents in $I \setminus J$.

Since the aggregate rules x^J and x^{-J} belong to the convex cone generated by the strategies of agents in J and $I \setminus J$ respectively, condition **R4** is sufficient to guarantee that they can never be equal, irrespective of the wealth distribution. Notice that, as long as individual rules are all different and $K \geq I$, condition **R4** is satisfied for any group J. **R4** is also satisfied when the D has full rank and $\pi^i \neq \pi^j$ for all $i, j \in I$.

Assumption **R4** combined with the absence of arbitrages is sufficient to prove that, for all t, the ratio w_t^J / w_t^{-J} increases or decreases with positive probability. Equivalently, group J relative wealth might grow faster or slower than group $I \setminus J$ relative wealth.

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[20]Long-run heterogeneity is possible in log economies but, first, it is non-generic in that it occurs when agents' beliefs have the same relative entropy with respect to the truth, and, second, it is not robust to a perturbation of beliefs, see Blume and Easley (2009).

Lemma 4.1. *Under* **R4**, *if market equilibrium prices p_t and assets payoffs r_{t+1} do not admit arbitrages, then for all groups J and all $t \in \mathbb{N}_0$ there exist $\epsilon > 0$, $a_{J,t} > \epsilon$, and $b_{J,t} > \epsilon$ such that*

$$\text{Prob}\left\{\frac{w_{t+1}^J}{w_t^J} > \frac{w_{t+1}^{-J}}{w_t^{-J}} \,\middle|\, \Im_t\right\} = a_{J,t} \quad and \quad \text{Prob}\left\{\frac{w_{t+1}^J}{w_t^J} < \frac{w_{t+1}^{-J}}{w_t^{-J}} \,\middle|\, \Im_t\right\} = b_{J,t}. \quad (18)$$

In order to characterize the relative performance of group J, we use the difference between the conditional expected log-growth rate of agents in J and the conditional expected log-growth rate of the other agents. Corollary 3.1 implies that this quantity depends on the history of states prior to period t only through the wealth distribution $\boldsymbol{w}$ at period t. Formally

$$\mu_t^J(\boldsymbol{w}) = \mathrm{E}^{\,\mathrm{P}}\left[\log\frac{w_{t+1}^J}{w_t^J} - \log\frac{1 - w_{t+1}^J}{1 - w_t^J}\,\middle|\,\Im_t \text{ s.t. } \boldsymbol{w}_t = \boldsymbol{w}\right]$$

$$(19)$$

$$= \mathrm{E}^{\,\pi}\left[\log\frac{\sum_{k=1}^K \beta_k^J(\boldsymbol{w}_t; x) r_k(\boldsymbol{w}_t, ; x, \delta, D)}{\sum_{k=1}^K \beta_k^{-J}(\boldsymbol{w}_t; x) r_k(\boldsymbol{w}_t; x, \delta, D)}\right].$$

The sign of $\mu_t^J(\boldsymbol{w})$ reveals if the aggregate wealth of the agents in J grows or shrinks, in expectations. It turns out that sufficient conditions for survival or dominance of group J can be derived studying the sign of $\mu_t^J(\boldsymbol{w})$ when its relative wealth w^J is very large or very small.

For a proper subset J and for all $v \in [0, 1]$ consider the quantity

$$\overline{\mu}^J(v) = \max\left\{\mu^J(\boldsymbol{w}) \;\mid\; \boldsymbol{w} \in \Delta^K, \, w^J = v\right\},$$

and

$$\underline{\mu}^J(v) = \min\left\{\mu^J(\boldsymbol{w}) \;\mid\; \boldsymbol{w} \in \Delta^K, \, w^J = v\right\}.$$

The definition is meaningful because the function μ^J is continuous in $\boldsymbol{w}$ and the extrema are computed on compact sets. Since these sets are continuous in v (both upper and lower hemicontinuous) the quantities $\underline{\mu}^J$ and $\overline{\mu}^J$ are continuous function of their argument.

The next Proposition exploits the Martingale Converge Theorem (see the proposition's proof for details) to characterize long-run survival and dominance.

Proposition 4.1. *Consider an exchange economy with I agents using rules obeying* **R1 − R3** *and trading K assets satisfying* **D1 − D4**:

   i) *If $\underline{\mu}^J(0) > 0$, then group J survives.*

   ii) *If $\overline{\mu}^J(1) < 0$, then group $I \setminus J$ survives.*

Moreover, if **R4** *is satisfied:*

iii) *If $\underline{\mu}^J(0) > 0$ and $\underline{\mu}^J(1) > 0$, then group J dominates;*

iv) *If $\overline{\mu}^J(0) < 0$ and $\overline{\mu}^J(1) < 0$, then group J vanishes.*

v) *If $\underline{\mu}^J(0) > 0$ and $\overline{\mu}^J(1) < 0$, then both groups survive and for both groups $G = J, I \setminus J$*

$$\text{Prob}\{\liminf_{t \to \infty} w_t^G = 0 \text{ and } \limsup_{t \to \infty} w_t^G = 1\} = 1$$

Under *i)* and *ii)*, both group J and $I \setminus J$ survive and the market exhibits log-run heterogeneity. If moreover the rules used by each group are always different, **R4**, then, due to Lemma 4.1, also *v)* holds. The relative wealth shares keep fluctuating in the interval $(0, 1)$ and assets' prices keep fluctuating in between the two groups evaluations. Proposition 4.1 is only a sufficient condition because by considering all possible wealth distributions w one is in principle also taking into consideration wealth distributions that cannot be realized with positive probability. It is however clear that if the conditions on the conditional drift in Proposition 4.1 are realized, they are *a forziori* true for (almost) all possible trajectories of the system. As we will show in the next Section, when we consider the case $I = 2$, Proposition 4.1 is sufficient to classify all possible asymptotic outcome of the economy.

It is important to note that in order for *iii)*, *iv)* and *v)* to be proved, we have also assumed that rules are 'separated', or **R4**. The requirement, together with the fact that there are no arbitrages when agents use generalized Kelly rules, ensures that Lemma 4.1 holds and each agent's relative wealth increases and decreases with positive probability. In cases *iii)* and *iv)*, this is important because even if the asymptotic drift conditions point to dominance of a trader, a limited arbitrage in favor of the trader with asymptotically "worst" portfolio rule could occur when $w \in (0, 1)$, thus preventing the trader with asymptotically "better" rules to dominate. In case *v)*, knowing that the relative wealth keep fluctuating implies that its effective domain is the interval $(0, 1)$ and that asset prices keep fluctuating too between the two groups evaluations (which by assumptions are separated).

4.1   2-agent economy

In a two-agent economy there is only one proper partition of I so that we can limit the study to the relative wealth dynamics of one agent, say agent 1. Given the wealth normalization, the conditional drift (19) can be written as function of $w_t^1 = w$ only, giving for $\boldsymbol{w} = (w, 1 - w)$

$$\mu_t^1(\boldsymbol{w}) = \mu(w) = E^\pi \left[\log \frac{\sum_{k=1}^K \beta_k^1(\boldsymbol{w}; x) r_{k,s}(\boldsymbol{w}; x, \delta, D)}{\sum_{k=1}^K \beta_k^2(\boldsymbol{w}; x) r_{k,s}(\boldsymbol{w}; x, \delta, D)} \right].$$

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As a result, $\mu(w) = \underline{\mu}^1(w) = \overline{\mu}^1(w)$ for all $w \in [0,1]$. A straightforward application of Proposition 4.1 lead to the following set of sufficient conditions for the long-run outcomes of a two-agent economy.

Corollary 4.1. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1 − R4** *and trading K assets satisfying* **D1 − D4***:*

i) *If $\mu(0) > 0$ and $\mu(1) > 0$, then agent 1 dominates and 2 vanishes;*

ii) *If $\mu(0) < 0$ and $\mu(1) < 0$, then agent 2 dominates and 1 vanishes;*

iii) *If $\mu(0) > 0$ and $\mu(1) < 0$, then both agents survive and for all assets $k \in K$*

$$\text{Prob} \left\{ \liminf_{t \to \infty} p_{k,t} = \min_{i=1,2} \{ \mathrm{E}^{\pi^i}[d_k] \} \text{ and } \limsup_{t \to \infty} p_{k,t} = \max_{i=1,2} \{ \mathrm{E}^{\pi^i}[d_k] \} \right\} = 1 \,.$$

Long-run heterogeneity occurs when both agents have a higher wealth growth rates at the returns determined by the other agent. As we shall discuss in Section 5 the result can be given in term of "effective" beliefs and log-optimal rules. Long-run heterogeneity occurs when both agents have more accurate "effective" beliefs at the returns determined by the other agent.

As in Proposition 4.1, long-run heterogeneity amounts to a relative wealth that keeps fluctuating when agents portfolio rules can be separated (otherwise, if **R4** does not hold agents have the same demand for assets and their relative wealth is constant). With only two agents the result has direct implications for the asset prices dynamics: prices keep fluctuating between the two agents evaluations. Moreover, contrary to Proposition 4.1, $\mu(0)$ and $\mu(1)$ can be computed easily, making the 2-agent economy particularly tractable and amenable to applications.

Another advantage of a two-agent market is that no other cases than those of Corollary 4.1 can occur as we shall show in the following.

Proposition 4.2. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1 − R4** *and trading K assets satisfying* **D1 − D4***, then $\mu(0) > \mu(1)$ so that, provided both $\mu(0)$ and $\mu(1)$ have a definite sign, one among i), ii), iii) of Corollary 4.1 occurs.*

The proposition states that, leaving out the non generic cases when asymptotic drifts are zero, only three cases are possible. If an agent has a favorable drift when she has most of the wealth, then she has a favorable drift also when she has little wealth. Conversely, if she faces an unfavorable drift when she has little wealth, the drift would be against her also if she possessed almost all the wealth. As a result $\mu(1) > 0$ ($\mu(0) < 0$) is sufficient to prove that agent 1 (2) dominates. Proposition 4.2 thus excludes the case in which both agents are better off, in

expectations, when they have most of the wealth.[21]. The third possibility is that $\mu(0) > 0$ and $\mu(1) < 0$, in this case both agent 1 and 2 survive, and none dominates. We concentrate on proving that such cases do always exist and are robust to perturbations of the beliefs in the next section.

The proposition has also an implication for the survival of an agent whose beliefs are such that the portfolio rule she uses is the 'closest' to the Generalized Kelly rules derived under correct beliefs, confirming the results of Bektur (2013). To make the point precise, denote Kullback Leibler distance, or relative entropy, of rules x^i with respect to the reference rule x^* with $x_k^* = \mathrm{E}^\pi[d_k]$ as

$$D(x^*||x^i) = \mathrm{E}^{x^*}\left[\log\left(\frac{x^*}{x^i}\right)\right].$$

and the difference of relative entropies as

$$\Delta_{x^*}(x^2||x^1) = D(x^*||x^2) - D(x^*||x^1).$$

From the proof of Proposition 4.2 it holds that, any given D,

$$\mu(0) > (1 - \delta)\Delta_{x^*}(x^2||x^1) > \mu(1) \tag{20}$$

Together with condition i) of Corollary 4.1, the former implies that if agents beliefs are such that rule x^1 is closer, as measured by the relative entropy, to the rule x^* than rule x^2, then agent 1 survives.

Whether agent 1 also dominates, or both agents survive, it depends on the sign of $\mu(1)$. As also shown by Evstigneev et al. (2008), we find that if agent 1 has correct beliefs, then he dominates against any other generalized Kelly trader. Summarizing, we have the following.

Corollary 4.2. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1** − **R4** *and trading K assets satisfying* **D1** − **D4**. *If agents beliefs are such that*
$$D(x^*||x^2) > D(x^*||x^1),$$
then agent 1 survives. If, moreover, agent 1 beliefs are correct so that $x^1 = x^*$, then agent 1 dominates.

In particular when all assets are anti-correlated, $D = I_S$, the above inequality involves directly beliefs and becomes

$$D(\pi||\pi^2) > D(\pi||\pi^1),$$

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[21]This is in contrast with what has been shown in Bottazzi and Dindo (2014) for a market of short-lived assets. The key difference is that Bottazzi and Dindo allow portfolio rules to depend also on prices.

In this case agent 1 survives when she has more accurate beliefs. With more general dividend matrices D and, possibly, incomplete markets it is the relative entropy of rules, rather than of beliefs that guarantees survival to the most "accurate" trader. Having correct beliefs is instead always sufficient for dominance.[22]

4.2   Long-run Heterogeneity

Having defined sufficient conditions for long-run heterogeneity we turn to generality and existence. First, we show that when long-run heterogeneity occurs it is also generic, in that perturbations of beliefs do not lead to dominance of any of the surviving agent. Second, we show that for any asset structure D there exist beliefs for which heterogeneity is indeed the long-run outcome. In both cases, we restrict our analysis to an economy with 2 agents and assume that both agents do not know the truth, $\pi^i \neq \pi$.

The next proposition states that if the sufficient conditions for persistent heterogeneity of Proposition 4.1 apply, then there exist perturbations of agents' beliefs such that heterogeneity is still the long-run outcome.

Proposition 4.3. *If an economy with 2 agents having beliefs $\bar{\pi}^1, \bar{\pi}^2$ and rules satisfying $\mathbf{R1} - \mathbf{R4}$ exhibits long-run heterogeneity, then there exist vectors $\epsilon^1, \epsilon^2 \in \mathbb{R}^S$ with components $\epsilon_s^i \in [-\varepsilon, \varepsilon]$, $\varepsilon > 0$, and $\sum_{s=1}^{S} \epsilon_s^i = 0$ for $i = 1, 2$, such that under beliefs $\bar{\pi}^1 + \epsilon^1$ and $\bar{\pi}^2 + \epsilon^2$ the economy still exhibits long-run heterogeneity.*

The idea behind the proof is that since sufficient conditions for long-run heterogeneity involve strict inequalities, and conditional drift are continuous functions of beliefs (via the rules), then due to the preservation of the sign there exists perturbation of beliefs, such that conditional drift have still the right sign.

Having shown that heterogeneity, when occurs, is generic we address a different issue. Given Generalized Kelly traders and a market for long-lived assets satisfying $\mathbf{D1} - \mathbf{D4}$, is it always possible to find some beliefs such that heterogeneity occurs?

At this purpose (20) together with condition iii) of Corollary 4.1 imply that if two agents have beliefs such that the corresponding rules have the same relative entropy with respect to x^*, then they both survive. Thus, in order to prove the existence of long-run heterogeneity for every admissible choice of the matrix D, we have find beliefs for agents 1 and 2 such that the corresponding Generalized Kelly rules have the same relative entropy. At this purpose, define $\bar{\Delta}$ as the open set $\{x' : \pi' \in \Delta_+^S\} \subset \Delta^K$ and call $\partial(\bar{\Delta})$ its frontier.

Proposition 4.4. *Given a asset dividend matrix D and beliefs of agent 1 $\pi^1 \neq \pi$ such that $D(x^* \| x^1) < \mathcal{K}$ with $\mathcal{K} = \min_{\pi'} \{D(x^* \| x')\} \, s.t. \, x' \in \partial(\bar{\Delta})\}$, there exists*

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[22]The dominance of the agent with correct beliefs holds also for I-agents economies, as shown by Evstigneev et al. (2008).

a non-empty set of beliefs $\Pi \subset \bar{\Delta}$ with $\pi^1 \in \Pi$ such that for all $\pi^2 \in \Pi$ the asset market economy with generalized Kelly traders having beliefs π^1 and π^2 exhibits long-run heterogeneity.

The fundamental ingredients for proving Proposition 4.4 are the properties of the relative entropy. Its continuity, strict convexity and the fact that it has a minimum equal to zero in x^* are sufficient to show the existence of Π. Indeed, to build Π it is enough to fix π^1 and take the set of beliefs such that the Generalized Kelly rules they generate have all the same relative entropy with respect to x^*.[23] Note that, thanks to Proposition 4.3, one can also expand such set including the neighborhood of all these beliefs.

Finally note that equality of beliefs relative entropy implies long-run heterogeneity also in market economies where agents are expected utility maximizers and assets markets were (dynamically) complete, see e.g. Blume and Easley (2009). There is however one important differences between the two models. Whereas, as we have shown in Proposition 4.3, heterogeneity is generic in our market, it is non-generic in the complete market case. Any small perturbation of an agent beliefs will break the tie of beliefs relative entropy and thus lead to dominance of the agent whose beliefs turn out to be "closer" to the truth.

5 Discussion and Examples

We begin this section by providing an intuition on the source of long-run heterogeneity based on the comparison between Generalized Kelly rules and log-optimal rules. Then we explore market selection outcomes in I-agent economies for specific choices of the dividend matrix D, see also Section 2.1.

5.1 Effective Beliefs

What is the intuition behind the occurrence of long-run heterogeneity? In a similar asset market economy, Sandroni (2000), Blume and Easley (2006), and Yan (2008) find that if agents demand are log-optimal and the asset market is complete, then the agent with the most accurate beliefs dominates. However, we find that when agents use generalized Kelly rules the accuracy of beliefs is not directly related to dominance. Provided D is diagonal, Corollary 4.2 proves only a weaker result, that is, accuracy of beliefs is sufficient for survival. Since a generalized Kelly

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[23] The fact that $\Pi \subset \bar{\Delta}$ depends on the technical condition $D(x^*\|x^1) < \mathcal{K}$. Otherwise the set Π could encompass rules that are not generated by any belief.

rule is log-optimal in the limit when the agent using it has most of the wealth, failure to dominate must be caused by the portfolio of the agent with less accurate beliefs being particularly "good" in this limit. Non accuracy of beliefs and non-log-optimality of the portfolio rule must compensate each other so that the agent with less accurate beliefs does not vanish and long-run heterogeneity occurs.

In order to establish how, and when, the non accuracy of beliefs and non log-optimality of rules compensate each other, we use the concept of "effective beliefs". Given asset prices in t and payoffs in $t+1$, we define an agent i effective beliefs in t, $\bar{\pi}_t^i$, as the beliefs such that the generalized Kelly rule x^i derived from π^i is log-optimal in t. More specifically, to compute "effective beliefs" we proceed as follows. Given agents beliefs, discount factors, and a dividend matrix D, for every value of the relative wealth distribution $\boldsymbol{w}_t$ there correspond both a vector of prices p_t and a payoff matrix r_{t+1} (see Section 3.3 for details). Thus for every $\boldsymbol{w}_t$ one can find the "effective beliefs" of agent i as those beliefs $\bar{\pi}_t^i$ such that the portfolio rule x^i is log-optimal given prices p_t and payoffs r_{t+1}.[24] As a result, for each agent i, we derive a function $\bar{\pi}^i : \Delta^I \to \Delta_+^S$ such that $\bar{\pi}_t^i = \bar{\pi}^i(\boldsymbol{w}_t; \pi, \delta, D)$. Note that the function depends on all agents rules (and thus beliefs), on the discount factor δ, and on the dividend matrix D. Since a generalized Kelly rule is log-optimal in the limit of the agent using it having all the wealth, $\bar{\pi}^i((0, \ldots, w^i = 1, \ldots, 0); \pi, \delta, D) = \pi^i$ for all $i \in I$.

The construct of effective beliefs enable us to view the economy with generalized Kelly traders as an economy with log-optimal traders using effective beliefs. The general equilibrium literature tells us that, provided the asset market is complete, an agent survives only when her beliefs are, on average, accurate. As a result whenever we find that long-run heterogeneity is the long-run outcome, agents effective beliefs must be, on average, equally accurate. Moreover, along the lines of Proposition 4.1 and Corollary 4.1, one can prove that the sufficient conditions that characterize long-run outcomes can be given in terms of 'asymptotic' effective beliefs accuracy instead that in terms of 'asymptotic' growth rates μ. In fact, the following proposition shows that the relative accuracy of effective beliefs can be used to characterize the value of asymptotic drifts $\mu(0)$ and $\mu(1)$.[25]

Proposition 5.1. *Consider an exchange economy with $I = 2$ agents using rules obeying* **R1** − **R4** *and trading a (dynamically) complete set of assets with dividend*

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[24]Note that since prices in t and payoffs in $t+1$ rely on agents using fixed rules x both in t and in $t+1$, the equivalence of the market dynamics under generalized Kelly traders and under log-optimal traders with effective beliefs can only be established for every t (and thus in an infinite horizon economy).

[25]The proposition generalizes to I-agent economies by taking all the possible combinations of the two groups effective beliefs.

matrix D satisfying **D1** − **D4***, then*

$$\mu(0) = \Delta_\pi(\pi^2||\bar{\pi}^1((0,1);\delta,D)) \quad and \quad \mu(1) = \Delta_\pi(\bar{\pi}^2((1,0);\delta,D))||\pi^1) \,.$$

In a two-agent economy, long-run heterogeneity occurs when, for both $i = 1, 2$, agent i effective believes are more accurate than agent $j \neq i$ (effective) believes when agent j sets prices and payoffs.

Fig. 1 shows effective beliefs in a two-agent economy with complete markets, two states, and diagonal dividend matrix D. Effective beliefs depend on the value of w^1. Since a generalized Kelly rule is log-optimal in the limit of the agent using it having all the wealth, effective beliefs and beliefs coincide when an agent has most of the wealth. However, beliefs and effective beliefs differ when both agents have positive wealth, in that assets' payoffs are determined by both agents. In particular, given two agents, the effective beliefs of each agent are a combination of his beliefs with the beliefs of the other agent. The larger the wealth share of one agent, the larger her impact on equilibrium returns, the larger the weight of her beliefs in determining both agents effective beliefs. Discount rates are also important because, by setting the interest rate and thus the level of asset prices, they determine the relative importance of dividends and prices in the payoff matrix.
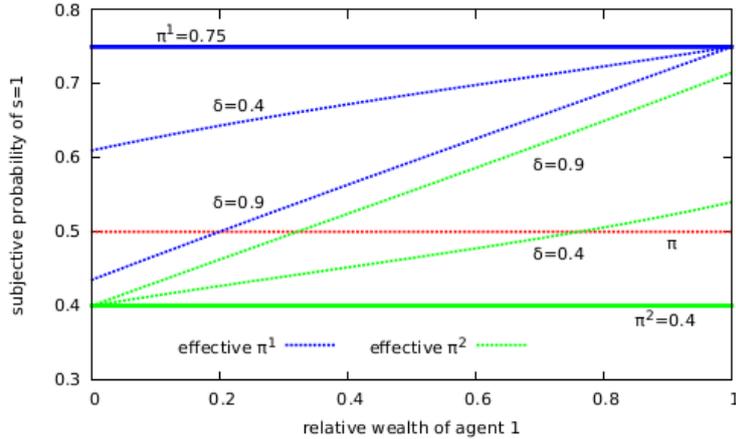

Figure 1: Effective beliefs of two agents with different values of discount rate δ in a complete markets of two assets with $D = I_2$. Since the truth is $\pi = (1/2, 1/2)$, the relative entropy is a symmetric function so that the euclidean vertical distance between a belief $\bar{\pi}^i$ and $\pi = (1/2, 1/2)$ can be used directly to appraise $D(\pi||\bar{\pi}^i)$.

In the example of Fig. 1, agents beliefs are at the opposite side of the truth and agent 2 has more accurate beliefs than agent 1. Since D is diagonal, by Corollary 4.2 agent 2 never vanishes. Whether she dominates or also agent 1 survives it depends on the discount rate, that is on how much non-accuracy of beliefs and

non-optimality of the generalized Kelly rule influence each others. Simple calculations (and our numerical exploration of Section 5.2) show that agent 2 dominates when $\delta = 0.4$ whereas both agents survive when $\delta = 0.9$. Effective beliefs confirm this outcome. When $\delta = 0.4$ agent 2 has better effective beliefs both when he has most of the wealth and when she has none, and thus dominates. In fact, she has most accurate effective beliefs for all possible wealth distributions. When $\delta = 0.9$, however, each agent has most accurate effective beliefs when the other agent sets assets returns, so that both agents survive.

The graphical representation clarifies also why, for all $\delta \in (0, 1)$, long-run heterogeneity is the long run outcome when rules have the same relative entropy, see Proposition 4.3. Assume that the beliefs of agent 1 are $\pi^2 = 0.6$ instead of $\pi^2 = 0.75$, so that $\Delta_\pi(\pi^2 || \pi^1) = 0$. Effective beliefs, by lying between the two agents beliefs, are such that the condition for long-run heterogeneity is satisfied for all $\delta > 0$.

5.2   2-Generalized Kelly agents economies

In what follows we numerically explore the occurrence of long-run heterogeneity. We start with 2-agent economies.

Diagonal assets   Consider an economy with two states of the world where two Generalized Kelly agents trade two assets. Assume $D = I_2$ and fix $\pi = (0.5, 0.5)$. We use the sufficient conditions of Corollary 4.1 to characterize long-run outcomes for different values of the economy parameters.

In the left plot of Figure 2, $\delta = 0.8$ and all possible combinations of agents' beliefs are considered. In the right plot we instead set $\pi^2 = (0.6, 0.4)$, and vary the beliefs of agent 1 and the value of δ.
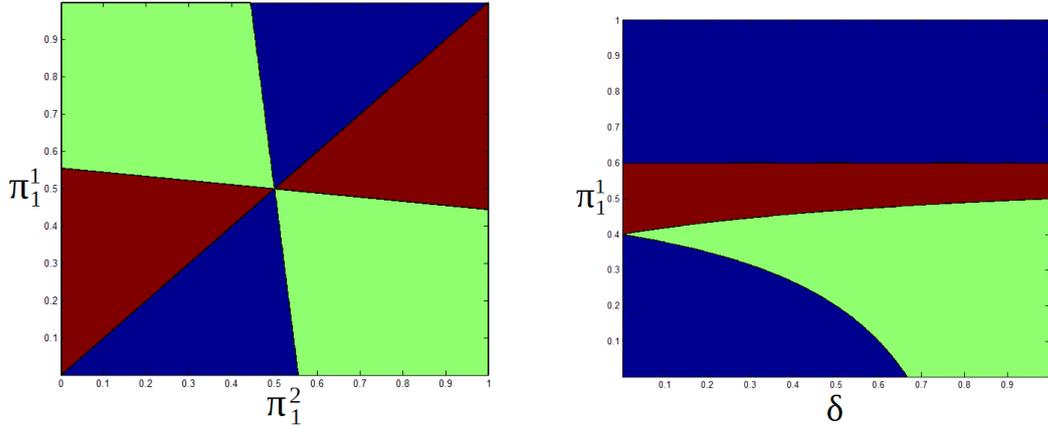
Figure 2: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity.

Consistently with the derivation of effective beliefs, long-run heterogeneity occurs only for beliefs that are anti-correlated, that is, when one agent believes that asset 1 pays with probability greater than $1/2$ while the other believes the opposite. The figure also confirms the resul of Corollary 4.2 for diagonal dividends matrices: the agent with beliefs farthest from the truth never dominates.

In the plot on the right one can notice how the area of long-run heterogeneity shrinks for low values of δ until it disappears when $\delta = 0$. In that limit "effective beliefs" coincide with beliefs for all values of the wealth distribution so that log-run heterogeneity is only a non-generic phenomenon that takes place when beliefs have the same relative entropy with respect to the truth.

To give an idea of how a particular trajectory of the stochastic system looks like, we keep $\delta = 0.8$ and $\pi = (0.5, 0.5)$ while we set $\pi^1 = (0.45, 0.55)$ and $\pi^2 = (0.6, 0.4)$. In Figure 3 we plot the evolution of wealth shares for $T = 1000$ periods when $w_0 = 0.5$. When the wealth share of an agent approaches low values then it is bounced back and, eventually, wealth shares are re-balanced. Asset prices (and thus state prices) follow a similar pattern where the bounds are not zero and one but each agent evaluation of the asset stream of dividends given by (8).
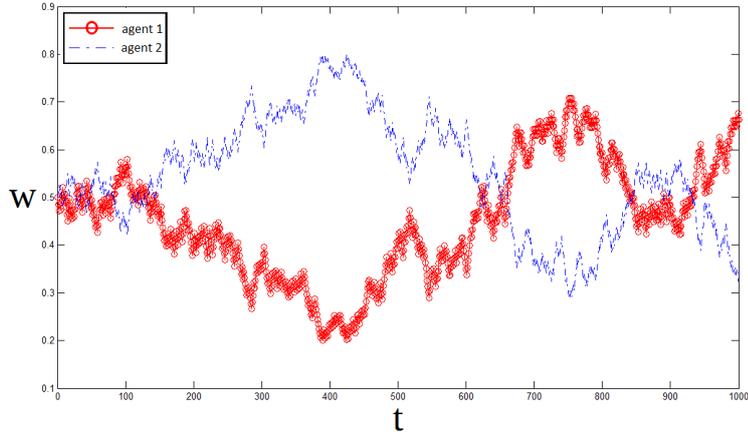
Figure 3: Simulation of the evolution of wealth shares for 1000 periods.

Geometric Random Walk   Consider now the case of a geometric random walk example of Section 2.1 with $r = g_d/g_u = 0.2$, $\pi = (0.5, 0.5)$. As before we can use our conditions to establish what happens for all the possible combinations of beliefs when $\delta = 0.8$, left panel of Figure 4, and for all possible combinations of δ and π^1 when $\pi^2 = (0.6, 0.4)$, right panel of Figure 4.

As one can notice from the comparison between Figure 2 and Figure 4, the plots are quite similar, the only difference is that with a Geometric Random Walk the areas of long-run heterogeneity slightly increase. Moreover, with non-diagonal assets there exist cases of long-run heterogeneity even in the limit of $\delta = 0$.
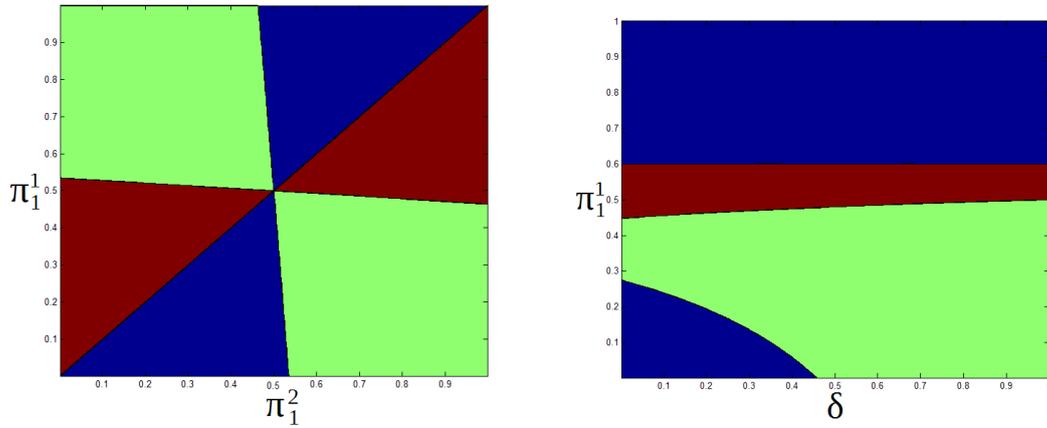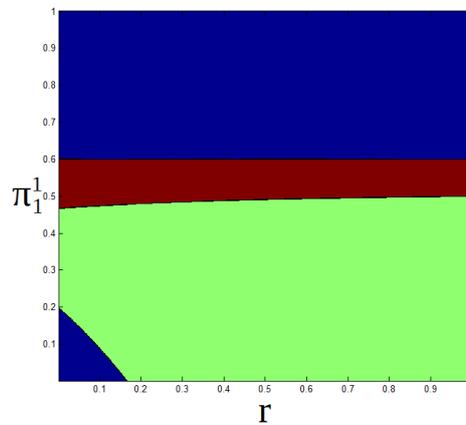
Figure 4: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. $r = 0.2$

In this example, the selection process is less sharp since investing in the second asset is a quite safe way to survive. To shed light, we investigate what happens when we change the value of r. In Figure 5 we plot the areas of dominance and survival for all the possible combinations of beliefs of agent 1 and the parameter r when $\delta = 0.5$ and $\pi^2 = (0.6, 0.4)$.

Figure 5: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. $\delta = 0.5$

We continue by exploring the outcomes of market selection with complete an incomplete market. For this purpose, we take the market structure with two

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assets and three states of the world showed in Section 2.1 with $r_u = r_m = 0.2$. We also choose $\pi = (1/3, 1/3, 1/3)$, $\delta = 0.5$, $\pi^1 = (3\pi^1_{1,2}/4, \pi^1_{1,2}/4, 1 - \pi^1_{1,2})$ and $\pi^2 = (\pi^2_{1,2}/4, 3\pi^2_{1,2}/4, 1 - \pi^2_{1,2})$.
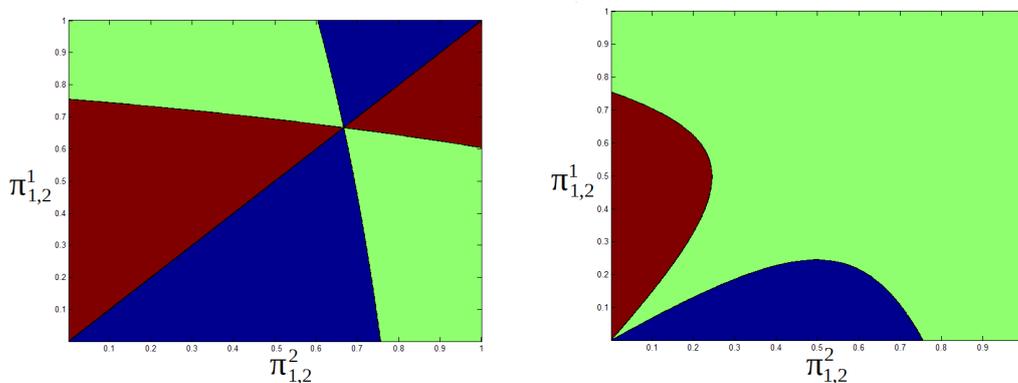

Figure 6: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. Left plot: incomplete markets. Right plot: complete markets.

In the left plot one can observe how the shape of the areas of dominance and survival is similar to those in the first plot of figure 4, the only difference is that now the truth corresponds to the sum of probabilities π_1 and π_2, hence $2/3$. Instead in the second plot (complete markets) the situation is dramatically different: the area of long-run heterogeneity occupies a large portion of the space. That is, completing the market offers a way for agents with anti-correlated beliefs for states 1 and 2 to speculate against each-others. This ends up in increasing the combinations of beliefs that produce long-run heterogeneity.

Obviously the role played by the choice of the belief structure is fundamental. To this regard consider a slightly different belief structure only for agent 1: $\pi^1 = (\pi^1_{1,2}/2, \pi^1_{1,2}/2, 1 - \pi^1_{1,2})$. In this situation agent 1 should be favored since he can distribute more evenly (hence in accordance with the underlying stochastic process) his wealth among assets. Indeed when $\pi^1_{1,2} = 2/3$, he plays the Generalized Kelly rule with correct beliefs, hence he dominates in the market no matter what is the value of $\pi^2_{1,2}$. Figure 7 confirms the intuition, the area where agent 1 dominates increases and occupies a large portion of the plot.
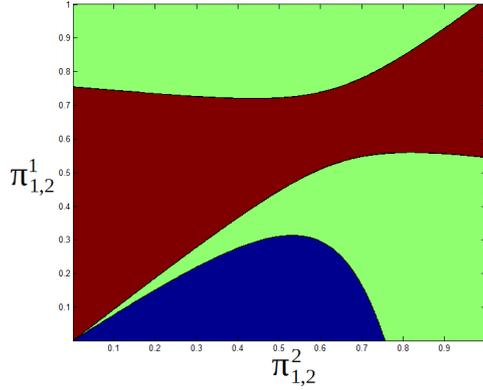
Figure 7: Areas of dominance and survival. Red: agent 1 dominates, blue: agent 2 dominates, green: long-run heterogeneity. Complete markets.

5.3   3-Generalized Kelly agents economy

In this section, we use our criteria to investigate the market selection outcomes in an economy with three states of the world, complete markets, and three Generalized Kelly agents . Assume $D = I_3$ and $\pi = (1/3, 1/3, 1/3)$. We fix $\pi^1 = (\pi_1^1, (1 - \pi_1^1)/2, (1 - \pi_1^1)/2)$, $\pi^2 = (1/4, 1/2, 1/4)$ and $\pi^3 = (1/4, 1/4, 1/2)$. As we have done in some previous examples we plot the outcome of our criteria for all possible combinations of δ and π_1^1.
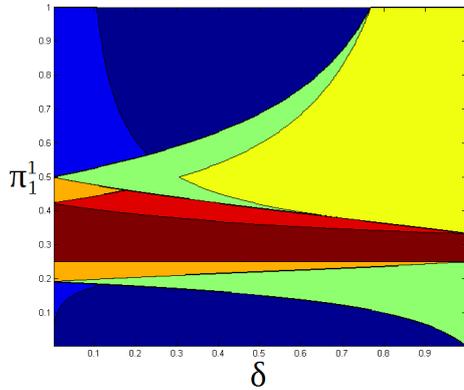

Figure 8: Areas of dominance and survival. Red: agent 1 survives, Dark Red: agent 1 dominates, Blue: agent 1 vanishes, Dark Blue: only agent 2 and agent 3 survive, Green: at least two agents survive, Yellow: all three agents survive, Orange: unknown.

Compared with two-agent economies, in three-agent economy our sufficient

conditions are not tight. Thus, there exists combinations of π_1^1 and δ for which we cannot characterize market selection long-run outcomes. Consider the red regions, for these combinations of π_1^1 and δ agent 1 survives, indeed choosing the group $J = \{1\}$ we have $\underline{\mu}^J(0) > 0$. In the darkest one we also have $\underline{\mu}^J(1) > 0$ so that agent 1 dominates. Notice how that region covers the area around the truth. In the blue areas, instead, it is $\overline{\mu}^J(0) < 0$ and $\overline{\mu}^J(1) < 0$ which implies that agent 1 vanishes. This basically means that the group $I \setminus J = \{2, 3\}$ dominates, however two possible situations are compatible with this outcome: both 2 and 3 survive or one of the two dominates. To see that consider now $J' = \{2\}$ and $J'' = \{3\}$, in the dark blue regions we have $\underline{\mu}^{J'}(0) > 0$ and $\underline{\mu}^{J''}(0) > 0$, thus in those areas we have that both agent 2 and agent 3 survive, while in the light blue regions nothing more can be said.

In the yellow area we have $\underline{\mu}^J(0) > 0$, $\underline{\mu}^{J'}(0) > 0$ and $\underline{\mu}^{J''}(0) > 0$, hence we know that all the agents survive. Again, notice how the survival of all agents is ensured by large enough values of δ and quite anti-correlated beliefs. In the green areas we only know that $\overline{\mu}^J(1) < 0$, $\overline{\mu}^{J'}(1) < 0$ and $\overline{\mu}^{J''}(1) < 0$, which implies that no one dominates. This is equivalent to say that at least two agents survive, but we do not know who these two agents are. Finally, there also exists a region, the orange one, where our criteria are not able to say anything about the market selection outcome.

From the previous figure, if we set $\pi^1 = (0.6, 0.2, 0.2)$, $\delta = 0.8$ and $w_0^1 = w_0^2 = w_0^3 = 1/3$ then all agents survive. Simulating the model for a particular realization of the underlying stochastic process we have an example of the dynamics of agents' relative wealth.
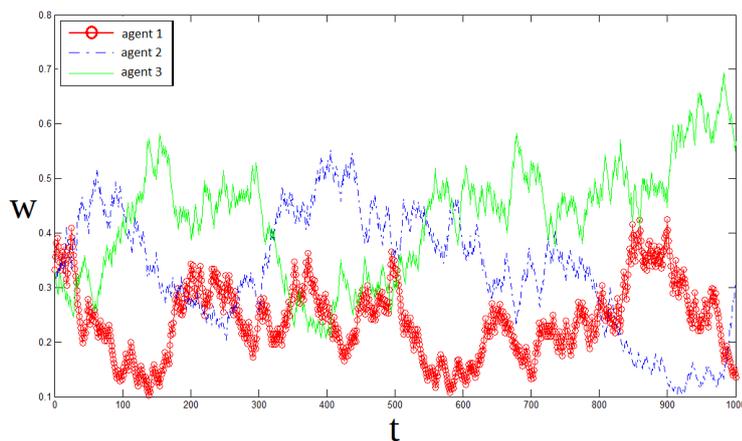

Figure 9: Simulation of the evolution of wealth shares for 1000 periods.

Figure 9 illustrates long-run heterogeneity in an economy with more than two agents. Around the period $t = 150$, and again at $t \approx 550$, agent 1 has lost almost all his wealth. However, sooner or later, he had still a substantial share of the aggregate output. Notice also that agents relative wealth have not a stable ordering.

6 Conclusion

In this paper we investigate the MSH in an exchange economy with long-lived assets as in Lucas (1978) where agents have homogeneous discount factors, heterogeneous beliefs, and employ Generalized Kelly rules. In this framework Evstigneev et al. (2008) proves that if there exists an agent with correct beliefs then she gains all the wealth in the long-run. Asset prices converge to those in the original Lucas' model where the representative agent has logarithmic preferences. We focus on an economy where agents have heterogeneous, and not correct, beliefs, and extended provide sufficient conditions for an agent to have a positive, null, or unitary fraction of wealth in the long-run when no agent in the economy has correct beliefs.

Our main finding is that there exist initial distributions of beliefs such that beliefs heterogeneity, rather then convergence to the most accurate beliefs, is the long-run outcome. Moreover this result is robust to local perturbation of beliefs. In order to investigate those issues we provided some examples that clearly show how the survival of heterogeneous agents is a robust and generic phenomenon. We show that our results are due to the non-optimality of fixed-mix rules in the limit of an agent having a negligible share of the total wealth. Non-optimality of beliefs and non-optimality of the rules balance each other and lead to survival instead than vanishing.

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A   Proofs of Theorems and Lemmas

A.1   Proof of Lemma 3.1

Proof. Let $\bar{\alpha} = \max_{i \in I}\{\alpha_h^i\}$ and $\bar{\delta}_h = \max_{i \in I}\{\sum_{k=1}^{K} \alpha_k^i\}$. From **R1** and **R2** it is immediate to see that $0 < A_{h,h} < \bar{\alpha} < 1$ and $0 < \sum_{k=1}^{K} A_{k,h} < \bar{\delta} < 1$. Then

$$\sum_{k=1}^{K} (I_K - A)_{k,h} = |I_K - A|_{h,h} - \sum_{k=1,k\neq h}^{K} |I_K - A|_{k,h}$$

but at the same time $\sum_{k=1}^{K} (I_K - A)_{k,h} = 1 - \sum_{k=1}^{K} A_{k,h} > 1 - \bar{\delta}_h > 0$ so that the matrix $I_k - A$ is column strictly diagonally dominant and, by the Levy-Desplanques theorem (Taussky, 1949), it is invertible. □

A.2   Proof of Proposition 3.1

Proof. The first part of the statement follows from Lemma 3.1 and from the derivation in the text before the proposition.

Regarding the absence of arbitrages consider the following. According to Stiemke's Lemma, if there exists a vector $q \in \mathbb{R}_{++}^S$ such that $R(\boldsymbol{W}; \alpha, D)q = P$ with $P = \sum_{i=1}^{I} \alpha^i W^i$ then $R(\boldsymbol{W}; \alpha, D)$ does not admit arbitrage. Using the definition of $R(\boldsymbol{W}; \alpha, D)$, the previous condition reads $Dq = (I_K A(\boldsymbol{W}; \alpha))P$. Computing the kth component of $(I_K A(\boldsymbol{W}; \alpha))P$ one finds $\sum_{i=1}^{I}[(1^i)\delta^i W^i i]x_k^i$. It follows that if for all $i \in I$ x^i belongs to the interior of the convex cone generated by the columns of D, also the vector $(I_K - A(\alpha, \boldsymbol{W}))P_t$ belongs to it and there are no arbitrages. Provided agents beliefs satisfy **R2** generalized Kelly rule do belong to the interior of the convex cone generated by the columns of D, so that the statement is proved. □

A.3   Proof of Lemma 4.1

See the proof of Lemma A.4.

A.4   Proof of Propositions 4.1 and Corollary 4.1

The present proof starts by showing that the stochastic process driving the dynamics of relative wealth satisfies the conditions of applicability of a set of general results reported in Bottazzi and Dindo (2015). Then it discusses how, in turn, those results imply the statements.

Consider the variable $z_t^J = \log w_t^J/(1 - w_t^J)$ such that $z_t^J = z_{t-1}^J + g^J(\sigma_t)$, with $g^J(\sigma_t) = \log G^J(\sigma_t)$ and

$$G^J(\sigma_t) = \frac{\sum_{k=1}^K r_{k,s_t}(\boldsymbol{w}_{t-1}; x, \delta, D) x_k^J(\boldsymbol{w}_{t-1}; x)/p_k(\boldsymbol{w}_{t-1}; x)}{\sum_{k=1}^K r_{k,s_t}(\boldsymbol{w}_{t-1}; x, \delta, D) x_k^{-J}(\boldsymbol{w}_{t-1}; x)/p_k(\boldsymbol{w}_{t-1}; x)} \ .$$

One has the following

Lemma A.1. *The process z_t^J has bounded increments, that is, there exists an $B \in \mathbb{R}$ such that $|z_t^J - z_{t-1}^J| < B$ P-a.s..*

Proof. By **R3** there exists a small enough $\varepsilon > 0$ such that $\varepsilon \leq x_k^J \leq 1 - \varepsilon \ \forall i, k$. Then for any asset k and any time t

$$\varepsilon \leq p_k(w_t; x) \leq 1 - \varepsilon$$

and for any agent i and state s

$$\frac{\varepsilon}{1 - \varepsilon} \leq \sum_{k=1}^K r_{k,s}(\boldsymbol{w}_{t-1}; x, \delta, D) \frac{x_k^J}{p_k(\boldsymbol{w}_{t-1}; x)} \leq \frac{1 - \varepsilon}{\varepsilon} \ .$$

By direct algebraic substitution it is straightforward to verify that

$$2 \log \frac{\varepsilon}{1 - \varepsilon} \leq z_t^J - z_{t-1}^J \leq 2 \log \frac{1 - \varepsilon}{\varepsilon}$$

and the statement is proven. $\square$

If the set of rules used by the agents belonging to group J and those used by the agents in $I \setminus J$ are non-overlapping, then the process z_t^J does not posses any deterministic fixed point.

Lemma A.2. *If the set of rules are not overlapping, **R4**, and if there are no redundant assets, **D4**, then z_t^J does not posses any deterministic fixed point, that is $\nexists z$ s.t. $P(z_t^J = z | z_{t'}^J = z) = 1 \ \forall t > t'$.*

Proof. Suppose that z is a deterministic fixed point and at a certain time $t - 1$ it is $z_{t-1}^J = z$. Then, by definition, it holds that $z_t^J - z_{t-1}^J = 0$ for all the possible states of the world $s = 1, 2, ..., S$, so that

$$\sum_{k=1}^K r_{k,s}(\boldsymbol{w}_{t-1}; x, \delta, D) \left(\beta_k^J(\boldsymbol{w}_{t-1}; x) - \beta_k^{-J}(\boldsymbol{w}_{t-1}; x) \right) = 0 \quad \forall s = 1, 2, ..., S \ .$$

that is

$$\left(\beta^J(\boldsymbol{w}_{t-1}; x) - \beta^{-J}(\boldsymbol{w}_{t-1}; x) \right) \left((I - \delta A(\boldsymbol{w}_{t-1}; x))^{-1} D \right) = \boldsymbol{0} \ ,$$

The trivial solution $\beta^J = \beta^{-J}$ is excluded by **R4** and according to Proposition 3.1 the kernel of $(I - \delta A(\boldsymbol{w}_{t-1}, x))^{-1} D$ is zero, implying that the system of equations has no solution and the statement is proven. $\square$

The absence of deterministic fixed points also implies that the process z_t^J has always a finite probability to have a finite jump.

Lemma A.3. *If the set of rules are not overlapping,* **R4**, *and if there are no redundant assets,* **D4**, *then the process z_t^J has a positive probability of having non-zero increments, that is, there exists a $\gamma > 0$ such that*

$$\text{Prob}\left\{|z_t^J - z_{t-1}^J| > \gamma | \mathfrak{I}_{t-1}\right\} > \gamma. \tag{21}$$

Proof. Notice that G^J depends on history σ_t thorough the wealth distribution w_t and the last realizes state s_t Given the distribution $\boldsymbol{m} \in \Delta^I$ define

$$\bar{G}^J(\sigma_t) = \max_{s=1,\ldots,S}\{|G^J(\boldsymbol{w}, s_t)|\},$$

which, being the upper envelope of continuous functions, is a continuous function on the compact set Δ^I. Then, by the Weierstrass theorem, it has a minimum $\underline{G}$. Moreover it is $\underline{G} > 0$ because, otherwise, z_t^J would have a deterministic fixed point, which is not possible by Lemma A.2. Then

$$\text{Prob}\left\{|z_t^i - z_{t-1}^i| \geq \underline{g} \,|\, \mathfrak{I}_{t-1}\right\} \geq \rho = \min\{\pi_1, \ldots, \pi_S\}.$$

and by taking $\gamma = \min\{\underline{g}, \rho\}/2$ the assertion is proved. □

The lack of weak arbitrage implies, in turn, that the process cannot have a deterministic drift.

Lemma A.4. *If the set of rules are not overlapping,* **R4**, *if there are no redundant assets,* **D4** *and if the process does not admit weak arbitrage, then $G^J((\sigma_{t-1}, s')) > 1$ for at least one s' and $G^J((\sigma_{t-1}, s'')) < 1$ for at least one s''.*

Proof. Let us consider the first statement. If it is wrong then

$$\sum_{k=1}^K (\beta_k^J - \beta_k^{-J}) \, r_{k,s}(\boldsymbol{w}_{t-1}; x, \delta, D) \geq 0 \quad \forall s$$

and since the process does not admit any deterministic fixed point (c.f. Lemma A.2), the inequality is strict for some s'. For construction it is

$$\sum_{k=1}^K (\beta_k^J - \beta_k^{-J}) p_k(\boldsymbol{w}_{t-1}; x) = 0,$$

so that $\beta^J(\boldsymbol{w}_{t-1}; x) - \beta^{-J}(\boldsymbol{w}_{t-1}; x)$ would be a weak arbitrage, which contradicts the hypotheses. The second statement is proved along the same lines. □

Lemma A.3 and Lemma A.4 together imply that the process z_t^J jumps to the left and to the right of a finite amount with a finite probability.

We can now move to prove the statements. Fist of all notice that $E[g(\sigma_t)] = \underline{\mu}^J(w_t)$. Concerning the first point, if $\underline{\mu}^J(0) > 0$, given the continuity of the function, then there is a neighborhood of $-\infty$ in which, almost surely, $\underline{\mu}^J(w_t) > \underline{\mu}^J(0) > 0$. Since the process z_t^J has bounded increments, we can apply the following Theorem from Bottazzi and Dindo (2015)

Theorem A.1. *Consider a finite increments process x_t for which $|x_{t+1} - x_t| < B$ P-a.s.. If there exist $M > B$ and $\epsilon > 0$ such that, P-a.s., $E[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ for all $x < -M$, then $\text{Prob}\{\limsup_{t \to \infty} x_t > -M\} = 1$.*

As a result $\text{Prob}\{\limsup_{t \to \infty} z_t^J > -\infty\} = 1$ and the statement follows. The same applies to the second statement.

Concerning the further three points, in this case by hypothesis the process z_t^J has finite increments (in both directions). The condition $\underline{\mu}^J(0) > 0$ and $\underline{\mu}^J(1) > 0$ implies that $\mu^J(w) > 0$ for sufficiently small and sufficiently large values of w. In this case, we can use this other Theorem from Bottazzi and Dindo (2015)

Theorem A.2. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. and such that for all t $\text{Prob}\{x_{t+1} - x_t > \gamma|\Im_t\} > \gamma$ for some $\gamma > 0$. If there exist $M > B$ and $\epsilon > 0$ such that, P-a.s., $E[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ for all $x > M$ and $E[x_{t+1}|x_t = y, \Im_t] > x + \epsilon$ for all $x < -M$, then the process is transient and $\text{Prob}\{\lim_{t \to \infty} x_t = +\infty\} = 1$.*

As a result, we can infer that $\text{Prob}\{\lim_{t \to \infty} z_t^J = +\infty\} = 1$ so that the considered group dominates. Conversely, from the condition $\overline{\mu}^J(0) < 0$ and $\overline{\mu}^J(1) < 0$ we have that $\mu^J(w) < 0$ for sufficiently small and large values of w. In this case, using Theorem A.2 applied to the process $-z_t^J$, we can infer that $\text{Prob}\{\lim_{t \to \infty} z_t^J = -\infty\} = 1$ and the group vanishes. The last point is a trivial consequence of the second point and the absence of deterministic fixed point from Lemma A.2.

A.5 Proof of Proposition 4.2

Proof. From the definition of conditional drift

$$\mu(0) = \sum_{s=1}^{S} \pi_s \log\left(\delta + (1-\delta)\sum_{k=1}^{K} d_{k,s}\frac{x_k^1}{x_k^2}\right),$$

$$\mu(1) = -\sum_{s=1}^{S} \pi_s \log\left(\delta + (1-\delta)\sum_{k=1}^{K} d_{k,s}\frac{x_k^2}{x_k^1}\right).$$

Exploiting the concavity of the logarithmic function and considering that $0 \leq d_{k,s} \leq 1$ for all s, k and that $\sum_{k=1}^{K} d_{k,s} = 1$ for all s, we can see that

$$\mu(0) > (1-\delta) \sum_{s=1}^{S} \pi_s \log \left(\sum_{k=1}^{K} d_{k,s} \frac{x_k^1}{x_k^2} \right) \geq (1-\delta) \sum_{s=1}^{S} \pi_s d_{k,s} \sum_{k=1}^{K} \log \left(\frac{x_k^1}{x_k^2} \right) .$$

At the same time

$$\mu(1) < -(1-\delta) \sum_{s=1}^{S} \pi_s \log \left(\sum_{k=1}^{K} d_{k,s} \frac{x_k^2}{x_k^1} \right) \leq (1-\delta) \sum_{s=1}^{S} \pi_s d_{k,s} \sum_{k=1}^{K} \log \left(\frac{x_k^1}{x_k^2} \right) .$$

Putting together the two inequalities proves that $\mu(0) > \mu(1)$. The second part of the statement follows from the inequality and from Corollary 4.1. □

A.6 Proof of Corollary 4.2

The proof of the survival of agent 1 follows from inequality (20) and Corollary 4.1. When agent 1 beliefs are correct, so that $x^1 = x^*$, she also dominates since, exploiting the strict convexity of $-\log(\cdot)$,

$$\mu(1) = \sum_s \pi_s \left(-\log \left(\delta + (1-\delta) \sum_{k=1}^{K} d_{k,s} \frac{x_k^2}{x_k^*} \right) \right) >$$

$$-\log \left(\delta + (1-\delta) \sum_{k=1}^{K} \frac{x_k^2}{x_k^*} \sum_s \pi_s d_{k,s} \right) = -\log \left(\delta + (1-\delta) \sum_{k=1}^{K} \frac{x_k^2}{x_k^*} x_k^* \right) = 0 .$$

A.7 Proof of Proposition 4.3

Notice that the conditional drift in the case of Generalized Kelly agents is a continuous function of agents' belief, that is

$$\mu(w^1) = \mu(w^1; \pi^1, \pi^2) .$$

Choose $\theta = (\mu(0; \bar{\pi}^1, \bar{\pi}^2) - (1-\delta)\Delta_{x^*}(x^2 || x^1))/2$, by continuity there exist an $\varepsilon_1 > 0$ and an $\varepsilon_2 > 0$ such that for all $\pi^1, \pi^2 \in \Delta^2$ with Euclidian distances $\|\pi^1 - \bar{\pi}^1\| < \varepsilon_1$ and $\|\pi^2 - \bar{\pi}^2\| < \varepsilon_2$ it is $|\mu(0; \pi^1, \pi^2) - \mu(0; \bar{\pi}^1, \bar{\pi}^2)| < \theta$. Hence

$$\mu(0; \pi^1, \pi^2) > (1-\delta)\Delta_{x^*}(x^2 || x^1) = 0 .$$

In a similar way, choose $\theta' = ((1-\delta)\Delta_{x^*}(x^2 || x^1) - \mu(1; \bar{\pi}^1, \bar{\pi}^2))/2$, by continuity there exist an $\varepsilon_1' > 0$ and an $\varepsilon_2' > 0$ such that for all $\pi^1, \pi^2 \in \Delta^2$ with Euclidian distances $\|\pi^1 - \bar{\pi}^1\| < \varepsilon_1'$ and $\|\pi^2 - \bar{\pi}^2\| < \varepsilon_2'$ it is $|\mu(1; \pi^1, \pi^2) - \mu(1; \bar{\pi}^1, \bar{\pi}^2)| < \theta'$. Hence

$$\mu(1; \pi^1, \pi^2) < (1-\delta)\Delta_{x^*}(x^2 || x^1) = 0 .$$

In order to complete the proof, choose $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_1', \varepsilon_2, \varepsilon_2'\}/\sqrt{S}$.

A.8 Proof of Proposition 4.4

The statement follows from the properties of the function $D(x^*||x) : \Delta^K \to \mathbb{R}_+$, $x \mapsto D(x^*||x)$. In particular it is a continuous strictly convex function with a minimum equal to zero in $x = x^*$. Thus it is defined over the compact set $\partial(\bar{\Delta})$ and a minimum over this set exists because of the Weirstrass theorem. The strict convexity of $D(x^*||x)$ implies that it is also strictly quasi convex. This property together with the fact that $x^* \in \bar{\Delta}$ implies $\{x : D(x^*||x) < \mathcal{K}\} \subseteq \bar{\Delta}$. Hence, it is possible to choose a $\pi^1 \neq \pi$ such that $D(x^*||x^1) = m < \mathcal{K} - \epsilon$ with $\epsilon > 0$ and small enough. Then, one can easily define the set $\Pi = \{\pi' : \pi' \in \Delta_+^S, D(x^*||x') = m\}$ which has always at least two elements[26]. Choosing x^1 and x^2 such that $\pi^1, \pi^2 \in \Pi$ it is $\Delta_{x^*}(x^2||x^1) = 0$.

A.9 Proof of Proposition 5.1

An asset market economy with agents trading according to generalized Kelly rules and agents maximizing expected log-utilities under effective beliefs has, by construction, the same relative wealth dynamics. When markets are dynamically complete and agents maximize an expected log-utility, there is no loss of generality, in assuming that they are trading all possible contingent commodities in date zero. In fact, all assets structure, as long as they are complete, allow agents to achieve the same consumption allocation, so that the relative wealth dynamics does not on the asset structure. Under time-zero trading it is well know that agents allocate in each commodity a fraction of wealth proportional to its likelihood. In a two-agent economy, the relative wealth dynamics can thus be re-written in this form

$$\frac{w_{t+1}^1(\sigma_t, s_t)}{w_{t+1}^2(\sigma_t, s_t)} = \frac{\bar{\pi}_{s_t}^1(\boldsymbol{w}_t; \delta, D)}{\bar{\pi}_{s_t}^2(\boldsymbol{w}_t; \delta, D)} \frac{w_t^1(\sigma_t)}{w_t^2(\sigma_t)} \quad \forall \sigma_t, s_t, t \,.$$

Applying Theorem A.1 to the log of this process, and remembering that the relative wealth dynamics just found is the same as the original one with long-lived assets and generalized Kelly traders, leads to the result.

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[26]The worst case scenario is $S = 2$ and in this case the cardinality of Π' is two.